

Bayesian Approximation Techniques for Kumaraswamy Distribution

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Abstract

The present study is concerned with the estimation of shape parameter of Kumaraswamy Distribution using various Bayesian approximation techniques like normal approximation, Lindley's Approximation, Tierney and Kadane (T-K) Approximation. Different informative and non-informative priors are used to obtain the Baye's estimate of parameter of Kumaraswamy Distributions under different approximation techniques. For comparing the efficiency of the obtained results a simulation study is carried out using R-software.

Keywords: Bayesian Estimation, Prior Distribution, Normal Approximation, Lindley's Approximation, T-K Approximation.

1. Introduction:

Kumaraswamy (1980) developed a general probability density function for double bounded random processes, which is known as Kumaraswamy's distribution. This distribution is applicable to many natural phenomena whose outcomes have lower and upper bounds, such as the heights of individuals, scores obtained on a test, atmospheric temperatures, hydrological data, etc. The probability distribution function and the cumulative distribution function of Kumaraswamy distribution is given as

$$f(x;\eta;\gamma)=\eta\,\gamma\, x^{\eta-1}(1-x^{\eta})^{\gamma-1} \quad ;0<x<1; \tag{1.1}$$

$$F(x)=1-(1-x^{\eta})^{\gamma-1}$$

where $\eta,\gamma>0$ are the shape parameters of the distribution. The pdf of kumaraswamy distribution can be unimodal, increasing, decreasing or constant, depending on the values of the parameters. The Kumaraswamy distribution is very similar to the Beta distribution, but has the important advantage of an invertible closed form cumulative distribution function. Nadarajah (2008) has pointed out that Kumaraswamy's distribution is a special case of the three parameter beta distribution. Jones (2009) explored the genesis of the Kumraswamy distribution and made some similarities and differences between the beta and Kumaraswamy distributions. It has many of the same properties as the beta distribution but has some advantages in terms of tractability. Sundar and Subbiah (1989), Fletcher and Ponnambalam (1996), Seifi et al. (2000), Ponnambalam et al. (2001) considered this distribution for interests in hydrology and related areas.

Gholizadeh et al. (2011a, 2011b) studied classical and Bayesian estimators of the kumaraswamy distribution using grouped and un-grouped data, also studied Bayesian and non-Bayesian estimators for the shape parameter, reliability and failure rate functions of the Kumaraswamy distribution in the cases of progressively type II censored samples. Mostafa et al. (2014) produced a study in Estimation for parameters of the Kumaraswamy distribution based on general progressive type II censoring. These estimates are derived using the maximum likelihood and Bayesian approaches. Mustafa et al. (2012a, 2012b) reviewed some results that have been derived on record values for based on *m* records from Kumaraswamy's distribution and derived estimators for the two parameters using the maximum likelihood and Bayesian approaches and also studied classical and Bayesian estimation of *P(Y < X)* for it.

The kumaraswamy distribution has not been discussed in detail under the Bayesian approach. Our present study aims to obtain the Bayesian estimators for the shape parameter of the kumaraswamy distribution based on Bayesian approximation techniques. A simulation study has also been conducted along with concluding remarks.

2. Normal Approximation:

If the posterior distribution $P(\gamma \mid x)$ is unimodal and roughly symmetric, it is convenient to approximate it by a normal distribution centered at the mode, yielding the approximation

$$P(\gamma \mid x) \sim N\left(\hat{\gamma}, [I(\hat{\gamma})]^{-1}\right)$$

where $$I(\hat{\gamma}) = -\frac{\partial^2 \log P(\gamma \mid y)}{\partial \gamma^2} \tag{2.1}$$

If the mode, $\hat{\gamma}$ is in the interior parameter space, then $I(\gamma)$ is positive; if $\hat{\gamma}$ is a vector parameter, then $I(\gamma)$ is a matrix.

Some good sources on the topic is provided by Ahmad et.al (2007, 2011) discussed Bayesian analysis of exponential distribution and gamma distribution using normal and Laplace approximations. Sultan et al. (2015) obtained the Baye's estimates under different informative and non-informative priors of shape parameter of Topp-Leone Distribution using Bayesian approximation techniques

In our study the normal approximations of kumaraswamy distribution under different priors is be obtained as under:

The likelihood function of (1.1) for a sample of size n is given as

$$L(\underline{x} \mid \gamma) \propto (\gamma)^n \, e^{-\gamma \sum_{i=1}^{n} \ln(1-x_i^{\eta})^{-1}} \tag{2.2}$$

Under Jeffrey's prior $g(\gamma) \propto 1/\gamma$, the posterior distribution for γ is as

$$P(\gamma \mid x) \propto \gamma^{n-1} e^{-\gamma T} \text{ where } T = -\sum_{i=1}^{n} \ln(1-x^{\eta}) \tag{2.3}$$

Log posterior is $\ln P(\gamma \mid x) = \ln \text{constant} + (n-1)\ln \gamma - \gamma T$

