

## S- ITERATION PROCESS FOR CONVERGENCE OF TWO ASYMPTOTICALLY NON-EXPANSIVE MAPPING IN CAT (0) SPACE

Sanchita Pandey \*, Anil Rajput \*\* and Abha Tenguria \*\*\*

\* *Research scholar, Govt. PG College, Sehore*

\*\* *Department of mathematics, CSA Govt. PG College, Nodal Sehore*

\*\*\* *Department of mathematics, Govt. M.L.B college, Bhopal*

### Abstract

In this paper we establish some strong convergence theorem for two asymptotically non expansive mapping by modified S-iteration process under suitable conditions.

### Introduction

Fixed point theory in CAT(0) space has been first studied by Kirk (see[15, 16]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. It is worth mentioning that the results in CAT (0) spaces can be applied to any CAT(k) space with  $k \leq 0$  since any CAT(k) space is a CAT( $k'$ ) space for every  $k \geq k'$  (see,e.g., [2]).

A CAT(0) space plays a fundamental role in various areas of mathematics (see Bridson and Haefliger [2], “Metric spaces of non-positive curvature”). A metric space X is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as ‘thin’ as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. The complex Hilbert ball with a hyperbolic metric is a CAT(0) space (see [10]). Other examples include pre-Hilbert spaces, R-trees (see [2]) and Euclidean buildings (see [3]).

Let  $(X, d)$  be a metric space. A geodesic path joining  $x \in X$  to  $y \in Y$  (or, more briefly, a geodesic from  $x$  to  $y$ ) is a map  $c$  from a closed interval  $[0, l] \subset \mathbb{R}$  to  $X$  such that  $c(0) = x$ ,  $c(l) = y$ , and let  $d(c(t), c(t')) = |t - t'|$  for all  $t, t' \in [0, l]$ . In particular,  $c$  is an isometry, and  $d(x, y) = l$ . The image  $\alpha$  of  $c$  is called a geodesic (or metric) segment joining  $x$  and  $y$ .

A geodesic triangle  $\Delta(x_1, x_2, x_3)$  in a geodesic metric space  $(X, d)$  consists of three points in  $X$  (the vertices of  $\Delta$ ) and a geodesic segment between each pair of vertices (the edges of  $\Delta$ ). A comparison triangle for geodesic triangle  $\Delta(x_1, x_2, x_3)$  in  $(X, d)$  is a triangle  $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$  in  $R^2$  such that  $d_{R^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$  for  $i, j, \in \{1, 2, 3\}$ . Such a triangle always exists (see [2]).

Let  $\Delta$  be a geodesic triangle in  $X$ , and let  $\bar{\Delta} \subset R^2$  be a comparison triangle for  $\Delta$ . Then  $\Delta$  is said to satisfy the CAT(0) inequality if for all  $x, y \in \Delta$  and all comparison points  $\bar{x}, \bar{y} \in \bar{\Delta}$ .

$$d(x, y) \leq d_{R^2}(\bar{x}, \bar{y}) \tag{9} \quad (a)$$

Complete CAT(0) spaces are often called Hadamard spaces (see [12]). If  $x, y_1, y_2$  are points of a CAT(0) space and  $y_0$  is the mid point of the segment  $[y_1, y_2]$  which we will denote by  $(y_1 + y_2)/2$ , then the CAT(0) inequality implies

$$d^2\left(x, \frac{y_1+y_2}{2}\right) \leq \frac{1}{2} d^2(x, y_1) + \frac{1}{2} d^2(x, y_2) - \frac{1}{4} d^2(y_1, y_2). \tag{10} \quad (b)$$

The inequality (9) is the (CN) inequality of Bruhat and Tits [4]. The above inequality has been extended in [6] as

$$d^2(z, \alpha x \oplus (1 - \alpha)y) \leq \alpha d^2(z, x) + (1 - \alpha)d^2(z, y) - \alpha(1 - \alpha)d^2(x, y) \quad (11) \quad (c)$$

For any  $\alpha \in [0,1]$  and  $x, y, z \in X$ .

