Full Details of Solving Initial Value Problems by Reproducing Kernel Hilbert Space Method

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Abstract: In this paper we solve in full details an initial value problem by reproducing kernel Hilbert space method and we notice that this solution is close to the exact solution.

Key words: Reproducing kernel Hilbert space method, Differential Equation, Initial value problem.

1- Introduction

The theory of reproducing kernels was used for the first time at the beginning of the 20[th] century by S. Zaremba in his work on boundary value problems for harmonic and biharmonic functions and introduced the kernel corresponding to a class of functions with its reproducing property but he did not develop any theory and did not give any particular name to the kernels he introduced [3].

After two years in 1909, J. Mercer examined the functions which satisfy reproducing property in the theory of integral equations developed by Hilbert and he called these functions as "Positive definite kernel" in which they have nice properties among all continuous kernels of integral equations [3],[6].

The main idea of reproducing kernel appeared in the dissertations of three Berlin mathematicians G. Szego (1921), S. Bochner (1922) and S. Bergman (1922).

In particular, Bergman introduced reproducing kernels in one and several variables for the class of harmonic and analytic functions and called them "kernel functions" [3].

In 1948, the general theory of reproducing kernels was introduced by N. Aronszajn and since that time, the reproducing kernel Hilbert space has played a major role in operator theory and applications [3].

Reproducing Kernel Hilbert Spaces have been found useful in many fields such as differential equation, numerical analysis, probability and statistics and so on [4].

2- Basic Concept

Definition (Roughly Speaking) (1) [2]: A Reproducing Kernel Hilbert Space builds on a Hilbert space $\mathcal{H}$ and requires that all Dirac evaluation functional in $\mathcal{H}$ are bounded and continuous.

Definition (2) [2]: A Dirac functional at x is a functional $\delta_x \in \mathcal{H}$ such that $\delta_x(f) = f(x)$.

Note that δ_x is bounded if $\exists M > 0$ such that $\|\delta_x f\|_{\mathbb{R}} \leq M\|f\|_{\mathcal{H}}, \forall f \in \mathcal{H}$.

Definition (3) [5]: A function $K: X \times X \to \mathbb{R}$ is called kernel on X if $\exists$ a Hilbert space $\mathcal{H}$ and a map $\psi: X \to \mathcal{H}$, such that $K(x,y) =< \psi(x), \psi(y) > \forall x, y \in X$.

Definition (4) [3]: Let X be an arbitrary set. A symmetric function $K: X \times X \to \mathbb{C}$ is called positive definite kernel if $\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_i K(x_i, x_j) \geq 0 \;\; \forall n \in \mathbb{N}, x_1, x_2, \dots, x_n \in X$ and $\alpha_i \in \mathbb{C}, i = 1,2,\dots,n$.

Now, we will try to define a reproducing kernel Hilbert space.

Definition (Reproducing Kernel) (5) [3]: Let $\mathcal{H}$ be a Hilbert space of functions $f: X \to F$ on a set X. A function $K: X \times X \to \mathbb{C}$ is a reproducing kernel of $\mathcal{H}$ iff :

1) $K(.,x) \in \mathcal{H}, \;\; \forall x \in X$.
2) $f(x) = \langle f, K(.,x)\rangle, \;\; \forall f \in \mathcal{H}, \forall x \in X$.

The last condition is called "the reproducing property".

A Hilbert space which possesses a reproducing kernel is called a Reproducing Kernel Hilbert Space (RKHS).

Basic Properties of Reproducing Kernels [3]:

1. If a reproducing kernel K exists, then it is unique.
2. The reproducing kernel $K(x,y)$ is symmetric.
3. $K_x(x) \geq 0$, for any fixed $x \in [a,b]$.

Theorem (1) [3]: For a Hilbert space $\mathcal{H}$ of functions on X, there exists a reproducing kernel K for $\mathcal{H}$ if and only if for every $x \in X$, the evaluation linear functional $I: f \to f(x)$ is bounded linear functional.

