

SOLVING HYBRID FUZZY FRACTIONAL DIFFERENTIAL EQUATIONS BY IMPROVED EULER METHOD

S. Ruban Raj¹, M. Saradha²

1. Associate Professor, Department of Mathematics, St. Joseph College, Trichirappalli, Tamilnadu

Email: ruban946@gmail.com

2. Research Scholar, Department of Mathematics, St. Joseph College, Trichirappalli, Tamilnadu

Email: isuharisri88@gmail.com

Abstract

In this paper we study numerical methods for hybrid fuzzy fractional differential equations and the iteration method is used to solve the hybrid fuzzy fractional differential equations with a fuzzy initial condition. We consider a differential equation of fractional order and we compared the results with their exact solutions in order to demonstrate the validity and applicability of the method. We further give the definition of the Degree of Sub element hood of hybrid fuzzy fractional differential equations with examples.

1. Introduction

The origins of fractional calculus go back to 1695 when Leibniz considered the derivative of order $1/2$. Miller and Ross [8] and Oldham and Spanier [9] provide historical details on the fractional calculus. Many applications have been found for fractional calculus, some of which are discussed in Debnath [5] and Podlubny [12]. In particular, fractional differential equations have received much attention and a number of recent works concern their numerical solution (see Ford and Connolly [6] and others). As another development, hybrid systems are dynamical systems that progress continuously in time but have formatting changes called modes at a sequence of discrete times. Some recent papers about hybrid systems include [4, 7, 13].

When the continuous time dynamics of a hybrid system comes from fuzzy fractional differential equations the system is called a hybrid fuzzy fractional differential system or a hybrid fuzzy fractional differential equation. This is one of the first papers to study hybrid fractional differential equations. The aim of this paper is to study their numerical solution.

Keywords: hybrid fuzzy fractional differential equations, degree of sub element hood

With the rapid development of linear and nonlinear science, many different methods such as the variational iteration method (VIM) [1] were proposed to solve fuzzy differential equations. Fuzzy initial value problems for fractional differential equations have been considered by some authors recently [2, 3]. To study some

dynamical processes, it is necessary to take into account imprecision, randomness or uncertainty.

This paper is organized as follows. In Section 2, we provide some background on fuzzy fractional differential equations and hybrid fuzzy fractional differential equations. In Section 3 we discuss the numerical solution of hybrid fuzzy fractional differential equations by Improved Euler method. The method given uses piecewise application of a numerical method for fuzzy fractional differential equations. In Section 4, as an example, we numerically solve the Degree of Sub element hood of hybrid fuzzy fractional differential equations. The objective of the present paper is to extend the application of the iteration method, to provide approximate solutions for fuzzy initial value problems of differential equations of fractional order, and to make comparison with that obtained by an exact fuzzy solution.

2. HYBRID FUZZY FRACTIONAL DIFFERENTIAL EQUATIONS

Preliminaries

In this section the most basic notations used in fuzzy calculus are introduced. We start with defining a fuzzy number.

We now recall some definitions needed through the paper. The basic definition of fuzzy numbers is given by R , we denote the set of all real numbers. A fuzzy number is a mapping $u : R \rightarrow [0; 1]$ with the following properties:

- (a) u is upper semi-continuous,
- (b) u is fuzzy convex, i.e., $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x); u(y)\}$ for all $x; y \in R; \lambda \in [0; 1]$,
- (c) u is normal, i.e., $\exists x_0 \in R$ for which $u(x_0) = 1$,
- (d) $\text{supp } u = \{x \in R / u(x) > 0\}$ is the support of the u , and its closure $\text{cl}(\text{supp } u)$ is compact. Let E be the set of all fuzzy number on R . The r -level set of a fuzzy number $u \in E, 0 \leq r \leq 1$, denoted by $[u]_r$, is defined as

$$[u]_r = \begin{cases} \{x \in R / u(x) \geq r\} & \text{if } 0 < r \leq 1 \\ \text{cl}(\text{supp } u) & \text{if } r = 0 \end{cases}$$