Let us recall that a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality (see [2, p.163]). Moreover, if  $X$  is a CAT(0) metric space and  $x, y, \in X$ , then for any  $\alpha \in [0,1]$ , there exists a unique point  $\alpha x \oplus (1 - \alpha)y \in [x, y]$  such that

$$d(z, \alpha x \oplus (1 - \alpha)y) \leq \alpha d(z, x) + (1 - \alpha)d(z, y) \quad (12) \quad (d)$$

for any  $z \in X$  and  $[x, y] = \{ \alpha x \oplus (1 - \alpha)y : \alpha \in [0,1] \}$ .

A subset  $C$  of a CAT(0) space  $X$  is convex if for any  $x, y, \in C$ , we have  $[x, y] \in C$ .

The well known Mann and Ishikawa iteration process are given below :

- 1) The Mann iteration process is defined by the sequence  $\{x_n\}$ ,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1 \end{cases}$$

where  $\{\alpha_n\}$  is a sequence in  $(0,1)$ .

- 2) Further, the Ishikawa iteration process is defined by the sequence  $\{x_n\}$ ,

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n \end{cases} \quad n \geq 1,$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are a sequence in  $(0,1)$ . This iteration process reduces to the Mann iteration process when  $\beta_n = 0$  for all  $n \geq 1$ .

- 3) In 2007, Agarwal, O'Regan and Sahu [1] introduced the S-iteration process in Banach space,

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1 - \alpha_n)T x_n + \alpha_n T y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n \end{cases} \quad n \geq 1,$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are a sequence in  $(0,1)$ . Note that (3) is independent of (2) (and hence (1)). They showed that their process independent of those of Mann and Ishikawa and converges faster than both of these (see [[1], Proposition 3.1]).

- 4) Schu [20], in 1991, considered the modified Mann iteration process which is a generalization of the Mann iteration process,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1 \end{cases}$$

where  $\{\alpha_n\}$  is a sequence in  $(0,1)$ .

- 5) Tan and Xu [22], in 1994, studied the modified Ishikawa iteration process which is a generalization of the Ishikawa iteration process,

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n \end{cases} \quad n \geq 1,$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are a sequence in  $(0,1)$ . This iteration process reduces to the Mann iteration process when  $\beta_n = 0$  for all  $n \geq 1$ .

- 6) In 2007, Agarwal, O'Regan and Sahu [1] introduced the modified S- iteration process in Banach space,

$$\begin{cases} x_1 \in K \\ x_{n+1} = (1 - \alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n = (1 - \beta_n)x_n + \beta_n T^n x_n \end{cases} \quad n \geq 1,$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are a sequence in  $(0,1)$ . Note that (6) is independent of (5) (and hence (4)). Also (6) reduces to (3) when  $T^n = T$  for all  $n \geq 1$ .

Very recently, Sahin and Basarir [19] modified the iteration process (6) in a CAT(0) space as follows:

Let  $K$  be a nonempty closed convex subset of a complete CAT(0) space  $X$  and  $T: K \rightarrow K$  be an asymptotically nonexpansive mapping with  $F(T) \neq \emptyset$ . Suppose that  $\{x_n\}$  is a sequence generated iteratively by

$$7) \quad \begin{cases} x_1 \in K \\ x_{n+1} = (1 - \alpha_n)T^n x_n \oplus \alpha_n T^n y_n, \\ y_n = (1 - \beta_n)x_n \oplus \beta_n T^n x_n \end{cases} \quad n \geq 1,$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are a sequences such that  $0 \leq \alpha_n, \beta_n \leq 1$  for all  $n \geq 1$ .

They studied modified S-iteration process and established some strong convergence results under some suitable conditions which generalize some results of Khan and Abbas [13].