The first derivative is

$$\frac{\partial \ln P(\gamma \mid x)}{\partial \gamma} = \frac{n-1}{\gamma} - T$$

from which the posterior mode is obtained as $\hat{\gamma} = \dfrac{n-1}{T}$

The second derivative of the log-posterior density is given as

$$\frac{\partial^2 \ln P(\gamma \mid x)}{\partial \gamma^2} = -\frac{n-1}{\gamma^2}$$

Therefore, negative of Hessian $I(\hat{\gamma}) = -\dfrac{\partial^2 \ln P(\gamma \mid x)}{\partial \gamma^2} = \dfrac{T^2}{n-1} \Rightarrow [I(\hat{\gamma})]^{-1} = \dfrac{n-1}{T^2}$

Thus, the posterior distribution can be approximated as

$$P(\gamma \mid x) \sim N\left(\frac{n-1}{T}; \frac{n-1}{T^2}\right) \tag{2.4}$$

Under modified Jeffrey's prior $g(\gamma) \propto \left(\frac{1}{\sqrt{\gamma^3}}\right)$, the posterior distribution for γ is as

$$P(\gamma \mid x) \propto \gamma^{n-3/2} e^{-\gamma T} \tag{2.5}$$

Log posterior $\ln P(\gamma \mid x) = \ln \text{constant} + (n - 3/2)\ln \gamma - \gamma T$

The first derivative is

$$\frac{\partial \ln P(\gamma \mid x)}{\partial \gamma} = \frac{n-3/2}{\gamma} - T$$

from which the posterior mode is obtained as $\hat{\gamma} = \frac{n-3/2}{T}$

The second derivative of the log-posterior density is given as

$$\frac{\partial^2 \ln P(\gamma \mid x)}{\partial \gamma^2} = -\frac{n-3/2}{\gamma^2}$$

Therefore, negative of Hessian $I(\lambda\hat{\gamma}) = -\frac{\partial^2 \ln P(\gamma \mid x)}{\partial \gamma^2} = \frac{n-3/2}{\hat{\gamma}^2} = \frac{T^2}{n-3/2}$

$$[I(\hat{\gamma})]^{-1} = \frac{n-3/2}{T^2}$$

Thus, the posterior distribution can be approximated as

$$P(\hat{\gamma} \mid x) \sim N\left(\frac{n-3/2}{T}; \frac{n-3/2}{T^2}\right) \tag{2.6}$$

Under gamma prior $g(\gamma) \propto \gamma^{a-1} e^{-b\gamma}$; $a, b > 0; \gamma > 0$ where a, b are the known hyper parameters. The posterior distribution for γ is as

$$P(\gamma \mid x) \propto \gamma^{n+a-1} e^{-\gamma(b+T)} \tag{2.7}$$

Log posterior is $\ln P(\gamma \mid x) = \ln \text{constant} + (n + a - 1)\ln \gamma - \gamma(b + T)$

The first derivative is

$$\frac{\partial \ln P(\gamma \mid x)}{\partial \gamma} = \frac{n+a-1}{\gamma} - (b+T)$$

$$\therefore \text{ posterior mode } \hat{\gamma} = \frac{n+a-1}{(b+T)}$$

The second order derivative of the log posterior density is given as

$$\frac{\partial^2 \ln P(\gamma \mid x)}{\partial \gamma^2} = -\frac{n+a-1}{\gamma^2}$$

Therefore negative of Hessian $I(\hat{\gamma}) = -\frac{\partial^2 \ln P(\gamma \mid x)}{\partial \gamma^2} = \frac{[T+b]^2}{n+a-1}$

$$[I(\hat{\gamma})]^{-1} = \frac{n+a-1}{[T+b]^2}$$

Thus, the posterior distribution can be approximated as

$$P(\gamma \mid x) \sim N\left(\frac{n+a-1}{T+b}; \frac{n+a-1}{[T+b]^2}\right) \quad (2.8)$$

Under the Inverse Levy prior $g(\gamma) \propto \gamma^{-1/2} e^{-\frac{\gamma r}{2}}; r > 0; \gamma > 0$, where *r* is the known hyper parameter, thus the posterior distribution for γ is as

$$P(\gamma \mid x) \propto \gamma^{n-1/2} e^{-\gamma(T+\frac{r}{2})} \quad (2.9)$$

Log posterior is $\ln P(\gamma \mid x) = \ln \text{constant} + \left(n - \frac{1}{2}\right)\ln \gamma - \gamma\left(T + \frac{r}{2}\right)$

The first derivative is

$$\frac{\partial \ln P(\gamma \mid x)}{\partial \gamma} = \frac{n-1/2}{\gamma} - \left(T + \frac{r}{2}\right)$$

$$\therefore \text{ posteriormode } \hat{\gamma} = \frac{n-1/2}{(T+r/2)}$$

The second order derivative of the log posterior density is given as

$$\frac{\partial^2 \ln P(\gamma \mid x)}{\partial \gamma^2} = -\frac{n-1/2}{\gamma^2}$$