It is clear that the r -level set of a fuzzy number is a closed and bounded interval $[\underline{u}(r); \bar{u}(r)]$, where $\underline{u}(r)$ denotes the left-hand endpoint of $[u]_r$ and $\bar{u}(r)$ denotes the right-hand endpoint of $[u]_r$. Since each $y \in R$ can be regarded as a fuzzy number Y defined by

$$Y(t) = \begin{cases} 1 & \text{if } t = y \end{cases}$$

0 if $t \neq y$

Definition 2.1.

A fuzzy number (or an interval) u in parametric form is a pair (\underline{u}, \bar{u}) of functions $\underline{u}(r), \bar{u}(r), 0 \leq r \leq 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0, 1]$ and right continuous at 0.
2. $\bar{u}(r)$ is a bounded non-decreasing left continuous function in $(0, 1]$ and right continuous at 0.
3. $\underline{u}(r) \leq \bar{u}(r), 0 \leq r \leq 1$.

2.2. Analytical solution of Hybrid Fuzzy Fractional Differential Equations:

Let us consider the following fractional differential equation:

$$\begin{aligned}
 {}_c D_a^\beta x(t) &= f(t, x(t), \lambda_k(x_k)), & t \in [t_k, t_{k+1}] \\
 x(t_k) &= x_k
 \end{aligned} \tag{1}$$

Where, $0 \leq t_0 \leq t_1 \leq \dots \leq t_k \rightarrow \infty$

$$f \in C[\mathbb{R}^+ \times E \times E, E], \lambda_k \in C[E, E]$$

Here we assume that the existence and uniqueness of solution of the hybrid system hold on each $[t_k, t_{k+1}]$ to be specific the system would look like:

$${}_c D_a^\beta x(t) = \left\{ \begin{aligned}
 &{}_c D_a^\beta x_0(t) = f(t, x_0(t), \lambda_0(x_0)), x(t_0) = x_0, t \in [t_0, t_1] \\
 &{}_c D_a^\beta x_1(t) = f(t, x_1(t), \lambda_1(x_1)), x(t_1) = x_1, t \in [t_1, t_2] \\
 &\cdot \\
 &\cdot \\
 &\cdot \\
 &{}_c D_a^\beta x_k(t) = f(t, x_k(t), \lambda_k(x_k)), x(t_k) = x_k, t \in [t_k, t_{k+1}]
 \end{aligned} \right.$$

By the solution of (1) we mean the following function:

$$x(t) = x(t, t_0, x_0) = \begin{cases} x_0(t), t \in [t_0, t_1] \\ x_1(t), t \in [t_1, t_2] \\ \cdot \\ \cdot \\ x_k(t), t \in [t_k, t_{k+1}] \\ \cdot \\ \cdot \end{cases}$$

We note that the solutions of (1) are piecewise differentiable in each interval for $t \in [t_k, t_{k+1}]$ for a fixed $x_k \in E$ and $k = 0, 1, 2, \dots$

We can also represent a fuzzy numbers $x \in E$ by a pair of functions,

$$\begin{aligned} {}_c D_a^\beta x(t) &= {}_c D_a^\beta [\underline{x}(t; r), \bar{x}(t; r)] \\ &= [{}_c D_a^\beta \underline{x}(t), {}_c D_a^\beta \bar{x}(t)] \end{aligned}$$

Using a representation of fuzzy numbers we may represent $x \in E$ by a pair of functions $(\underline{x}(r), \bar{x}(r))$, $0 \leq r \leq 1$, such that:

- $\underline{x}(r)$ is bounded, left continuous and non decreasing,
- $\bar{x}(r)$ is bounded, left continuous and non increasing and
- $\underline{x}(r) \leq \bar{x}(r)$, $0 \leq r \leq 1$