We now further modify (7) for two mappings in a CAT(0) space as follows:  
 Let  $K$  be a nonempty closed convex subset of a complete CAT(0) space  $X$  and  $S, T: K \rightarrow K$   
 be two asymptotically -nonexpansive mappings with  $F(S, T) = F(S) \cap F(T) \neq \emptyset$  Suppose that  $\{x_n\}$   
 is a sequence generated iteratively by

$$8) \quad \begin{cases} x_1 \in K \\ x_{n+1} = (1 - \alpha_n)T^n x_n \oplus \alpha_n S^n y_n, \\ y_n = (1 - \beta_n)x_n \oplus \beta_n T^n x_n \end{cases} \quad n \geq 1,$$

where  $\{\alpha_n\}$  and  $\{\beta_n\}$  are a sequences such that  $0 \leq \alpha_n, \beta_n \leq 1$  for all  $n \geq 1$ .

If we take  $S = T$ , then (8) reduces to (7). Recently, Nanjaras and Panyanak [17] proved a  $\Delta$ -convergence theorem of the Krasnosel'skii-Mann iterations for asymptotically nonexpansive mappings in CAT(0) spaces.

In this paper, motivated by the above results, we prove strong convergence theorems of the  $S$  modified iterative schemes for asymptotically nonexpansive mappings in the CAT(0) space setting.

We will try to establish some strong convergence results under some suitable conditions for two asymptotically non expansive mapping by modified  $S$ -iteration process (8)

## 2. Preliminaries and Lemmas

This class of asymptotically nonexpansive mappings was to introduced by Goebel and Kirk[9] . He proved that if  $K$  is a non empty bounded closed convex subset of a real uniformly convex Banach space and  $T$  is an asymptotically nonexpansive self-mapping of  $K$ , then  $T$  has a fixed point. Chidume et al.[5] further generalized the class of asymptotically nonexpansive mappin- gs introduced by Goebel and Kirk [9], and proposed the concept of nonself asymptotically nonexpansive mapping The iterative approximation problem for asymptotically nonexpansive mapping and asymptotically nonexpansive type mapping were studied by many authors (see, e.g.[, 14, 20 ,28 ) in a Banach space and a CAT(0) space.

**Definition 2.1.** Let  $(X, d)$  be a metric space and  $K$  be its subset. Then  $T: K \rightarrow K$  is said to be nonexpansive if  $d(Tx, Ty) \leq d(x, y)$  for all  $x, y \in K$  and asymptotically nonexpansive if there exists a sequence  $\{u_n\} \subset [0, \infty)$  with  $\lim_{n \rightarrow \infty} u_n = 0$  such that  $d(T^n x, T^n y) \leq (1 + u_n)d(x, y)$  for all  $y, x \in K$  and  $n \geq 1$

We need the following useful lemmas to prove our main results in this paper.

**Lemma 2.1.** (See [18]) Let  $X$  be a CAT(0) space.

- (i) For  $x, y \in X$  and  $t \in [0, 1]$ , their exists a unique point  $z \in [x, y]$  such that  $d(x, z) = t d(x, y)$  and  $d(x, z) = d(y, z) = (1-t) d(x, y)$ . (A)  
 We use the notation  $(1 - t)x \oplus ty$  for the unique point  $z$  satisfying (A).

- (ii) For  $x, y \in X$  and  $t \in [0, 1]$ , we have  
 $d((1 - t)x \oplus ty, z) \leq (1 - t)d(x, z) + td(y, z)$  .

**Lemma 2.2.** (See [21]) Let ,  $\{a_n\}$  ,  $\{b_n\}$  and  $\{\delta_n\}$  be sequences of nonnegative real numbers satisfying the inequality.

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n, \quad n \geq 1$$

If  $\sum_{n=1}^{\infty} \delta_n < \infty$  and  $\sum_{n=1}^{\infty} b_n < \infty$ , then  $\lim_{n \rightarrow \infty} a_n$  exists. In particular, if  $\{a_n\}$  has a subsequence converging to zero, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

### MAIN RESULTS

In this section, we establish some strong convergence results of newly defined modified S-iteration scheme (8) to converge to common fixed point for two asymptotically nonexpansive mappings in the setting of CAT(0) space.

