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Solution of Partial Integro-Differential Equations by Double Elzaki Transform Method

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Abstract

Partial integro-differential equations (PIDE) occur naturally in various fields of science, engineering and social sciences. The main purpose in this paper for solving partial integro-differential equations (PIDE) by using double Elzaki transform , we convert the proposed PIDE to an algebraic equation , Solving this algebraic equation & applying double inverse Elzaki transform we obtain the exact solution **.**

Keywords Double Elzaki transform, Inverse Elzaki transform, Partial integro-differential equation, Partial derivatives.

1. Introduction

Elzaki transform [11-16], whose fundamental properties are presented in this paper, is based on fourier transform . it introduced by tarig Elzaki (2010).

The increasing attempts in applied mathematics to model real world phenomena often lead to integral and integro-differential equations [8], [9], [10]. This explains a growing interest in the applied mathematics community to integro-differential equations, and in particular, to partial integro-differential equations. They frequently arise and play an important role in many areas of mathematics, physics, engineering, biology, and other sciences. Therefore it is very important to know methods to solve such partial differential equations. One of the most known methods to solve partial differential equations is the integral transform method [1], [2]. Eltayeb and Kilicman [3], [4] have established and studied the relationship between the double Sumudu transform and the double Laplace transform and their applications to differential equations. Jyoti Thorwe & Sachin Bhalekar [5] in 2012, used single Laplace transform method to solve linear partial integro-differential equations (PIDE).

In the resent years, Eltayeb and Kilicman [6] have applied the double Laplace transform to solve general linear telegraph and partial integro-differential equations. Ranjit R. Dhunde1, G. L. Waghmare [7] Solving Partial Integro-Differential Equations Using Double Laplace Transform Method.

In this paper we solve (PIDE) using double Elzaki transform method, we directly convert PIDE into an algebraic equation instead of converting to ODE. Solving this algebraic equation & applying double inverse Elzaki transform we obtain the exact solution. This method is illustrated by giving examples of various types already described in [5,7].

Definition 1.

Elzaki Transform. Given a function $f(t)$ defined for all $t \geq 0$, Elzaki transform of f is the function T defined as follow:

$$E[f(t)\,,v] = T(v) = v\int_0^t f(t)e^{-\frac{t}{v}}dt \quad , \quad v \in (k_1\,,k_2)$$

for all values of s for which the improper integral converges.

Theorem 1. The Convolution Theorem

Let $f(t)$ and $g(t)$ be defined in A . having Elzaki transform $M(v)$ and $N(v)$ then the single Elzaki transform of the convolution of $f(t)$ and $g(t)$ is,

$$E[(f*g)(t)] = \frac{1}{v}\,M(v)N(v)$$

Where

$$(f*g)(t) = \int_0^t f(x-\tau)g(\tau)d\tau$$

whenever the integral is defined. Using double Elzaki transform convolution theorem becomes

$$E_2[(f * g)(x\,,t);(u\,,v)] = \frac{1}{uv}\,M(u\,,v)N(u\,,v)$$

2. Double Elzaki Transform [16]:

Theorem 2:

Let $f(x\,,t)\,,t\,,x\,\in R^+$ be a function which can be expressed as a convergent infinite series, then, its double Elzaki transform, given by

$$E_2[f(x\,,t)\,,u\,,v] = T(u\,,v) = uv\int_0^\infty\int_0^\infty f(x\,,t)e^{-\left(\frac{x}{u}+\frac{t}{v}\right)}dx\,dt\,,\; x\,,t > 0 \tag{1}$$

Where u ,v are complex values.