Therefore negative of Hessian $I(\hat{\gamma}) = -\frac{\partial^2 \ln P(\gamma \mid x)}{\partial \gamma^2} = \frac{[T+r/2]^2}{n-1/2}$

$$[I(\hat{\gamma})]^{-1} = \frac{n-1/2}{[T+r/2]^2}$$

Thus, the posterior distribution can be approximated as

$$P(\lambda \mid x) \sim N\left(\frac{n-1/2}{T+r/2}; \frac{n-1/2}{[T+r/2]^2}\right) \quad (2.10)$$

3. Lindley's Approximation:

Sometimes, the integrals appearing in Bayesian estimation can't be reduced to closed form and it becomes tedious to evaluate of the posterior expectation for obtaining the Baye's estimators. Thus, we propose the use of Lindley's approximation method (1980) for obtaining Baye's estimates. Lindley developed an asymptotic approximation to the ratio

$$I = \frac{\int_{\Omega} h(\gamma) e^{L(\gamma)+U(\gamma)} d\gamma}{\int_{\Omega} e^{L(\gamma)+U(\gamma)} d\gamma} \quad (3.1)$$

where $\gamma = (\gamma_1,....,\gamma_m), L(\gamma)$ is the logarithmic of likelihood function, $h(\gamma)$ & $U(\gamma)$ are arbitrary functions of γ & Ω represents the space range of γ. Thus $I = E\{h(\gamma) | x\}$ can be evaluated as

$$I \cong h(\hat{\gamma}) + \frac{1}{2}[h_2(\hat{\gamma}) + 2h_1(\hat{\gamma})U'(\hat{\gamma})]\hat{\phi}^2 + \frac{1}{2}[L_3(\hat{\gamma})h_1(\hat{\gamma})](\hat{\phi}^2)^2 \quad (3.2)$$

In particular, if $h(\gamma) = \theta; h'(\gamma) = 1; h''(\gamma) = 0$

Thus $E(\gamma | x) = \hat{\gamma} + \frac{\partial}{\partial \lambda}(U(\hat{\gamma}))\hat{\phi}^2 + \left(\frac{1}{2}L_3(\hat{\gamma})\right)(\hat{\phi}^2)^2$ (3.3)

where $\hat{\phi}^2 = (-L_2(\hat{\gamma}))^{-1}; U(\gamma) = \ln g(\gamma)$

Thus, for Kumaraswamy Distribution Lindley's approximation for shape parameter γ under Jeffrey's prior, modified Jeffrey's prior, gamma prior and inverse levy prior can be obtained as

From (2.2) $\hat{\gamma} = \frac{n}{T}$ where $T = -\sum_{i=1}^{n} \ln(1 - x^{\eta})$

$$L_2(\hat{\gamma}) = \frac{\partial^2 \ln(\gamma | x)}{\partial \gamma^2} = -\frac{n}{\gamma^2} = \frac{-T^2}{n}; \; L_3(\hat{\gamma}) = \frac{\partial^3 \ln(\gamma | x)}{\partial \gamma^3} = \frac{2n}{\gamma^3} = \frac{2T^3}{n^2}$$

$$\hat{\phi}^2 = [-L_2(\hat{\gamma})]^{-1} = \frac{n}{T^2}$$

Under Jeffrey's prior $g(\gamma) \propto 1/\gamma$, $U(\gamma) = -\ln \gamma; U'(\hat{\gamma}) = -T/n$

If $h(\gamma) = \theta; h'(\gamma) = 1; h''(\gamma) = 0$

Thus Lindley's approximation for γ from (4.3) is obtained as

$$E(\gamma | x) = \hat{\gamma} = \frac{n}{T} \quad (3.4)$$

Under the extension of Jeffrey's prior $g(\gamma) \propto \left(\frac{1}{\sqrt{\gamma^3}}\right)$

$$U(\gamma) = \ln g(\gamma) = -\frac{3}{2}\ln \gamma; U'(\hat{\gamma}) = \frac{-3T}{2n}$$

Thus Lindley's approximation for γ from (4.3) is obtained as

$$E(\gamma \mid x) = \frac{2n-1}{2T} \tag{3.5}$$

Under gamma prior $g(\gamma) \propto \gamma^{a-1} e^{-b\gamma}$; $a, b > 0; \gamma > 0$ where a, b are the known hyper parameters.

$$U(\gamma) = \ln g(\gamma) = -b\gamma + (a-1)\ln\gamma; U'(\hat{\gamma}) = -b + (a-1)\frac{T}{n}$$

Thus $E(\gamma \mid x) = \frac{n+a}{T} - \frac{bn}{T^2}$ (3.6)