**Theorem 3.1.** Let  $K$  be a nonempty closed convex subset of a complete CAT(0) space  $X$  and let  $S, T: K \rightarrow K$  be two uniformly asymptotically nonexpansive mappings with sequence  $\{u_n\} \subset [0, \infty)$  such that  $\sum_{n=1}^{\infty} u_n < \infty$ . Suppose that  $F(S, T) \neq \emptyset$ . Let  $\{x_n\}$  be defined by the iteration process (8). If  $\liminf_{n \rightarrow \infty} d(x_n, F(S, T)) = 0$  or  $\limsup_{n \rightarrow \infty} d(x_n, F(S, T)) = 0$  where  $d(x, F(S, T)) = \inf_{p \in F(S, T)} d(x, p)$ , then the sequence  $\{x_n\}$  converges strongly to a point in  $F(S, T)$ .

Proof. Let  $p \in F(S, T)$ . From (8) and Lemma 2.1(ii), we have

$$\begin{aligned} d(y_n, p) &= d((1 - \beta_n)x_n \oplus \beta_n T^n x_n, p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n d(T^n x_n, p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n((1 + u_n)d(x_n, p)) \\ &\leq (1 + u_n)d(x_n, p) \end{aligned} \tag{9}$$

Again using (7), (d) and Lemma 2.1(ii), we have

$$\begin{aligned} d(x_{n+1}, p) &= d((1 - \alpha_n)T^n x_n \oplus \alpha_n S^n y_n, p) \\ &\leq (1 - \alpha_n) d(T^n x_n, p) + \alpha_n d(S^n y_n, p) \\ &\leq (1 - \alpha_n)\{(1 + u_n)d(x_n, p)\} + \alpha_n\{(1 + u_n)d(y_n, p)\} \\ &\leq (1 - \alpha_n)(1 + u_n)d(x_n, p) + \alpha_n(1 + u_n)(1 + u_n)d(x_n, p) \\ &\leq (1 + u_n)^2 d(x_n, p) \\ &\leq (1 + A_n) d(x_n, p) \end{aligned} \tag{10}$$

Where  $A_n = 2u_n + u_n^2$ . Since by hypothesis of the theorem  $\sum_{n=1}^{\infty} u_n < \infty$ , it follows that  $\sum_{n=1}^{\infty} A_n < \infty$ . This gives

$$d(x_{n+1}, F(S, T)) \leq (1 + A_n) d(x_n, F(S, T)) \tag{11}$$

Since by hypothesis  $\sum_{n=1}^{\infty} A_n < \infty$  by Lemma 2.2 and  $\liminf_{n \rightarrow \infty} d(x_n, F(S, T)) = 0$  or  $\limsup_{n \rightarrow \infty} d(x_n, F(S, T)) = 0$  gives that

$$\lim_{n \rightarrow \infty} d(x_n, F(S, T)) = 0 \tag{12}$$

Now, we show that  $\{x_n\}$  is a Cauchy sequence in  $K$ . With the help of inequality  $1 + x \leq e^x, x \geq 0$ . For any integer  $m \geq 1$ , we have from (9) that

$$\begin{aligned}
 d(x_{n+m}, p) &\leq (1 + A_{n+m-1}) d(x_{n+m-1}, p) \\
 &\leq e^{A_{n+m-1}} d(x_{n+m-1}, p) \\
 &\leq e^{A_{n+m-1}} e^{A_{n+m-2}} d(x_{n+m-2}, p) \\
 &\leq \dots \\
 &\leq \left( e^{\sum_{k=n}^{n+m-1} A_k} \right) d(x_n, p) \\
 &\leq \left( e^{\sum_{n=1}^{\infty} A_n} \right) d(x_n, p) \\
 &= M d(x_n, p)
 \end{aligned} \tag{13}$$