Therefore, we may replace (1) by an equivalent system equation (2):

$$\begin{cases} {}_c D_a^\beta \underline{x}(t) = \underline{f}(t, x, \lambda_k(x_k) \equiv F_k(t, \underline{x}, \bar{x}), \underline{x}(t_k) = \underline{x}_k \\ {}_c D_a^\beta \bar{x}(t) = \bar{f}(t, x, \lambda_k(x_k) \equiv G_k(t, \underline{x}, \bar{x}), \bar{x}(t_k) = \bar{x}_k \end{cases} \dots\dots (2)$$

This possesses a unique solution (\underline{x}, \bar{x}) which is a fuzzy function. That is for each t , the pair $[\underline{x}(t; r), \bar{x}(t; r)]$ is a fuzzy number, where $\underline{x}(t; r), \bar{x}(t; r)$ are respectively the solutions of the parametric form given by Equation (3):

$$\left\{ \begin{aligned} {}_c D_a^\beta \underline{x}(t) &= F_k(t, \underline{x}(t; r), \bar{x}(t; r), \underline{x}(t_k; r) = \underline{x}_k(r) \end{aligned} \right.$$

$${}_c D_a^\beta \bar{x}(t) = G_k(t, \underline{x}(t; r), \bar{x}(t; r)), \bar{x}(t_k; r) = \bar{x}_k(r) \quad \dots\dots (3)$$

for $r \in [0, 1]$

3. Improved Euler Method

For a hybrid fuzzy fractional differential equation we develop the Improved Euler method with harmonic mean when f and λ_k in (1) can be obtained via the Zadeh extension principle from:

$$f \in [\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}, \mathbb{R}] \text{ and } \lambda_k \in C[\mathbb{R}, \mathbb{R}]$$

we assume that the existence and uniqueness of solutions of (1) hold for each $[t_k, t_{k+1}]$. For a fixed r , to integrate the system in (3) $[t_0, t_1], [t_1, t_2], \dots, [t_k, t_{k+1}] \dots\dots$ we replace each interval by a set of N_{k+1} discrete equally spaced grid points (including the end points) at which the exact solution $x(t; r) = (\underline{x}(t; r), \bar{x}(t; r))$ is approximated by some $(\underline{y}(t; r), \bar{y}(t; r))$. For the chosen grid

points on $[t_k, t_{k+1}]$ at $t_{k,n} = t_k + nh_k$, $h_k = \frac{t_{k+1} - t_k}{N_k}$, $0 \leq n \leq N_k$.

Let $(\underline{Y}_k(t; r), \bar{Y}_k(t; r)) \equiv (\underline{x}_k(t; r), \bar{x}_k(t; r))$, $(\underline{y}_k(t; r), \bar{y}_k(t; r))$ and $(\underline{y}_k(t; r), \bar{y}_k(t; r))$ may be denoted respectively by $(\underline{Y}_{k,n}(t; r), \bar{Y}_{k,n}(t; r))$ and $(\underline{y}_{k,n}(t; r), \bar{y}_{k,n}(t; r))$.

We allow N_k 's to vary over the $[t_k, t_{k+1}]$'s so that the h_k 's may be comparable.

The Euler method for (1) is given by:

$$(\underline{Y}_k(t; r), \bar{Y}_k(t; r)) \equiv (\underline{x}_k(t; r), \bar{x}_k(t; r)), (\underline{y}_k(t; r), \bar{y}_k(t; r))$$

and

$$\left. \begin{aligned} \bar{Y}_{k+1}(t; r) &= \bar{y}_k + h_k f(\bar{x}_k(t; r), \bar{y}_k(t; r)) \equiv \bar{y}'_{k+1} \\ \underline{Y}_{k+1}(t; r) &= \underline{y}_k + h_k f(\underline{x}_k(t; r), \underline{y}_k(t; r)) \equiv \underline{y}'_{k+1} \end{aligned} \right\} \dots\dots(4)$$