Under inverse levy prior $g(\gamma) \propto \gamma^{-1/2} e^{-\frac{\gamma r}{2}}; r > 0; \gamma > 0$, where *r* is the known hyper parameter

$$U(\gamma) = \ln g(\gamma) = \frac{-\ln\gamma}{2} - \frac{\gamma r}{2}; U'(\gamma) = -\frac{T}{2n} - \frac{r}{2}$$

Thus $E(\gamma \mid x) = \frac{2n+1}{2T} - \frac{rn}{2T^2}$ (3.7)

4. T-K Approximation:

Laplace's method uses asymptotic arguments in the development of new simulation techniques. From (4.2) it may be observed that Lindley's approximation requires evaluation of third order partial derivatives of likelihood function which may be cumbersome to compute when the parameter γ is a vector valued parameter. Tierney and Kadane (1986) gave Laplace method to evaluate $E(h(\lambda) \mid x)$ as

$$E(h(\gamma) \mid x) \cong \frac{\hat{\phi}^*}{\hat{\phi}} \frac{\exp\{-nh^*(\hat{\gamma}^*)\}}{\exp\{-nh(\hat{\gamma})\}} \tag{4.1}$$

where $-nh(\hat{\gamma}) = \ln P(\gamma \mid x)$; $-nh^*(\hat{\gamma}^*) = \ln P(\gamma \mid x) + \ln h(\gamma)$;

$$\hat{\phi}^2 = -[-nh''(\hat{\gamma})]^{-1}; \hat{\phi}^{*2} = -[-nh''^*(\hat{\gamma}^*)]^{-1}$$

Thus, for Topp-Leone Distribution Laplace approximation for shape parameter γ can be calculated as

Under uniform prior $g(\gamma) \propto 1/\gamma$, the posterior distribution for γ is given in (2.3)

$$\therefore\ -nh(\gamma) = (n-1)\ln\gamma - T\gamma\ ;\ -nh'(\gamma) = \frac{n-1}{\gamma} - T\ ;\ \Rightarrow \hat{\gamma} = \frac{n-1}{T}$$

Also $-nh''(\hat{\gamma}) = -\frac{T^2}{n-1}$

Therefore $\hat{\phi}^2 = -[-nh''(\hat{\gamma})]^{-1} = \frac{n-1}{T^2}$ or $\hat{\phi} = \frac{\sqrt{n-1}}{T}$

now $-nh^*(\gamma^*) = -nh(\gamma) + \ln h(\gamma) = (n)\ln\gamma^* - \gamma^* T$

$$-n\,h'^{*}(\gamma^{*}) = \frac{n}{\gamma^{*}} - T \;\Rightarrow \hat{\gamma}^{*} = \frac{n}{T}$$

Also $-n\,h''^{*}(\hat{\gamma}^{*}) = -\frac{T^{2}}{n} \;\Rightarrow \hat{\phi}^{*} = \frac{\sqrt{n}}{T}$

Thus using (4.1) we have

$$E(\lambda \mid x) = \frac{(n)^{1/2}}{(n-1)^{1/2}} \frac{\exp\{(n)\ln \hat{\gamma}^{*} - \hat{\gamma}T\}}{\exp\{(n-1)\ln \hat{\gamma} - \hat{\gamma}T\}} = \left(\frac{n}{n-1}\right)^{1/2} \frac{\hat{\gamma}^{*n}}{\hat{\gamma}^{n-1}} e^{-\hat{\gamma}^{*}T + \hat{\gamma}T}$$

$$= \frac{n-1}{T}\left(\frac{n}{n-1}\right)^{n+1/2} e^{-1} \tag{4.2}$$

Similarly $E(\gamma^{2} \mid x) = \frac{\hat{\phi}^{*}}{\hat{\phi}} \frac{\exp\{-n\,h^{*}(\hat{\gamma}^{*})\}}{\exp\{-n\,h(\hat{\gamma})\}}; here\; -n\,h^{*}(\gamma^{*}) = \log(\gamma^{2}) - n\,h(\gamma)$

$$E(\lambda^{2} \mid x) = \left(\frac{n+1}{n-1}\right)^{n+1/2} \frac{(n+1)(n-1)}{T^{2}} e^{-2}$$

$$\therefore Variance = \left(\frac{n+1}{n-1}\right)^{n+1/2} \frac{(n+1)(n-1)}{T^{2}} e^{-2} - \left[\left(\frac{n+1}{T}\right)\left(\frac{n+1}{n}\right)^{n+1/2} e^{-1}\right]^{2}$$

Under extension of Jeffrey's prior $g(\gamma) \propto \left(\frac{1}{\sqrt{\gamma^{3}}}\right)$, the posterior distribution for γ is given in (2.5)

$$\therefore -n\,h(\gamma) = (n-3/2)\ln \gamma - \gamma T \;;\; -n\,h'(\gamma) = \frac{n-3/2}{\gamma} - T \;\Rightarrow \hat{\gamma} = \frac{n-3/2}{T}$$