Where  $M = e^{\sum_{n=1}^{\infty} A_n}$

Since  $\lim_{n \rightarrow \infty} d(x_n, F(S, T)) = 0$ , without loss of generality, we may assume that a subsequence  $\{x_{nk}\}$  of  $\{x_n\}$  and a sequence  $\{p_{nk}\} \subset F(S, T)$  such that  $d(x_{nk}, p_{nk}) \rightarrow 0$  as  $k \rightarrow \infty$ . Then for any  $\varepsilon > 0$  there exists  $k_\varepsilon > 0$  such that

$$d(x_{nk}, p_{nk}) < \frac{\varepsilon}{2M} \text{ for all } k \geq k_\varepsilon \tag{14}$$

For any  $m \geq 1$  and for all  $n \geq n_{k_\varepsilon}$ , by (13) and (14), we have

$$\begin{aligned}
 d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_{nk}) + d(x_n, p_{nk}) \\
 &\leq M d(x_{nk}, p_{nk}) + M d(x_{nk}, p_{nk}) \\
 &= 2M d(x_{nk}, p_{nk}) \\
 &< 2M \frac{\varepsilon}{2M} = \varepsilon
 \end{aligned} \tag{15}$$

This proves that  $\{x_n\}$  is a Cauchy sequence in  $K$ . Thus, the completeness of  $X$  implies that  $\{x_n\}$  must be convergent. Assume that  $\lim_{n \rightarrow \infty} x_n = q$ . Since  $K$  is closed, therefore  $q \in K$ . Next, we show that  $q \in F(S, T)$ . Now  $\lim_{n \rightarrow \infty} d(x_n, F(S, T)) = 0$  gives that  $d(q, d(x_n, F(S, T))) = 0$ . Since  $F(S, T)$  is closed,  $q \in F(S, T)$ . This completes the proof.

**Theorem 3.2** Let  $K$  be a nonempty closed convex subset of a complete CAT(0) space  $X$  and let  $S, T: K \rightarrow K$  be two uniformly asymptotic non expansive mapping with sequence  $\{u_n\} \subset [0, \infty)$  such that  $\sum_{n=1}^{\infty} u_n < \infty$ . Suppose that  $F(S, T) \neq \emptyset$ . Let  $\{x_n\}$  be defined by the iteration process (8). If  $S$  and  $T$  satisfy the following conditions:

(i)  $\lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0$  and  $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$

(ii) If the sequence  $\{z_n\}$  in  $K$  satisfies  $\lim_{n \rightarrow \infty} d(z_n, Sx_n) = 0$  and  $\lim_{n \rightarrow \infty} d(z_n, Tx_n) = 0$ , then  $\liminf_{n \rightarrow \infty} d(z_n, F(S, T)) = 0$  or  $\limsup_{n \rightarrow \infty} d(z_n, F(S, T)) = 0$ .

Then the sequence  $\{x_n\}$  converges strongly to a point of  $F(S, T)$ .

Proof. It follows from the hypothesis that

$$\lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0 \text{ and } \lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0.$$

From (ii),

$$\liminf_{n \rightarrow \infty} d(x_n, F(S, T)) = 0 \text{ or } \limsup_{n \rightarrow \infty} d(x_n, F(S, T)) = 0$$

Therefore the sequence  $\{x_n\}$  must converges to a point of  $F(S, T)$  by theorem 3.1. This completes the proof.

**Theorem 3.3 2** Let  $K$  be a nonempty closed convex subset of a complete CAT(0) space  $X$  and let  $S, T: K \rightarrow K$  be two uniformly generalized asymptotic non expansive mapping with sequence  $\{u_n\} \subset [0, \infty)$  such that  $\sum_{n=1}^{\infty} u_n < \infty$ . Suppose that  $F(S, T) \neq \emptyset$ . Let  $\{x_n\}$  be defined by the iteration process (8). If either  $S$  or  $T$  is semicompact and  $\lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0$  or  $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ , then the sequence  $\{x_n\}$  converges strongly to a point of  $F(S, T)$ .