Next we define:

The Improved Euler Method for (1) using (4) is given by:

$$(\underline{Y}_k(t; r), \bar{Y}_k(t; r)) \equiv (\underline{x}_k(t; r), \bar{x}_k(t; r)), (\underline{y}_k(t; r), \bar{y}_k(t; r))$$

and

$$\left. \begin{aligned} S_k[t_{k,n}, \underline{y}_{k,n}(r), \bar{y}_{k,n}(r)] &\equiv \underline{S}_{k+1}(t; r) = (h_k)[\underline{y}'_k + \underline{y}'_{k+1}] \\ & \dots\dots \end{aligned} \right\}$$

$$T_k [t_{k,n}, \underline{y}_{k,n}(r), \bar{y}_{k,n}(r)] \equiv \bar{T}_{k+1}(t; r) = (h_k)[\bar{y}'_k + \bar{y}'_{k+1}] \quad \dots\dots\dots (5)$$

Where

$$\left\{ \begin{array}{l} \underline{y}'_k \equiv {}_c D_a^\beta \underline{x}(t) = F_k(t, \underline{x}(t; r), \bar{x}(t; r)), \underline{x}(t_k; r) = \underline{x}_k(r) \\ \bar{y}'_k \equiv {}_c D_a^\beta \bar{x}(t) = G_k(t, \underline{x}(t; r), \bar{x}(t; r)), \bar{x}(t_k; r) = \bar{x}_k(r) \\ \text{for } r \in [0, 1] \end{array} \right. \quad \dots\dots\dots (6)$$

and

$$\left\{ \begin{array}{l} \underline{y}'_{k+1} \equiv \underline{Y}_{k+1}(t; r) = f(\underline{x}_k(t; r) + \frac{h_k}{2}, \underline{y}_k(t; r) + \frac{h_k}{2} f(\underline{x}_k(t, r), \underline{y}_k(t, r))) \\ \bar{y}'_{k+1} \equiv \bar{Y}_{k+1}(t; r) = f(\bar{x}_k(t; r) + \frac{h_k}{2}, \bar{y}_k(t; r) + \frac{h_k}{2} f(\bar{x}_k(t, r), \bar{y}_k(t, r))) \end{array} \right. \quad \dots\dots\dots (7)$$

The exact solution at $t_{k,n+1}$ is given by:

$$\left\{ \begin{array}{l} F_{k,n+1}(r) = \underline{Y}_{k,n}(r) + S_k [t_{k,n}, \underline{y}_{k,n}(r), \bar{y}_{k,n}(r)], \\ G_{k,n+1}(r) = \bar{Y}_{k,n}(r) + T_k [t_{k,n}, \underline{y}_{k,n}(r), \bar{y}_{k,n}(r)]. \end{array} \right. \quad \dots\dots\dots(4)$$

3.1.Degree of Sub Element hood:

Let X be a Universal, U be a set of parameters and let $(F_{k,n+1})$ and $(G_{k,n+1})$ are two fuzzy elements of X. Then the degree of sub element hood denoted by

$\mathfrak{S}(F_{k,n+1}, G_{k,n+1})$ is defined as,

$$\mathfrak{S}(F_{k,n+1}, G_{k,n+1}) = \frac{1}{|(F_{k,n+1})|} \left\{ |(F_{k,n+1})| - \sum \max\{0, (F_{k,n+1}) - (G_{k,n+1})\} \right\}$$

Where $|F_{k,n+1}| = \sum_{e_j \in A} \exp(F_{k,n+1})$

and

$$\mathfrak{S}(G_{k,n+1}, F_{k,n+1}) = \frac{1}{|(G_{k,n+1})|} \left\{ |(G_{k,n+1})| - \sum \max\{0, (G_{k,n+1}) - (F_{k,n+1})\} \right\}.$$

Example:

Consider the following linear hybrid fuzzy fractional differential equation:

$${}_c D_a^\beta X(t) = Y - Y^2 + 1 \quad \dots\dots (7)$$

$$X(0) = X_0,$$

where $\beta \in (0,1], t > 0$

This problem is a generalization of the following hybrid fuzzy fractional differential equation:

$${}_c D_a^\beta x(t) = y - y^2 + 1 = [\underline{y}(t; r), \bar{y}(t; r)] - [\underline{y}(t; r), \bar{y}(t; r)]^2 + 1 \quad \dots\dots (8)$$

$$x(t) = x_0,$$

Where $\beta \in (0,1], t > 0, \alpha$ is the step size and x_0 is a real number.

We can find the solution of the hybrid fractional fuzzy differential equation, by the method of Improved Euler Method. We compared & generalized the hybrid fractional fuzzy differential equation solution with the exact solution in the following table; also we illustrated the figure for this generalization by using Matlab.

TABLE : Numerical Solution of Example

α	$Y_{k,n+1}$	$F_{k,n+1}$	$G_{k,n+1}$
0.0	0.0	0.0	0.0
0.5	1.3125	1.0929	1.5089
1.0	1.5586	1.4207	1.6190
1.5	1.6216	1.6017	1.6179
2.0	1.6176	1.6197	1.6180
2.5	1.6181	1.6178	1.6180
3.0	1.6180	1.6181	1.6180
3.5	1.6180	1.6180	1.6180
4.0	1.6180	1.6180	1.6180
4.5	1.6180	1.6180	1.6180
5.0	1.6180	1.6180	1.6180

$$|F_{k,n+1}| = 15.4429$$

$$|G_{k,n+1}| = 16.0718$$

$$\mathfrak{S}(F_{k,n+1}, G_{k,n+1}) = \frac{1}{|(F_{k,n+1})|} \left\{ |(F_{k,n+1})| - \sum \max\{0, (F_{k,n+1}) - (G_{k,n+1})\} \right\}$$

$$= 0.99989$$

$$\cong 1$$

$$\mathfrak{S}(G_{k,n+1}, F_{k,n+1}) = \frac{1}{|(G_{k,n+1})|} \left\{ |(G_{k,n+1})| - \sum \max\{0, (G_{k,n+1}) - (F_{k,n+1})\} \right\}.$$