Also $-n\,h''(\hat{\gamma}) = -\frac{T^{2}}{n-3/2} \;\Rightarrow \hat{\phi} = \frac{(n-3/2)^{1/2}}{T}$

now $-n\,h^{*}(\gamma^{*}) = -n\,h(\gamma) + \ln h(\gamma) = (n-1/2)\ln \gamma^{*} - \gamma^{*}T$

$$\therefore -n\,h'^{*}(\gamma^{*}) = \frac{n-1/2}{\gamma^{*}} - T \;\Rightarrow \hat{\gamma}^{*} = \frac{n-1/2}{T}$$

and $-n\,h''^{*}(\hat{\gamma}^{*}) = -\frac{T^{2}}{n-1/2} \;\Rightarrow \hat{\phi}^{*} = \frac{(n-1/2)^{1/2}}{T}$

Thus $E(\gamma\,|\,x)=\left(\dfrac{n-1/2}{n-3/2}\right)^{1/2}\dfrac{\exp\{(n-1/2)\ln\hat{\gamma}^{*}-\hat{\gamma}^{*}T\}}{\exp\{(n-3/2)\ln\hat{\gamma}-\hat{\gamma}T\}}=\left(\dfrac{n-1/2}{n-3/2}\right)^{1/2}\dfrac{\hat{\gamma}^{*n-1/2}}{\hat{\gamma}^{n-3/2}}e^{-\hat{\gamma}^{*}T+\hat{\gamma}^{*}T}$

$$=\sqrt{\frac{(n-3/2)^3}{n-1/2}}\left(\frac{n-1/2}{n-3/2}\right)^{n+1/2}\frac{e^{-1}}{T} \qquad (4.3)$$

Further $E(\gamma^2\,|\,x)=\left(\dfrac{n+1/2}{n-3/2}\right)^{n+1/2}\dfrac{\sqrt{(n+1/2)(n-3/2)^3}}{T^2}e^{-2}$

$$\therefore Variance=\left(\frac{n+1/2}{n-3/2}\right)^{n+1/2}\frac{\sqrt{(n+1/2)(n-3/2)^3}}{T^2}e^{-2}-\left[\sqrt{\frac{(n-3/2)^3}{n-1/2}}\left(\frac{n-1/2}{n-3/2}\right)^{n+1/2}\frac{e^{-1}}{T}\right]^2$$

Under Gamma prior $g(\gamma)\propto\gamma^{a-1}e^{-b\gamma}\ ;\ a,b>0;\gamma>0$; the posterior distribution for γ is given in (2.7)

$$\therefore -n\,h(\gamma)=(n+a-1)\ln\gamma-\gamma(T+b)\ ;\ -n\,h'(\gamma)=\frac{n+a-1}{\gamma}-(T+b)$$

$$\Rightarrow\hat{\gamma}=\frac{n+a-1}{(T+b)}$$

Also $-n\,h''(\gamma)=-\dfrac{(T+b)^2}{(n+a-1)}\ \Rightarrow\hat{\phi}=\dfrac{(n+a-1)^{1/2}}{(T+b)}$

now $-n\,h^{*}(\gamma^{*})=-n\,h(\gamma)+\ln h(\gamma)=(n+a)\ln\gamma^{*}-(T+b)\gamma^{*}$

$$-n\,h'^{*}(\gamma^{*})=\frac{(n+a)}{\gamma^{*}}-(T+b)\ \therefore\ \hat{\gamma}^{*}=\frac{(n+a)}{(T+b)}$$

also $-n\,h''^{*}(\hat{\gamma}^{*})=-\dfrac{(T+b)^2}{(n+a)}\ \Rightarrow\hat{\phi}^{*}=\dfrac{(n+a)^{1/2}}{(T+b)}$

Thus $E(\lambda\,|\,x)=\left(\dfrac{n+a}{n+a-1}\right)^{1/2}\dfrac{\exp\{(n+a)\ln\hat{\gamma}^{*}-\hat{\gamma}^{*}(T+b)\}}{\exp\{(n+a-1)\ln\hat{\gamma}-\hat{\gamma}(T+b)\}}$

$$=\left(\frac{n+a-1}{T+b}\right)\left(\frac{n+a}{n+a-1}\right)^{n+a-1/2}e^{-1} \qquad (4.4)$$

Further $E(\gamma^2\,|\,x)=\dfrac{(n+a+1)(n+a-1)}{(b+T)^2}\left(\dfrac{n+a+1}{n+a-1}\right)^{n+a+1/2}e^{-2}$

$$\therefore Variance = \frac{(n+a)^2-1}{(b+T)^2}\left(\frac{n+a+1}{n+a-1}\right)^{n+a+1/2} e^{-2} - \left[\left(\frac{n+a-1}{T+b}\right)\left(\frac{n+a}{n+a-1}\right)^{n+a-1/2} e^{-1}\right]^2$$