Definition (6) [5]: $W_2^m[a,b] := \{u|u^j$ is absolutely continuous, $j = 1,2,\dots,m-1$ and $u^{(m)} \in L^2[a,b]\}$.
Theorem (2) [6]: Function space $W_2^m[a,b]$ is a reproducing kernel space.
Theorem (3) [3],[6]: The function space $W_2^m[a,b]$ is a reproducing kernel space, that is, for each fixed $x \in [a,b]$ and for any $u(y) \in W_2^m[a,b], \exists K_x(y) \in W_2^m[a,b], y \in [a,b]$ such that $\langle u(y), K_x(y)\rangle = u(x)$ and $K_x(y)$ is called reproducing kernel function of $W_2^m[a,b]$.

3- Main Result

In this section we will solve the problem

$$y''(x) = 6x + 2$$

with the initial conditions (1)

$$y(0) = 1, \qquad y'(0) = 2$$

by reproducing kernel Hilbert space method.

We will try to represent the reproducing kernel functions in the space $W_2^3[0,1]$ by a piecewise polynomial of degree 5.

At first, we must to construct the reproducing kernel function to space $W_2^3[0,1]$ which is defined by: $W_2^3[0,1] = \{u: u, u', u''$are absolutely continuous and $u''' \in L^2[0,1], u(0) = u'(0) = 0\}$ with the inner product and norm of $W_2^3[0,1]$ are defined by:

$$\langle u(y), v(y)\rangle_{W_2^3} = u(0)v(0) + u'(0)v'(0) + u''(0)v''(0) + \int_0^1 u'''(y)v'''^{(y)}dy$$

and $\|u\|_{W_2^3} = \sqrt{< u, u >_{W_2^3}}$, where $u, v \in W_2^3[0,1]$.
Let $K_x(y)$ be the reproducing kernel of the space $W_2^3[0,1]$ .Then $\forall x \in [0,1]$ and $u(y) \in W_2^3[0,1], y \in [0,1]$, we have:

$$u(x) = \langle u(y), K_x(y)\rangle_{W_2^3} = u(0)K_x(0) + u'(0)K'_x(0) + u''(0)K''_x(0) + \int_0^1 u'''(y)K_x'''(y)dy$$

$$= u''(0)K''_x(0) + \int_0^1 u'''(y)K_x'''(y)dy. \qquad (2)$$

Through three integrations by parts for (2) we have:

$$u(x) = u''(0)\big(K_x''(0) - K_x'''(0)\big) + u''(1)K_x'''(1) - u'(1)K_x^{(4)}(1) + u(1)K_x^{(5)}(1)$$
$$- \int_0^1 u(y)K_x^{(6)}(y)dy\,. \qquad (3)$$

Since $K_x(0) \in W_2^3[0,1]$, it follows that
$K_x(0) = 0, \qquad K'_x(0) = 0.$
If $K_x''(0) - K_x'''(0) = 0, K_x'''(1) = 0, K_x^{(4)}(1) = 0$ and $K_x^{(5)}(1) = 0$.
Then (3) implies that $u(x) = \langle u(y), K_x(y)\rangle_{W_2^3} = -\int_0^1 u(y)K_x^{(6)}(y)dy$.
For $\forall x \in [0,1]$, if $K_x(y)$ satisfies $-K_x^{(6)}(y) = \delta(x-y)$ (4)
then

$$u(x) = -\int_0^1 u(y)K_x^{(6)}(y)dy = \int_0^1 u(y)\delta(x-y)dy.$$

Let $x \neq y$, then $-K_x^{(6)}(y) = 0 \Rightarrow K_x^{(6)}(y) = 0$. (5)

The characteristic equation of (5) is $\lambda^6 = 0 \Rightarrow \lambda = 0$ with multiplicity 6

$$K_x(y) = \begin{cases} \sum_{i=0}^{5} c_i(x)y^i, & y \le x \\ \sum_{i=0}^{5} d_i(x)y^i, & y > x \end{cases}.$$

Also, since $-K_x^{(6)}(y) = \delta(x-y)$, then, $K_x{}^{(i)}(x+0) = K_x{}^{(i)}(x-0), i = 0,1,2,3,4$.

Integrating (4) from $x-\epsilon$ to $x+\epsilon$ with respect to y and let $\epsilon \to 0$, we have the jump degree of $K_x^{(5)}(y)$ at $x = y$,

$K_x^{(5)}(x-0) - K_x^{(5)}(x+0) = 1$.