$$= 0.984053$$

$$\cong 1$$

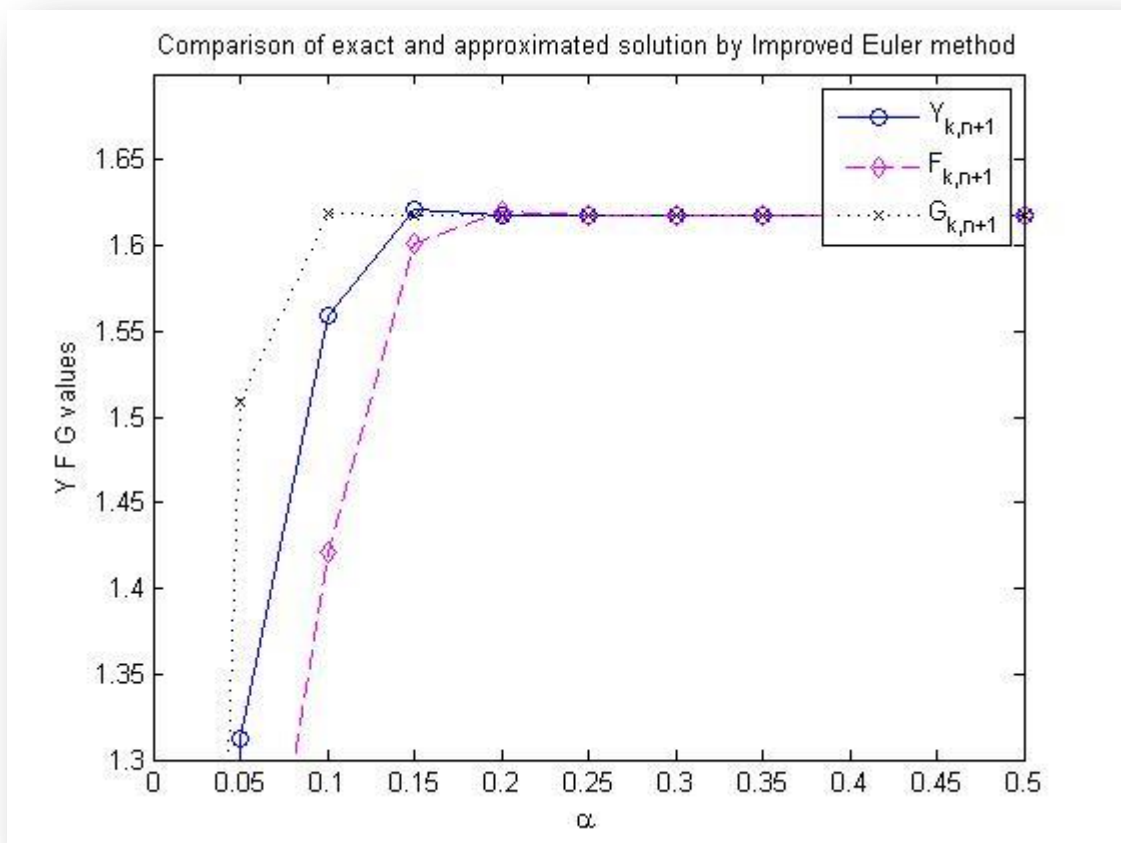


Fig.. Comparison of exact and approximated solution of Example

Conclusion

In this paper, we have studied a hybrid fuzzy fractional differential equation. Final results showed that the solution of hybrid fuzzy fractional differential equations approaches the solution of hybrid fuzzy differential equations as the fractional order approaches the integer order. The results of the study reveal that the proposed method with fuzzy fractional derivatives is efficient, accurate, and convenient for solving the hybrid fuzzy fractional differential equations.

References

- [1] Jafari, H, Saeidy, M, Baleanu, D: The variational iteration method for solving n-th order fuzzy differential equations. *Cent. Eur. J. Phys.* 10(1), 76-85 (2012)
- [2] Arshad, S, Lupulescu, V: Fractional differential equation with the fuzzy initial condition. *Electron. J. Differ. Equ.* 34, 1-8 (2011)
- [3] Agarwal, RP, Lakshmikanthama, V, Nieto, JJ: On the concept of solution for fractional differential equations with uncertainty. *Nonlinear Anal.* 72, 2859-2862 (2010)
- [4] G. Badowski and G. Yin, Stability of hybrid dynamic systems containing singularly perturbed random processes, *IEEE Trans. Automat. Control* 47 (2002), 2021–2032.
- [5] L. Debnath, Recent applications of fractional calculus to science and engineering, *Int. J. Math. Sci.* 2003 (2003), 3413–3442.
- [6] N. J. Ford and J. A. Connolly, Comparison of numerical methods for fractional differential equations, *Commun. Pure Appl. Anal.* 5 (2006), 289–306.
- [7] V. Lakshmikantham and X. Z. Liu, Impulsive hybrid systems and stability theory, *Intern. J. Nonl. Diff. Eqns.* 5 (1999), 9–17.
- [8] K. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, New York, 1993.
- [9] K. Oldham and J. Spanier, *The fractional calculus*, Academic Press, New York-London, 1974.
- [10] A. Oustaloup, From fractality to non integer derivation through recursivity, a property common to these two concepts: a fundamental idea for a new process control strategy, in: *Proceedings of the 12th IMACS World Congress, Paris, July 18-22, 1988*, vol. 3, pp. 203–208.
- [11] S. Pederson and M. Sambandham, Numerical solution to hybrid fuzzy systems, *Mathematical and Computer Modelling* 45 (2007), 1133–1144.