Under inverse levy prior $g(\gamma) \propto \gamma^{-1/2} e^{-\frac{\gamma r}{2}}; r > 0; \gamma > 0$ the posterior distribution for γ is given in (2.9)

$$\therefore -n\,h(\gamma) = -(T + r/2)\gamma + (n - 1/2)\ln\gamma \;; -n\,h'(\gamma) = \frac{(n-1/2)}{\gamma} - (T + r/2)$$

$$\Rightarrow \hat{\gamma} = \frac{(n-1/2)}{(T + r/2)}$$

also $-n\,h''(\hat{\gamma}) = -\frac{(T+r/2)^2}{(n-1/2)} \quad \Rightarrow \hat{\phi} = \frac{(n-1/2)^{1/2}}{(T+r/2)}$

now $-n\,h^*(\gamma^*) = -n\,h(\gamma) + \ln h(\gamma) = (n + 1/2)\ln\gamma^* - (T + r/2)\gamma^*$

$$\Rightarrow -n\,h'^*(\gamma^*) = \frac{(n+1/2)}{\gamma^*} - (T + r/2) \quad \therefore \hat{\gamma}^* = \frac{(n+1/2)}{(T+r/2)}$$

also $-n\,h''^*(\hat{\gamma}^*) = -\frac{(T+r/2)^2}{(n+1/2)} \quad \Rightarrow \hat{\phi}^* = \frac{(n+1/2)^{1/2}}{(T+r/2)}$

Therefore $E(\gamma\,|\,x) = \left(\frac{n+1/2}{n-1/2}\right)^{1/2} \frac{\exp\{(n+1/2)\ln\hat{\gamma}^* + \hat{\gamma}^*(T + r/2)\}}{\exp\{n\ln\hat{\gamma} + \hat{\gamma}(T + r/2)\}}$

$$= \left(\frac{n+1}{n-1/2}\right)^{n+1/2} \sqrt{(n+1/2)(n-1/2)}\,\frac{e^{-1}}{(T+r/2)} \qquad (4.5)$$

Also $E(\gamma^2\,|\,x) = \frac{\sqrt{(n+3/2)^3(n-1/2)}}{(T+r/2)^2}\left(\frac{n+3/2}{n-1/2}\right)^{n+1/2} e^{-2}$

$$\therefore Variance = \frac{e^{-2}}{(T+r/2)^2}\left[\sqrt{(n+3/2)^3(n-1/2)}\left(\frac{n+3/2}{n-1/2}\right)^{n+1/2} - \left(\frac{n+1}{n-1/2}\right)^{2n+1}(n+1/2)(n-1/2)\right]$$

Simulation study:

In our simulation study we have generated a sample of sizes n=25, 50, 100 to observe the effect of small, medium, and large samples on the estimators. The results are replicated 5000 times and the average of the results has been presented in the tables. To examine the performance of Bayesian estimates for shape parameter of kumaraswamy distribution under different approximation techniques, estimates are presented along with posterior standard deviation and MSE in case of Lindley’s approximation given in parenthesis in the below tables.

Table1: Posterior estimates and posterior standard deviation (in parenthesis) under normal approximation:

n	γ	Jeffrey's prior	Modified Jeffrey's prior	Gamma prior			Inverse levy prior		
				a1=b1=1	a1=b1=2	a1=b1=3	r=1	r=2	r=3
25	0.5	0.47631 (0.09722)	0.46639 (0.09620)	0.4865 (0.09730)	0.4963 (0.09733)	0.5057 (0.09732)	0.4767 (0.09632)	0.4676 (0.09448)	0.4589 (0.09271)
	1.0	0.9480 (0.19352)	0.9283 (0.19149)	0.9501 (0.19001)	0.9518 (0.18667)	0.9535 (0.18351)	0.9310 (0.18810)	0.8969 (0.18121)	0.8652 (0.17481)
	1.5	1.3228 (0.27002)	1.2952 (0.26719)	1.3059 (0.26119)	1.2907 (0.25314)	1.27702 (0.24576)	1.2798 (0.25856)	1.2163 (0.24573)	1.1587 (0.23410)
50	0.5	0.6085 (0.08693)	0.6023 (0.08649)	0.6133 (0.08674)	0.6180 (0.08654)	0.6226 (0.08634)	0.6072 (0.08630)	0.5998 (0.08526)	0.5926 (0.08424)
	1.0	1.1862 (0.16946)	1.1741 (0.16859)	1.1818 (0.16713)	1.1776 (0.16490)	1.1736 (0.16275)	1.1700 (0.16629)	1.1430 (0.16245)	1.1272 (0.15879)
	1.5	1.1762 (0.16946)	1.1752 (0.16859)	1.1718 (0.16713)	1.1686 (0.16490)	1.1670 (0.16275)	1.1670 (0.16629)	1.1489 (0.16245)	1.1172 (0.15879)
100	0.5	0.5060 (0.05086)	0.5035 (0.05073)	0.5085 (0.05085)	0.5110 (0.05085)	0.5135 (0.05084)	0.5060 (0.05073)	0.5034 (0.05047)	0.5009 (0.05022)
	1.0	1.1802 (0.11861)	1.1742 (0.11831)	1.1780 (0.11780)	1.1760 (0.11701)	1.1739 (0.11624)	1.1722 (0.11751)	1.1585 (0.11614)	1.1452 (0.11480)
	1.5	1.2474 (0.12536)	1.2411 (0.12505)	1.2443 (0.12443)	1.2413 (0.12351)	1.2384 (0.12262)	1.2381 (0.12412)	1.2228 (0.12259)	1.2080 (0.12110)