To find $c_i(x), d_i(x), i = 0,1, \dots, 5$ we will solve the following equations:

1) $K_x(0) = 0$ 2) $K_x'(0) = 0$
3) $K_x''(0) - K_x'''(0) = 0$ 4) $K_x'''(1) = 0$
5) $K_x^{(4)}(1) = 0$ 6) $K_x^{(5)}(1) = 0$
7) $K_x^{(i)}(x+0) = K_x^{(i)}(x-0)$ $i = 0,1,2,3,4$

8) $K_x^{(5)}(x+0) - K_x^{(5)}(x-0) = -1$

The reproducing kernel function of $W_2^3[0,1]$ is given by:

$$K_x(y) = \frac{-1}{120}\begin{cases} y^2\left(5xy^2 - y^3 - 10x^2(3+y)\right) & y \le x \\ x^2\left(5x^2y - x^3 - 10y^2(3+x)\right) & y > x \end{cases} \tag{6}$$

Also we can define five inner products with their reproducing kernels:

1) $\langle u(y), K_{1x}(y)\rangle_{W_2^3} = u(0)K_{1x}(0) + u(1)K_{1x}(1) + u''(0)K_{1x}''(0) + \int_0^1 u'''(y)K_{1x}'''(y)dy$

$$K_{1x}(y) = \frac{1}{4680}\begin{cases} f_1(x,y), & y \ge x \\ f_2(y,x), & y \le x \end{cases}$$

$$f_1(x,y) = y^2(30x^2(27-4x+x^2) + 10x^2(27-4x+x^2)y - 5x(39-12x-4x^2+x^3)y^2 + (39 - 12x^2 - 4x^3 + x^4)y^3)$$

$$f_2(y,x) = x^2(39x^3 - 195x^2y + 30(27+9x+x^2)y^2 + 10(-12-4x+x^2)y^3 - 5(-12-4x + x^2)y^4 + (-12-4x+x^2)y^5)$$

2) $< u(y), K_{2x}(y) >_{W_2^3} = u(0)K_{2x}(0) + u'(1)K_{2x}'(1) + u''(0)K_{2x}''(0) + \int_0^1 u'''(y)K_{2x}'''(y)dy$

$$K_{2x}(y) = \frac{1}{6720}\begin{cases} f_1(x,y), & y \ge x \\ f_2(y,x), & y \le x \end{cases}$$

$$f_1(x,y) = y^2(60x^2(16-4x+x^2) + 20x^2(16-4x+x^2)y - 5x(56-12x-4x^2+x^3)y^2 + 56y^3)$$

$$f_2(x,y) = x^2(56x^3 + 60y^2(16-4y+y^2) + 20xy^2(16-4y+y^2) - 5x^2y(56-12y-4y^2 + y^3))$$

3) $< u(y), K_{3x}(y) >_{W_2^3} = u(0)K_{3x}(0) + u'(0)K_{3x}'(0) + u(1)K_{3x}(1) + u'(1)K_{3x}'(1) + \int_0^1 u'''(y)K_{3x}'''(y)\,dy$

$$K_{3x}(y) = \frac{1}{71520}\begin{cases} f_1(x,y), & y \ge x \\ f_2(y,x), & y \le x \end{cases}$$

$$f_1(x,y) = y^2(-5x^2(1779+1192x-353x^2+22x^3)+5x(-596+353x+5x^3-2x^4)y^2 + 2(298-55x^2-5x^4+2x^5)y^3)$$

$$f_2(x,y) = x^2(-5y^2(1779+1192y-353y^2+22y^3)-5x^2y(596-353y-5y^3+2y^4) + 2x^3(298-55y^2-5y^4+2y^5))$$