Proof. Suppose  $T$  is semi-compact, there exists a subsequence  $x_{n_j} \rightarrow p \in K$ . By hypothesis of the theorem  $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ , we have  $\lim_{n_j \rightarrow \infty} d(x_{n_j}, Tx_{n_j}) = 0$ . Hence we have

$$\begin{aligned} d(p, Tp) &\leq d(p, x_{n_j}) + d(x_{n_j}, Tx_{n_j}) + d(Tx_{n_j}, Tp) \\ &\leq (1 + A_n) d(p, x_{n_j}) + d(Tx_{n_j}, Tp) \rightarrow 0 \end{aligned}$$

Thus  $p \in F(S, T)$ . By 14,

$$d(x_{n+1}, p) \leq (1 + A_n) d(x_n, p)$$

Since by hypothesis  $\sum_{n=1}^{\infty} A_n < \infty$ , by Lemma 2.2,  $\lim_{n \rightarrow \infty} d(x_n, p)$  exists and  $x_{n_j} \rightarrow p \in F(S, T)$ . This shows that  $\{x_n\}$  converges strongly to a point of  $F(S, T)$ . This completes the proof.

From the definition

A mapping  $T: K \rightarrow K$ , where  $K$  is a subset of a metric space  $(X, d)$ , is said to satisfy Condition (A) if there exists a nondecreasing function  $f: [0, \infty) \rightarrow [0, \infty)$  with  $f(0) = 0$  and  $f(t) > 0$  for all  $t \in (0, \infty)$  such that  $d(x, Tx) \geq f(d(x, F(T)))$  for all  $x \in K$  where  $d(x, F(T)) = \inf\{d(x, p): p \in F(S, T) \neq \emptyset\}$ .

We modify this definition for two mappings

Two mappings  $S, T: K \rightarrow K$ , where  $K$  is a subset of a metric space  $(X, d)$ , is said to satisfy Condition (B) if there exists a nondecreasing function  $f: [0, \infty) \rightarrow [0, \infty)$  with  $f(0) = 0$  and  $f(t) > 0$  all  $t \in (0, \infty)$  such that  $d(x, F(S, T)) = \inf\{d(x, p): p \in F(S, T) \neq \emptyset\}$  and  $a_1$  and  $a_2$  are two nonnegative real numbers such that  $a_1 + a_2 = 1$ . It is to be noted that Condition (B) is weaker than compactness of the domain  $K$ . Condition (B) reduces to Condition (A) when  $S = T$ .

As an application of Theorem 3.1, we establish another strong convergence result employing Condition (B) as follows.

**Theorem 3.4** Let  $K$  be a nonempty closed convex subset of a complete CAT(0) space  $X$  and let  $S, T: K \rightarrow K$  be two uniformly generalized asymptotic non expansive mapping with sequence  $\{u_n\} \subset [0, \infty)$  such that  $\sum_{n=1}^{\infty} u_n < \infty$ . Suppose that  $F(S, T) \neq \emptyset$ . Let  $\{x_n\}$  be defined by the iteration process (8). Assume that  $\lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0$  and  $\lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0$ . Let

$S$  and  $T$  satisfy Condition (B), then the sequence  $\{x_n\}$  converges strongly to a point of  $F(S, T)$ .

Proof. Since by hypothesis

$$(16) \quad \lim_{n \rightarrow \infty} d(x_n, Sx_n) = 0 \text{ and } \lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0.$$

From Condition (B) and (20), we get

$$\lim_{n \rightarrow \infty} d(x_n, F(S, T)) \leq a_1$$
$$\lim_{n \rightarrow \infty} d(x_n, Sx_n) + a_2 \cdot \lim_{n \rightarrow \infty} d(x_n, Tx_n) = 0,$$

i.e.  $\lim_{n \rightarrow \infty} d(x_n, F(S, T)) = 0$ . Since  $f: [0, \infty) \rightarrow [0, \infty)$  is a nondecreasing function satisfying  $f(0) = 0$ , and  $f(t) > 0$  all  $t \in (0, \infty)$ , therefore we have

$$\lim_{n \rightarrow \infty} d(x_n, F(S, T)) = 0.$$

Now all the conditions of Theorem 3.1 are satisfied, therefore by its conclusion  $\{x_n\}$  converges strongly to a point of  $F(S, T)$  This completes the proof.

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