Table2: Posterior estimates and MSE under Lindley's approximation:

n	γ	Jeffrey's prior	Modified Jeffrey's prior	Gamma prior			Inverse levy prior		
				a1=b1=1	a1=b1=2	a1=b1=3	r=1	r=2	r=3
25	0.5	0.4961 (0.00987)	0.4862 (0.00999)	0.5061 (0.00983)	0.5161 (0.01005)	0.5261 (0.01053)	0.5012 (0.00981)	0.4962 (0.00986)	0.4913 (0.00992)
	1.0	0.9875 (0.03917)	0.9678 (0.04005)	0.9881 (0.03916)	0.9885 (0.03915)	0.9890 (0.03914)	0.9878 (0.03916)	0.9683 (0.03902)	0.9488 (0.03864)
	1.5	1.3779 (0.09080)	1.3503 (0.09831)	1.3571 (0.09631)	1.3362 (0.10273)	1.3154 (0.10997)	1.3675 (0.09315)	1.3295 (0.00874)	1.2915 (0.00865)
50	0.5	0.6209 (0.02231)	0.6147 (0.02085)	0.6256 (0.02347)	0.6304 (0.02470)	0.6351 (0.02595)	0.6233 (0.02290)	0.6194 (0.02195)	0.6156 (0.02106)
	1.0	1.2104 (0.07356)	1.1983 (0.06862)	1.2053 (0.07144)	1.2002 (0.06938)	1.1951 (0.06736)	1.2079 (0.07252)	1.1932 (0.06662)	1.1785 (0.06116)
	1.5	1.8048 (0.15800)	1.7868 (0.14735)	1.7758 (0.14226)	1.7467 (0.12596)	1.7177 (0.11249)	1.7903 (0.14137)	1.7577 (0.12150)	1.7252 (0.11181)
100	0.5	0.5111 (0.00272)	0.5086 (0.00269)	0.5136 (0.00278)	0.5161 (0.00285)	0.5186 (0.00294)	0.5124 (0.00271)	0.5111 (0.00272)	0.5098 (0.00267)
	1.0	1.1921 (0.05110)	1.1961 (0.05093)	1.1898 (0.05022)	1.1875 (0.04935)	1.1852 (0.04849)	1.1909 (0.05064)	1.1838 (0.18298)	1.1767 (0.04542)
	1.5	1.2600 (0.07340)	1.2557 (0.07548)	1.2567 (0.07549)	1.2534 (0.07661)	1.2501 (0.07825)	1.2537 (0.07546)	1.2504 (0.07481)	1.2425 (0.07210)

Table3: Posterior estimates and posterior standard deviation (in parenthesis) under T-K approximation:

n	γ	Jeffrey's prior	Modified Jeffrey's prior	Gamma prior			Inverse levy prior		
				a1=b1=1	a1=b1=2	a1=b1=3	r=1	r=2	r=3
25	0.5	0.5773 (0.12722)	0.5673 (0.10620)	0.5865 (0.12730)	0.5953 (0.12733)	0.5147 (0.10732)	0.5757 (0.09732)	0.5646 (0.09548)	0.5539 (0.09371)
	1.0	1.2480 (0.21352)	1.2283 (0.20149)	1.2150 (0.21001)	1.2151 (0.20667)	1.2135 (0.20351)	1.2131 (0.18910)	1.2096 (0.18721)	1.2052 (0.17591)
	1.5	1.4238 (0.28012)	1.4052 (0.25929)	1.3159 (0.27219)	1.3007 (0.26315)	1.2870 (0.25578)	1.2898 (0.25866)	1.2463 (0.24683)	1.1687 (0.23411)
50	0.5	0.6185 (0.08899)	0.6123 (0.08655)	0.6183 (0.08774)	0.6189 (0.08755)	0.6128 (0.08659)	0.6091 (0.08650)	0.5898 (0.08546)	0.5828 (0.08439)
	1.0	1.1962 (0.17946)	1.1941 (0.17269)	1.1848 (0.17715)	1.1788 (0.17390)	1.1766 (0.17275)	1.1715 (0.16694)	1.1580 (0.16279)	1.1392 (0.15888)
	1.5	1.1882 (0.17976)	1.1841 (0.16369)	1.1899 (0.16819)	1.1795 (0.16780)	1.1786 (0.16376)	1.1720 (0.16739)	1.1530 (0.16354)	1.1274 (0.15888)
100	0.5	0.5960 (0.05188)	0.5935 (0.05073)	0.5885 (0.05170)	0.5770 (0.05099)	0.5735 (0.05085)	0.5562 (0.05078)	0.5634 (0.05057)	0.5409 (0.05033)
	1.0	1.1992 (0.11967)	1.1942 (0.11735)	1.1880 (0.11880)	1.1860 (0.11805)	1.1799 (0.11745)	1.1742 (0.11789)	1.1685 (0.11714)	1.1552 (0.11580)
	1.5	1.2574 (0.12636)	1.2489 (0.12505)	1.2458 (0.12548)	1.2443 (0.12454)	1.2375 (0.12582)	1.2361 (0.12422)	1.2218 (0.12269)	1.2078 (0.12210)