Further the double Elzaki transform of the first and second order partial derivatives are given by:

$$E_2\left\{\frac{\partial f}{\partial x}\right\} = \frac{1}{u}T(u\,,v) - uT(0\,,v)\;,\; E_2\left\{\frac{\partial^2 f}{\partial x^2}\right\} = \frac{1}{u^2}T(u\,,v) - T(0\,,v) - u\frac{\partial T(0\,,v)}{\partial x}$$

$$E_2\left\{\frac{\partial f}{\partial t}\right\} = \frac{1}{v}T(u\,,v) - \mathrm{v}T(u\,,0)\;,\; E_2\left\{\frac{\partial^2 f}{\partial t^2}\right\} = \frac{1}{v^2}T(u\,,v) - T(u\,,0) - v\frac{\partial T(u\,,0)}{\partial \mathrm{t}}$$

$$E_2\left\{\frac{\partial^2 f}{\partial x\,\partial t}\right\} = \frac{1}{\mathrm{v}}T(u\,,v) - \mathrm{v}T(u\,,0) - \frac{u}{v}\mathrm{T}(0\,,\mathrm{v}) + \mathrm{uvT}(0\,,0)$$

Now to illustrate the method, we consider the general linear partial integro-differential equation,

$$\sum_{i=0}^{m} a_i\frac{\partial^i u}{\partial t^i} + \sum_{i=0}^{n} b_i\frac{\partial^i u}{\partial x^i} + cu + \sum_{i=0}^{r} d_i\int_0^t k_i(t-s)\frac{\partial^i u(x,s)}{\partial x^i}ds + f(x,t) = 0 \tag{2}$$

(With prescribed conditions)

Where $f(x,t)$ and $k_i(t,s)$, are known functions, and a_i , b_i , d_i , c are constant or the function of x.

Taking double Elzaki transform of both sides of PIDE (2) with respect to t we get,

$$\sum_{i=0}^{m} a_iE_2\left\{\frac{\partial^i u}{\partial t^i}\right\} + \sum_{i=0}^{n} b_iE_2\left\{\frac{\partial^i u}{\partial x^i}\right\} + cE_2\{u\} + \sum_{i=0}^{r} d_iE_2\left\{\int_0^t k_i(t-s)\frac{\partial^i u(x,s)}{\partial x^i}ds\right\} + E_2\{f(x,t)\} = 0$$

Using **theorem 1** and **theorem 2** for Elzaki transform we get

$$\sum_{i=0}^{m} a_i\left\{\frac{\bar{u}(x\,,v)}{v^i} - \sum_{k=0}^{i-1} v^{2-i+k}E_x\left\{\frac{\partial^k}{\partial t^k}\bar{u}(x\,,0)\right\}\right\} + \sum_{i=0}^{m} b_i\left\{\frac{\bar{u}(x\,,v)}{u^i} - \sum_{j=0}^{i-1} u^{2-i+j}E_t\left\{\frac{\partial^j}{\partial x^j}\bar{u}(0\,,t)\right\}\right\} + c\bar{u}(x\,,v) +$$
$$\sum_{i=0}^{r} d_i\frac{1}{v}\bar{k}_i(v)\left\{\frac{\bar{u}(x\,,v)}{u^i} - \sum_{j=0}^{i-1} u^{2-i+j}E_t\left\{\frac{\partial^j}{\partial x^j}\bar{u}(0\,,t)\right\}\right\} + \bar{f}(x,v) = 0 \tag{3}$$

Where $\bar{u}(x\,,v) = E_2[u(x\,,v)]$, $\bar{f}(x,v) = E_2\left[\bar{f}(x\,,v)\right]$, $\bar{k}_i(v) = E_2\left[\bar{k}_i(t)\right]$.

Equation (3) is an algebraic equation in $\bar{u}(x,v)$. Solving algebraic equation and take inverse double Elzaki transform of $\bar{u}(x,v)$, we get a solution $u(x,t)$ of (2) .

3. Applications:

In this section we illustrate some examples to explain the presented the method, we chose examples have exact solutions.

Example 1: Consider the PIDE

$$u_{tt} = u_x + 2\int_o^t (t-y)u(x\,,y)dy - 2e^x \tag{4}$$

with initial condition

$$u(x\,,0) = e^x \quad , \quad u_t(x\,,0) = 0 \tag{5}$$

& boundary condition

$$u(o\,,t) = \cos t \tag{6}$$

Taking double Elzaki transform of equation (4)

$$\frac{1}{v^2}T(u,v)-T(u,0)-vT_t(u,0)=\frac{1}{u}T(u,v)-uT(0,v)+2v^2T(u,v)-2\frac{u^2v^2}{1-u} \qquad (7)$$

