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Mathematical Modeling Using Sturm-Liouville System of Differential Equations to Physical Problems

S. S. Raghuwanshi (Corresponding author)
HPG (Retd), AMD/DAE, Hyderabad, India
raghu2257@yahoo.com

Sameer Sharma
Department of Mathematics, DAV College , Jalandhar, Punjab, India

Aditya Patwardhan
Department of Electrical Engineering, VIT , Chennai, India

Abstract

A few exactly solvable electrical conductivities useful in electromagnetic induction studies for Maxwell equations without displacement current have been evaluated by using Sturm-Liouville system of differential equations of the second order. A few solvable conductivities for the well-known Bessel, Stokes equations have also been presented

Keywords: Bessel, Stokes, Maxwell, equations, conductivities, magnetotellurics, electric field, magnetic field, induction.

1. Introduction

Present approach of constructing families of solvable potentials for the Schrodinger equation was initiated by Bargmann (1949). Bhattacharjee and Sudarshan, (1962), Bose(1964) developed it in a systematic method in which wave equation could be solved in a closed form giving simplest nuclear potential models shedding deeper light in the physical behavior of the system under study. Bhattacharjee and Sudarshan (1962) used the Sturm-Liouvelle system of second order differential equations for this purpose and employed suitable transformations for evaluating solvable potentials. The same technique was applied to the electromagnetic and electrical problem in the earth sciences by Raghuwanshi and Singh (1983).

In the present work, an attempt has been made to derive exactly solvable electrical conductivities based on Bhattacharjee and Sudarshan (1962) approach. In the electromagnetic phenomena, the electrical and magnetic fields are governed by the electrical conductivity, permittivity, and magnetic susceptibility of the medium. In exploration earth science too, the study of electromagnetic phenomenon is used to derive the information about the electrical properties of the medium in which they travel.

2. Transformations for Maxwell equations

Consider a general second order linear differential equation of Sturm-Liouville system of differential equations in following way:

$$\frac{d^2u(z)}{dz^2}+P(z)\frac{du(z)}{dz}+Q(z).u(z)=0 \qquad (1)$$

where *u(z)* is dependent function, *z* is independent variable, P*(z)* and Q*(z)* are algebraic functions of *z*.

Using the following transformations

z = f(r) and *u(z) = g(r).F(r)* (2)

such that $g(r) \neq 0$ we get a new transformed differential equation as follows:

$$\frac{d^2F(r)}{dr^2}+A(r)\frac{dF(r)}{dr}+B(r).F(r)=0 \qquad (3)$$

where *A(r), B(r)* are given as follows:

$$A(r)=z\frac{g'}{g}+P(r)f'-\frac{f''}{f'}$$

$$B(r)=\frac{g''}{g}+Q(r)f'^2+{g'}/{g}\left[P(r)f'-{f''}/{f'}\right]$$

$$g'=\frac{dg(r)}{dr},\ f'=\frac{df(r)}{dr},g''=\frac{d^2g(r)}{dr^2},\ f''=\frac{d^2f(r)}{dr^2}$$

$$P=P(r)=P(f(r))=P(z) and Q=Q(r)=Q(f(r))=Q(z)$$

Here primes are denoting the derivatives of that function with respect to r.

2.1 Electromagnetic Induction: Necessary Conditions

In the electromagnetics, the Maxwell's equation for the electric field E(r) in the x-direction can be reduced in one dimensional simple differential equation of second order assuming that there are no displacement currents in the medium of electrical conductivity $\sigma(r)$ with permeability μ as follows:

$$\frac{d^2E_x(r)}{dr^2}-i\omega\mu\sigma(r)E_x(r)=0 \qquad (4)$$

and associated with the magnetic field *H(r)* in y-direction can be written in terms of electric field E(r) in the x-direction as follows:

$$\frac{dE_x(r)}{dr}=-i\omega\mu H_y(r) \qquad (5)$$

where ω is angular frequency of this electrical field in x-direction while magnetic field will be in y-direction in this medium.

For the transformed equation (3) to be of the form of the displacement equation following conditions should be fulfilled:

$$A(r) = 2\frac{g'}{g} + P(r)f' - \frac{f''}{f'} = 0$$
$$B(r) = \frac{g''}{g} + Q(r)f'^2 + (g'/g)\left[P(r)f' - f''/f'\right] = -i\mu w\sigma(r) \quad (6)$$

2.2 Bessel Equation and conductivity

The Bessel differential equation is given as follows

$$\frac{d^2u(z)}{dz^2} + \frac{1}{z}\frac{du(z)}{dz} + (1 - n^2/z^2)u(z) = 0 \quad (7)$$

where $P(f(r)) = 1/f(r)$ and $Q(f(r)) = 1 - n^2/f^2(r)$ (8)

For making a choice of $f(r) = i\,e^{-\alpha r}$ equation (6) will give

$$g(r) = \sqrt{-\alpha/M} \quad (9)$$

where M being a constant and for n = 0

$$\sigma(r) = \frac{\alpha^2}{C}e^{-2\alpha r} \quad (10)$$

where $C = i\omega\mu$ has been assumed

Since the argument $f(r) = i\,e^{-\alpha r}$ of the Bessel function u(z) is imaginary the general solution of electric field component Ex(r) will be in terms of modified functions I_0 and K_0

$$E(r) = \frac{u(z)}{g(r)} = A1I_o(e^{-\alpha r}) + B1K_o(e^{-\alpha r}) \quad (11)$$

where A_1 and B_1 are complex constants.