Conclusion:

In this paper the focus was to study the importance of Bayesian approximation techniques. We presented approximate to Bayesian integrals of Kumarswamy distribution depending upon numerical integration and simulation study and showed how to study posterior distribution by means of simulation study. From the findings of above tables it can be observed that the large sample distribution could be improved when prior is taken into account. In all cases normal approximation, Lindley's approximation, T-K approximation, Bayesian estimates under informative priors are better than those under non-informative priors especially the Inverse levy distribution proves to be efficient with minimum posterior standard deviation and mean square error in case of Lindley's approximation. In case of non-informative priors modified Jeffrey's prior proves to be efficient. We observe that under informative as well as non- informative priors, the normal approximation behaves well than T-K approximation, although the posterior variances in case of T-K approximation are very close to that of normal approximation. Further we conclude that the posterior standard deviation based on different priors tends to decrease with the increase in sample size. It implies that the estimators obtained are consistent. It can also be observed that the performance of Bayes estimates under informative priors (inverse levy) is better than non-informative prior.

References:

- Ahmad SP, Ahmad, A, Khan AA (2011). Bayesian Analysis of Gamma Distribution Using SPLUS and R-softwares. Asian J. Math. Stat., 4: 224-233.
- Ahmed AA, Khan AA, Ahmed SP (2007), Bayesian Analysis of Exponential Distribution in S-PLUS and R Softwares. Sri Lankan Journal of Applied Statistics, 8: 95-109.
- Fletcher SC, Ponnamblam K (1996) Estimation of reservoir yield and storage distribution using moments analysis. J Hydrol 182:259–275
- Jones. M.C., (2009): Kumaraswamy's distribution: A β-type distribution with some tractability advantages, statistical methodolody,6, 70-81.
- Kumaraswamy P (1980) A generalized probability density function for double-bounded random processes. J Hydrol 46:79–88
- Lindley,D.V.(1980) "Approximate Bayesian Method". Traabayos Estatistica 31, 223-237.
- Mostafa M. E, Nora .K, M. A.,(2014); Estimation of parameters of the Kumaraswamy distributionbased on general progressive type II censoring , American Journal of Theoretical and Applied Statistics, 3(6): 217-222.

- Mustafa N, Alexander. P, Fatih K.(2012a) Statistical analysis for Kumaraswamy's distribution based on record data. Springer-verlag doi: 10.1007/s00362-012-0432-7.
- Mustafa N, Alexander. P, Fatih K.(2012b): Classical and Bayesian estimation of P(Y < X) for Kumaraswamy's distribution, journal of statistical computation and simulation.1-25.
- Nadarajah .S., (2008) On the distribution of Kumaraswamy. J Hydrol 348:568–569.
- Ponnambalam K, Seifi A, Vlach J (2001) Probabilistic design of systems with general distributions of parameters. Int J Circuit Theory Appl 29:527–53
- Ramin.G. Aliakbar. M.S. Samira.M, (2011a) Classical and Bayesian estimators on the kumaraswamy distribution using grouped and un-groped data under different loss functions, journal of applied sciences. 11 (12) 2154-2162.
- Ramin.G. Mastane. K, Maghdoode. H, (2011b) Bayesian estimations in the Kumaraswamy distribution under progressively type II censoring data, International Journal of Engineering, Science and Technology, 3(9), 47-65.
- Seifi A, Ponnambalam K, Vlach J (2000) Maximization of manufacturing yield of systems with arbitrary distributions of component values. Ann Oper Res 99:373–383
- Sultan. H, Ahmad S.P (2015). Bayesian approximation techniques of Topp-leone distribution. Int. J. Stat. Math., 2(1): 066-070.
- Sundar V, Subbiah K (1989) Application of double bounded probability density function for analysis of ocean waves. Ocean Eng 16:193–200
- Tierney L, Kadane J (1986). Accurate approximations for posterior moments and marginal densities. Journal of the American Statistical Association, 81: 82-86.

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