4) $< u(y), K_{4x}(y) >_{W_2^3} = u(0)K_{4x}(0) + u(1)K_{4x}(1) + u'(0)K'_{4x}(0) + u'(1)K'_{4x}(1)$

$$+u''(0)K''_{4x}(0) + \int_0^1 u'''(y)K'''_{4x}(y)\,dy$$

$$K_{4x}(y) = \frac{1}{913320}\begin{cases} f_1(x,y), & y \ge x \\ f_2(y,x), & y \le x \end{cases}$$

$$f_1(x,y) = y^2(38055xy^2 - 7611y^3 + 30x^2(-3987-1329y-367y^2+26y^3) + 10x^3(3624 + 1208y - 367y^2 + 26y^3) - 5x^4(2202+734y-286y^2+41y^3) + x^5(780 + 260y - 205y^2 + 56y^3))$$

$$f_2(x,y) = x^2(7611x^3 - 38055x^2y + 30(3987+1329x+367x^2-26x^3)y^2 - 10(3624+1208x - 367x^2 + 26x^3)y^3 + 5(2202+734x-286x^2+41x^3)y^4 - (780+260x - 205x^2 + 56x^3)y^5.$$

5) $< u(y), K_{5x}(y) >_{W_2^3} = u(0)K_{5x}(0) + u(1)K_{5x}(1) + u'(0)K'_{5x}(0) + u'(1)K'_{5x}(1)$

$$+\int_0^1 (u''(y)K''_{5x}(y) + u'''(y)K'''_{5x}(y))\,dy$$

$$K_{5x}(y) = \frac{e^{-1-x-y}}{2(-1+e)(-43+53e)}\begin{cases} f_1(x,y), & y \ge x \\ f_2(y,x), & y \le x \end{cases}$$

$$f_1(x,y) = -19e^3 + 24e^4 - 19e^{1+2x} + 24e^{2+2x} - 24e^{2(x+y)} - 24e^{2+2y} + 29e^{3+2y} + 29e^{1+2x+2y} - 6e^{1+x+2y}(3-4x+x^2) - 6e^{3+x}(19-4x+x^2) + 2e^{2+x}(52-12x+3x^2) + e^{2+x+2y}(8-24x+6x^2) + e^{1+x+y}(3-4x+x^2)(6+6y-13y^2+y^3) + e^{4+y}(-24+24y-9y^2+4y^3) + e^{2+2x+y}(-24+24y-9y^2+4y^3) - 2e^{3+y}(5+24y-9y^2+4y^3) - 2e^{1+2x+y}(5+24y-9y^2+4y^3) + e^{2+y}(24 + 24y - 9y^2 + 4y^3) + e^{2x+y}(24+24y-9y^2+4y^3) + 4e^{2+x+y}(-28+24y + 9y^2 + 4y^3 + x(12-24y^2) + x^2(-3+6y^2)) + e^{3+x+y}(114-114y+3y^2 - 19y^3 + 4x(-6+6y+11y^2+y^3) - x^2(-6+6y+11y^2+y^3))$$

$$f_2(x,y) = -57e^3 + 72e^4 + 72e^{1+2x} - 216e^{2+2x} + 159e^{3+2x} - 72e^{2(x+y)} - 129e^{1+2y} + 216e^{2+2y} - 72e^{3+2y} + 87e^{1+2x+2y} - 18e^{1+x+2y}(3-4x+x^2) - 18e^{3+x}(19 - 4x + x^2) + 6e^{2+x+2y}(4-12x+3x^2) + 6e^{2+x}(52-12x+3x^2) + 3e^{4+y}(-24 + 24y - 9y^2 + 4y^3) + 3e^{2+2x+y}(-24+24y-9y^2+4y^3) - 6e^{3+y}(5+24y - 9y^2 + 4y^3) - 6e^{1+2x+y}(5+24y-9y^2+4y^3) + 3e^{2+y}(24+24y-9y^2 + 4y^3) + 3e^{2x+y}(24+24y-9y^2+4y^3) + 12e^{2+x+y}(-28+60x+8x^3-24y + 9y^2 - 4y^3 + 3x^2(-1-8y+2y^2)) + e^{3+x+y}(342-53x^3-24y+9y^2-4y^3 - 3x^2(-6-47y+11y^2+y^3) + 3x(-130+24y-9y^2+4y^3)) + e^{1+x+y}(54 - 43x^3 + 312y - 117y^2 + 52y^3 + 3x^2(6+49y-13y^2+y^3) - 3x(110+24y - 9y^2 + 4y^3)).$$