Using relation (5), we can write the magnetic field in the y-direction as below in terms of E(r):

$$Hy(r) = \frac{1}{i\omega\mu}\left[B1K_1(e^{-\alpha r}) - A1I_1(e^{-\alpha r})\right] \quad (12)$$

This is an exponential variation of the conductivity in the medium and is referred in literature quite often eg. Kao and Rankin (1982).

Let us take another choice of $f(r)$ as

$$f(r) = \frac{2}{m+2}r^{\frac{m+2}{2}} \quad (13)$$

Then the conductivity function $\sigma(r)$ varying with a power of the depth(r) can be derived using above relations as follows

$$\sigma(r) = \frac{k}{i\omega\mu} r^n \qquad \text{For n} = \frac{1}{m+2} \tag{14}$$

where *k* is taken as a constant.

If n = 1 is chosen then $f(r) = \frac{2}{3} r^{\frac{3}{2}}$ expression for the conductivity function becomes linearly dependent on depth as follows:

$$\sigma(r) = \frac{kr}{i\omega\mu} \tag{15}$$

For which the solution for electric field *Ex(r)* will be then as follows

$$Ex(r) = \frac{u(z)}{g(r)} = \sqrt{r}[AJ_{1/3}(\frac{2}{3} r^{3/2}) + B\underline{J}_{1/3}(\frac{2}{3} r^{3/2})] \tag{16}$$

which shows that electric field component is combination of Airy functions (Kao and Rankin,1982) .

Still for another choice $f(r) = 2i\sqrt{r}$ one can derive exactly solvable conductivity σ(r) which is combination of inverse and inverse square laws of depth(r) as follows

$$\sigma(r) = \frac{k}{i\omega\mu}\left[\frac{1}{r} + \frac{n^2 - 1}{4r^2}\right] \tag{17}$$

If we set n = 1 in the above relation, then exactly solvable conductivity σ(r) becomes simply an inverse function of depth(r) a profile found in the planets

$$\sigma(r) = \frac{k}{i\omega\mu r} \tag{18}$$

For this case the general solution for the electric field $Ex(r)$ becomes (because n = 1)

$$Ex(r) = \frac{u(z)}{g(r)} = \sqrt{r}[AJ_1(2i\sqrt{r}) + B\underline{J}_1(2i\sqrt{r})] = = \sqrt{r}[iAI_1(2\sqrt{r}) + BK_1(2\sqrt{r})] \tag{19}$$

Thus a various classes of $\sigma(r)$ can be constructed by choosing $f(r)$ as shown above.

2.3. Stokes Equation and conductivity

This equation has the following standard form

$$\frac{d^2u(z)}{dz^2} + k^2.z.u(z) = 0 \tag{20}$$

where $k^2 = i\omega\sigma\mu$

whose general solution is given as

$$u(z) = \sqrt{z}[AJ_{1/3}(\frac{2}{3}kz^{3/2}) + B\underline{J}_{1/3}(\frac{2}{3}kz^{3/2})] \tag{21}$$

Here P(r) = 0

$$Q(r) = k^2 f(r) \tag{22}$$

For a simplest choice of z = *f(r)* = r we get

$$g^2(r) = Mf'(r) = M \tag{23}$$

where M is constant of integration.

So, the condition (6) yields,

$$\sigma(r) = -\frac{k^2 r}{i\omega\mu} \tag{24}$$

So, that for this conductivity function, corresponding electrical field component *Ex(r)* shall be

$$Ex(r) = \sqrt{r}[AJ_{1/3}(\frac{2}{3}kr^{3/2}) + B\underline{J}_{1/3}(\frac{2}{3}kr^{3/2})] \tag{25}$$

and using equation (8)

$$Hy(r) = \frac{1}{i\omega\mu}\frac{d}{dr}\{\sqrt{r}[AJ_{1/3}(\frac{2}{3}kr^{3/2}) + B\underline{J}_{1/3}(\frac{2}{3}kr^{3/2})]\} \tag{26}$$

But the expression in {} braces is also called Airy function $A_i(-r)$.

Hence

$$Hy(r) = \frac{1}{i\omega\mu} A_i(-r) \tag{27}$$

3. Application and further scope and conclusion

The present approach of solving the differential equation offers a solution and may be helpful in a variety of physical and engineering problems where this kind of equations is dealt with. This approach can therefore be used in solving even the complex configurations of the layered earth which can be made equivalent to the electrical transmission line system and other complex electrical grid networks in engineering problems.

This technique can be dexterously used to evaluate, monitor and control losses in the electrical grid by chain-linking the transmission lines, to prevent Corona losses and differential heating. Moreover, effective grounding strategies can be achieved by altering the effective conductivity at each junction's cross-section. Further, the designing of circuit breakers which work on the flux principle, achieved by the employment of current transformers, to monitor over - current situation, can further be improved.

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