And single elzaki transforms of initial conditions (5) & boundary condition (6) are given by

$$T(u,0)=\frac{u^2}{1-u} \quad , \ T_t(u,0)=0 \quad , \ T(0,v)=\frac{v^2}{1+v^2}$$

Then equation (7) becomes,

$$\frac{1}{v^2}T(u,v)-\frac{u^2}{1-u}=\frac{1}{u}T(u,v)-\frac{uv^2}{1+v^2}+2v^2T(u,v)-2\frac{u^2v^2}{1-u}$$

$$\left(\frac{1}{u}+2v^2-\frac{1}{v^2}\right)T(u,v)=\frac{uv^2}{1+v^2}-\frac{u^2}{1-u}+2\frac{u^2v^2}{1-u}$$

$$\left(\frac{v^2+2uv^4-u}{uv^2}\right)T(u,v)=\frac{(1-u)uv^2-u^2(1+v^2)+2u^2v^2(1+v^2)}{(1+v^2)(1-u)}$$

$$\left(\frac{v^2+2uv^4-u}{uv^2}\right)T(u,v)=\frac{u(v^2+2uv^4-u)}{(1+v^2)(1-u)}$$

$$T(u,v)=\frac{u^2v^2}{(1+v^2)(1-u)}=\frac{u^2}{(1-u)}\frac{v^2}{(1+v^2)} \qquad (8)$$

Applying inverse double Elzaki transform of equation (8) , we get exact solution

$$u(x,t)=e^x\cos t\,.$$

Example 2 : Consider the PIDE

$$u_t+u_{ttt}+\int_o^t \sinh(t-y)u_{xxx}(x,y)dy=0 \qquad (9)$$

$$u(x,0)=0 \quad , \quad u_t(x,0)=\mathrm{x} \ , \ u_{tt}(x,0)=0 \qquad (10)$$

$$u(0,t)=0 \quad , \quad u_x(0,\mathrm{t})=\sin\mathrm{t} \ , \ u_{xx}(0,\mathrm{t})=0 \qquad (11)$$

Taking double Elzaki transform of equation (9)

$$\frac{1}{v}T(u,v)-\mathrm{v}T(u,0)+\frac{1}{v^3}T(u,v)-\frac{1}{\mathrm{v}}T(u,0)-T_t(u,0)-\mathrm{v}T_{tt}(u,0)-\frac{\mathrm{u}^3}{1-\mathrm{u}^2}\Big[\frac{1}{u^3}T(u,v)-\frac{1}{\mathrm{u}}T(0,v)-T_x(0,v)-\mathrm{u}T_{xx}(0,v)\Big]=0 \qquad (12)$$

And single elzaki transforms of equations (10) , (11) we get

$$T(u,0)=0 \quad , \ T_t(u,0)=\mathrm{u}^3 \quad , \ T_{tt}(\mathrm{u},0)=0$$

$$T(0,\mathrm{v})=0 \quad , \ T_x(0,\mathrm{v})=\frac{\mathrm{v}^3}{1+\mathrm{v}^2} \quad , \ T_{xx}(0,v)=0$$

Then equation (12) becomes,

$$\frac{1}{v}T(u,v)+\frac{1}{v^3}T(u,v)-\mathrm{u}^3-\frac{1}{1-\mathrm{u}^2}T(u,v)+\frac{\mathrm{u}^3\mathrm{v}^3}{(1-\mathrm{u}^2)(1+\mathrm{v}^2)}=0$$

$$\left(\frac{1}{v}+\frac{1}{v^3}-\frac{1}{1-\mathrm{u}^2}\right)T(u,v)=\mathrm{u}^3-\frac{\mathrm{u}^3\mathrm{v}^3}{(1-\mathrm{u}^2)(1+\mathrm{v}^2)}$$

$$\left(\frac{\mathrm{v}^2-\mathrm{u}^2\mathrm{v}^2+1-\mathrm{u}^2-\mathrm{v}^3}{\mathrm{v}^3(1-\mathrm{u}^2)}\right)T(u,v)=\frac{\mathrm{u}^3(1+\mathrm{v}^2-\mathrm{u}^2-\mathrm{u}^2\mathrm{v}^2-\mathrm{v}^3)}{(1-\mathrm{u}^2)(1+\mathrm{v}^2)}$$