Also, we need to find the reproducing kernel of $W_2^1[0,1] = \{u: u \text{ is absolutely continuous}, u, u' \in L^2[0,1]\}$ with the inner product and norm of $W_2^1[0,1]$ as:
$W_2^1[0,1] = \{u: u \text{ is absolutely continuous}, u, u' \in L^2[0,1]\}$, with inner product
$\langle u(y), R_x(y)\rangle_{W_2^1} = \int_0^1 (u(y)R_x(y) + u'(y)R_x{}'(y))dy$ and norm $\|u\|_{W_2^1} = \sqrt{\langle u, u\rangle_{W_2^1}}$
In [1], Li and Cui proved that $W_2^1[0,1]$ is a reproducing kernel Hilbert space and its reproducing kernel is given by

$$R_x(y) = \frac{1}{2\sinh(1)}[\cosh(x + y - 1) + \cosh|x - y| - 1]. \qquad (7)$$

To solve this problem by using reproducing kernel method we must homogenization the initial conditions as follows:

let $u(x) = y(x) + ax + b$ (8)
$u(0) = 0 = y(0) + b \Longrightarrow b = -1$
$u'(0) = 0 = y'(0) + a \Longrightarrow a = -2$
$\therefore u(x) = y(x) - 2x - 1$ (9)
Then $u''(x) = 6x + 2$
with the homogenize initial conditions (10)
$u(0) = 0, \quad u'(0) = 0.$
The exact solution of problem (1) is

$$y(x) = x^3 + x^2 + 2x + 1. \qquad (11)$$

Define the operator L: $W_2^3[0,1] \to W_2^1[0,1]$ such that:
$Lu(x) = u''(x) = f(x)$
where $f(x) = 6x + 2, x \in [0,1], u(x) \in W_2^3[0,1]$ and $f(x) \in W_2^1[0,1]$
with $u(0) = 0, u'(0) = 0, L$ is a bounded linear operator.
Let $\phi_i(x) = R_{x_i}(x)$ and $\psi_i(x) = L^*\phi_i(x) = \langle L^*\phi_i(x), K_x(y)\rangle_{W_2^3} = \langle \phi_i(x), LK_x(y)\rangle_{W_2^1}$
$= \langle R_{x_i}(x), LK_x(y)\rangle_{W_2^1} = L_{x_i}K_x(x_i)$, where L^* is the adjoint operator of L.

Theorem (4) [5]: Assume that the inverse operator L^{-1} exists. Then if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_i\}_{i=1}^{\infty}$ is the complete function system of the space $W_2^3[0,1]$.

Now, we will form an orthonormal function $\{\overline{\psi_i}(x)\}_{i=1}^{\infty}$ of the space W_2^3 by Gram- Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ as follows:

$$\overline{\psi_i}(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad i = 1,2, \ldots$$

where $\beta_{11} = \frac{1}{\|\psi_1\|}$, $\beta_{ii} = \frac{1}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1}(\overline{C_{ik}})^2}}$, $\beta_{ik} = \frac{-\sum_{j=k}^{i-1}\overline{C_{ij}}\beta_{jk}}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1}(\overline{C_{ik}})^2}}, k < i$ and

$$\overline{C_{ij}} = \langle \psi_i, \overline{\psi_j}\rangle_{W_2^3} = \sum_{k=1}^{i} \beta_{jk}\, c_{ik}.$$

Theorem (5) [5]: $\forall u(x) \in W_2^3[0,1]$, the series $\sum_{i=1}^{\infty}\langle u(x), \overline{\psi_i}(x)\rangle \overline{\psi_i}(x)$ is convergent in sense of the norm of $W_2^3[0,1]$. Moreover, if $\{x_i\}_{i=1}^{\infty}$ is a countable dense set in $[0,1]$, then the solution of (1) is unique and given by:

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi_i}(x)$$

The nth term of the solution is given by:

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi_i}(x).$$

Theorem (6) [5]: The approximate solution $u_n(x)$ and its derivative $u_n'(x), u_n''(x)$ are uniformly convergent.