- [12] I. Podlubny, Fractional differential equations- An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, Academic Press, San Diego, CA, 1999.
- [13] G. Yin and Q. Zhang, Stability of nonlinear hybrid systems, in: New trends in nonlinear dynamics and control, and their applications, Lecture Notes in Control and Inform. Sci., 295.
- [14] Ekhtiar Khodadadi and Ercan Çelik, The variational iteration method for fuzzy fractional differential equations with uncertainty, *Fixed Point Theory and Applications* 2013, **2013**:13
- [15] S. Pederson and M. Sambandham, Numerical Solution Of Hybrid Fractional Differential Equations, *Communications In Applied Analysis* 12 (2008), No. 4, 429–440.
- [16] R. Gorenflo and F. Mainardi, “Fractional calculus: integral and differential equations of fractional order,” in *Fractals and Fractional Calculus in Continuum Mechanics*, pp. 223–276, Springer, New York, NY, USA, 1997.
- [17] M. Z. Ahmad and M. K. Hasan, “Numerical methods for fuzzy initial value problems under different types of interpretation: a comparison study,” in *Informatics Engineering and Information Science*, vol. 252 of *Communications in Computer and Information Science*, pp. 275–288, Springer, Berlin, Germany, 2011.
- [18] M. T. Mizukoshi, L. C. Barros, Y. Chalco-Cano, H. Rom án Flores, and R. C. Bassanezi, “Fuzzy differential equations and the extension principle,” *Information Sciences*, vol. 177, no. 17, pp.3627–3635, 2007.
- [19] S. Ruban Raj, M. Saradha, Properties of Fuzzy Soft Set, *International Journal for Basic Sciences and Social Sciences (IJBSS)*, ISSN:2319-2968, 2(1) (2013), pp.112-118.
- [20] G.V. Welland, On the fractional differentiation of a function of several variables, *Trans. Amer. Math. Soc.* 132 (1968) 487500.
- [21] X. Yang, F.Gao, Fundamentals of local fractional iteration of the continuously non differentiable functions derived from local fractional calculus, In: *Proc. of CSIE2011*, Springer, 2011.
- [22] C. Yu, G. Guozhu Gao, Some results on a class of fractional functional differential equations, *Common. Appl. Nonlinear Anal.* 3 (2004) 67-75.
- [23] X. Zhang, Some results of linear fractional order time-delay system, *Appl. Math. Comp.* 1 (2008) 407-411.
- [24] HJ. Zimmermann, Fuzzy programming and linear programming with several objective function, *Fuzzy Set and System* 1 (1987) 45-55.

The IISTE is a pioneer in the Open-Access hosting service and academic event management. The aim of the firm is Accelerating Global Knowledge Sharing.

More information about the firm can be found on the homepage:

<http://www.iiste.org>

CALL FOR JOURNAL PAPERS

There are more than 30 peer-reviewed academic journals hosted under the hosting platform.

Prospective authors of journals can find the submission instruction on the following page: <http://www.iiste.org/journals/> All the journals articles are available online to the readers all over the world without financial, legal, or technical barriers other than those inseparable from gaining access to the internet itself. Paper version of the journals is also available upon request of readers and authors.

MORE RESOURCES

Book publication information: <http://www.iiste.org/book/>

Academic conference: <http://www.iiste.org/conference/upcoming-conferences-call-for-paper/>

IISTE Knowledge Sharing Partners

EBSCO, Index Copernicus, Ulrich's Periodicals Directory, JournalTOCS, PKP Open Archives Harvester, Bielefeld Academic Search Engine, Elektronische Zeitschriftenbibliothek EZB, Open J-Gate, OCLC WorldCat, Universe Digital Library, NewJour, Google Scholar