$$T(u,v)=\frac{\mathrm{u}^3\mathrm{v}^3}{1+\mathrm{v}^2} \qquad (13)$$

Applying inverse double Elzaki transform of equation (13), we get exact solution

$$u(x,t)=\mathrm{x}\sin\mathrm{t} \qquad (14)$$

Example 3 : Consider the PIDE

$$u_t - u_{xx} + +\mathrm{u} + \int_o^t \mathrm{e}^{\mathrm{t-y}} u\,(x\,,y)dy = (x^2+1)e^t - 2 \qquad (15)$$

$$u(x\,,0) = x^2 \quad , \quad u_t(x\,,0) = 1 \ , \qquad (16)$$

$$u(0\,,t) = \mathrm{t} \quad , \quad u_x(0\,,\mathrm{t}) = 0 \qquad (17)$$

Taking double Elzaki transform of equation

$$\frac{1}{v}T(u\,,v) - \mathrm{v}T(u\,,0) - \frac{1}{u^2}T(u\,,v) + T(0\,,v) + \mathrm{u}T_x(0\,,v) + T(u\,,v) + \frac{\mathrm{v}}{1-\mathrm{v}}T(u\,,v) = (2\mathrm{u}^2+1)\frac{\mathrm{u}^2\mathrm{v}^2}{1-\mathrm{v}} - 2\mathrm{u}^2\mathrm{v}^2 \qquad (18)$$

and single Elzaki transforms of equations (16) , (17) we get

$$T(u\,,0) = 2\mathrm{u}^4 \quad , \ T_t(u\,,0) = \mathrm{u}^2 \ \ \& \quad T(0\,,\mathrm{v}) = \mathrm{v}^3 \quad , \ T_x(0\,,\mathrm{v}) = 0$$

Then equation (18) becomes,

$$\frac{1}{v}T(u\,,v) - 2\mathrm{vu}^4 - \frac{1}{u^2}T(u\,,v) + \mathrm{v}^3 + T(u\,,v) + \frac{\mathrm{v}}{1-\mathrm{v}}T(u\,,v) = (2\mathrm{u}^2+1)\frac{\mathrm{u}^2\mathrm{v}^2}{1-\mathrm{v}} - 2\mathrm{u}^2\mathrm{v}^2$$

$$\left(\frac{1}{v} - \frac{1}{u^2} + 1 + \frac{\mathrm{v}}{1-\mathrm{v}}\right)T(u\,,v) = (2\mathrm{u}^2+1)\frac{\mathrm{u}^2\mathrm{v}^2}{1-\mathrm{v}} - 2\mathrm{u}^2\mathrm{v}^2 + 2\mathrm{vu}^4 - \mathrm{v}^3$$

$$\left(\frac{\mathrm{u}^2+\mathrm{v}^2-\mathrm{v}}{\mathrm{u}^2\mathrm{v}(1-\mathrm{v})}\right)T(u\,,v) = \frac{2\mathrm{u}^4\mathrm{v}-\mathrm{u}^2\mathrm{v}^2+2\mathrm{u}^2\mathrm{v}^3-\mathrm{v}^3+\mathrm{v}^4}{(1-\mathrm{v})}$$

$$T(u\,,v) = \frac{\mathrm{u}^2\mathrm{v}(2\mathrm{u}^2\mathrm{v}+\mathrm{v}^2)(\mathrm{u}^2+\mathrm{v}^2-\mathrm{v})}{(\mathrm{u}^2+\mathrm{v}^2-\mathrm{v})} = 2\mathrm{u}^4\mathrm{v}^2 + \mathrm{u}^2\mathrm{v}^3 \qquad (19)$$

Applying inverse double Elzaki transform of equation (19), we get exact solution

$$u\,(x\,,t) = \mathrm{x}^2 + \mathrm{t} \qquad (20)$$

4. Conclusions

PIDEs are used in modelling various phenomena in sciences, engineering and social sciences.The double Elzaki transform method technique is successfully used to convert PIDE into an algebraic equation. Solving this algebraic equation & applying double inverse Elzaki transform we obtain the exact solution.

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