We have reproducing kernels $K_{1x}(y) = \frac{-1}{120}y^2\left(5xy^2 - y^3 - 10x^2(3 + y)\right)$ for

$y \leq x$, $K_{2x}(y) = \frac{-1}{120}x^2\left(5x^2y - x^3 - 10y^2(3+x)\right)$ for $y > x$, if we choose $n = 5$, we have $x_i = \frac{i-1}{n-1}$, so $x_1 = 0,\ x_2 = 0.25,\ x_3 = 0.5,\ x_4 = 0.75,\ x_5 = 1$, then we have $K_{1x}(y)$ for x_1 and $K_{2x}(y)$ $x_2,\ x_3, x_4, x_5$.

$$L_y\, K_{1x}(y) = \frac{\partial^2 K_{1x}(y)}{\partial y^2} = \frac{-1}{2}xy^2 + \frac{1}{6}y^3 + \frac{1}{2}x^2 + \frac{1}{2}x^2 y$$

$$L_y\, K_{2x}(y) = \frac{\partial^2 K_{2x}(y)}{\partial y^2} = \frac{1}{2}x^2 + \frac{1}{6}x^3.$$

We have to find $\psi_i(x)$, $i = 1,2,3,4,5$

$\psi_1(x) = L_y\, K_{1x}(y)$ at $y = x_1 = 0 \Rightarrow \psi_1(x) = \frac{1}{2}x^2$

$\psi_2(x) = L_y\, K_{2x}(y)$ at $y = x_2 = 0.25 \Rightarrow \psi_2(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3$

$\psi_3(x) = L_y\, K_{2x}(y)$ at $y = x_3 = 0.5 \Rightarrow \psi_3(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3$

$\psi_4(x) = L_y\, K_{2x}(y)$ at $y = x_4 = 0.75 \Rightarrow \psi_4(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3$

$\psi_5(x) = L_y\, K_{2x}(y)$ at $y = x_5 = 5 \Rightarrow \psi_5(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3$

Let, $q_1(x,y) = L_y\, K(x,y) = \frac{\partial^2 K(x,y)}{\partial y^2}$.

Since $\psi_i(x) = L_y\, K(x,y)_{|y=x_i}$, $i = 1,2,3,4,5$ then defined another function as:

$$q_2(x,y) = \frac{\partial^2 q_1(x,y)}{\partial x^2}$$

Now, since the inner product is

$$C_{ij} = \langle \psi_i, \psi_j \rangle$$

so defined the inner product as:

$$C_{ij} = q_2(x_j, x_i)$$

The following is process of obtaining orthogonalization coefficients

$$\overline{\psi_i}(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x),\ \ i = 1,2,3,4,5$$

$$\overline{\psi_1}(x) = \beta_{11}\psi_1(x) = \frac{1}{\|\psi_1\|}\psi_1(x) = \frac{1}{\sqrt{\|\psi_1\|^2}}\psi_1(x)$$

$$\|\psi_1\|^2 = C_{11} = 1 + x_1 = 1$$

$$\overline{\psi_1}(x) = \frac{1}{2}x^2$$

$$\overline{\psi_2}(x) = \beta_{21}\overline{\psi_1}(x) + \beta_{22}\psi_2(x)$$

$$\beta_{21} = \frac{-\overline{C_{21}}\beta_{11}}{\sqrt{\|\psi_2\|^2 - (\overline{C_{21}})^2}}$$

$$\|\psi_2\|^2 = C_{22} = 1 + x_2 = 1.25,\ \overline{C_{21}} = \beta_{11}C_{21} = (1)(1 + x_1) = 1$$

$$\beta_{21} = \frac{-(1)(1)}{\sqrt{1.25 - 1}} = -2$$

$$\beta_{22} = \frac{1}{\sqrt{\|\psi_2\|^2 - (\overline{C_{21}})^2}} = \frac{1}{\sqrt{1.25 - 1}} = 2$$

$$\overline{\psi_2}(x) = -2\left(\frac{1}{2}x^2\right) + 2\left(\frac{1}{2}x^2 + \frac{1}{6}x^3\right) = \frac{1}{3}x^3$$

$$\overline{\psi_3}(x) = \beta_{31}\psi_1(x) + \beta_{32}\psi_2(x) + \beta_{33}\psi_3(x)$$

$$\beta_{31} = \frac{-\overline{C_{31}}\beta_{11} - \overline{C_{32}}\beta_{21}}{\sqrt{\|\psi_3\|^2 - (\overline{C_{31}})^2 - (\overline{C_{32}})^2}}$$

$$\overline{C_{31}} = \beta_{11}C_{31} = 1,\ \overline{C_{32}} = \beta_{21}C_{31} + \beta_{22}C_{32} = (-2)(1) + (2)(1.25) = 0.5$$

$$\|\psi_3\|^2 = C_{33} = 1.5$$

$$\beta_{31} = \frac{-1 - (0.5)(-2)}{\sqrt{1.5 - 1 - 0.25}} = 0$$

$$\beta_{32} = \frac{-\overline{C_{32}}\beta_{22}}{\sqrt{\|\psi_3\|^2 - (\overline{C_{31}})^2 - (\overline{C_{32}})^2}} = \frac{-(0.5)(2)}{\sqrt{0.25}} = -2$$

$$\beta_{33} = \frac{1}{\sqrt{\|\psi_3\|^2 - (\overline{C_{31}})^2 - (\overline{C_{32}})^2}} = 2$$

$$\overline{\psi_3}(x) = -2\left(\frac{1}{2}x^2 + \frac{1}{6}x^3\right) + 2\left(\frac{1}{2}x^2 + \frac{1}{6}x^3\right) = 0$$

$$\overline{\psi_4}(x) = \beta_{41}\psi_1(x) + \beta_{42}\psi_2(x) + \beta_{43}\psi_3(x) + \beta_{44}\psi_4(x)$$

$$\beta_{41} = \frac{-\overline{C_{41}}\beta_{11} - \overline{C_{42}}\beta_{21} - \overline{C_{43}}\beta_{31}}{\sqrt{\|\psi_4\|^2 - (\overline{C_{41}})^2 - (\overline{C_{42}})^2 - (\overline{C_{43}})^2}}$$

$$\overline{C_{41}} = \beta_{11}C_{41} = 1,\ \overline{C_{42}} = \beta_{21}C_{41} + \beta_{22}C_{42} = 0.5,\ \overline{C_{43}} = \beta_{31}C_{41} + \beta_{32}C_{42} + \beta_{33}C_{43} = 0.5$$

$$\|\psi_4\|^2 = C_{44} = 1.75$$

$$\beta_{41} = \frac{-1 + 1 - 0}{\sqrt{0.25}} = 0$$

$$\beta_{42} = \frac{-\overline{C_{42}}\beta_{22} - \overline{C_{43}}\beta_{32}}{\sqrt{\|\psi_4\|^2 - (\overline{C_{41}})^2 - (\overline{C_{42}})^2 - (\overline{C_{43}})^2}} = \frac{-1 + 1}{\sqrt{0.25}} = 0$$

$$\beta_{43} = \frac{-\overline{C_{43}}\beta_{33}}{\sqrt{\|\psi_4\|^2 - (\overline{C_{41}})^2 - (\overline{C_{42}})^2 - (\overline{C_{43}})^2}} = \frac{-1}{0.5} = -2$$

$$\beta_{44} = \frac{1}{\sqrt{\|\psi_4\|^2 - (\overline{C_{41}})^2 - (\overline{C_{42}})^2 - (\overline{C_{43}})^2}} = 2$$

$$\overline{\psi_4}(x) = -2\left(\frac{1}{2}x^2 + \frac{1}{6}x^3\right) + 2\left(\frac{1}{2}x^2 + \frac{1}{6}x^3\right) = 0$$

$$\overline{\psi_5}(x) = \beta_{51}\psi_1(x) + \beta_{52}\psi_2(x) + \beta_{53}\psi_3(x) + \beta_{54}\psi_4(x) + \beta_{55}\psi_5(x)$$

$$\beta_{51} = \frac{-\overline{C_{51}}\beta_{11} - \overline{C_{52}}\beta_{21} - \overline{C_{53}}\beta_{31} - \overline{C_{54}}\beta_{41}}{\sqrt{\|\psi_5\|^2 - (\overline{C_{51}})^2 - (\overline{C_{52}})^2 - (\overline{C_{53}})^2 - (\overline{C_{54}})^2}}$$

$$\overline{C_{51}} = \beta_{11}C_{51} = 1,\ \overline{C_{52}} = \beta_{21}C_{51} + \beta_{22}C_{52} = 0.5,\ \overline{C_{53}} = \beta_{31}C_{51} + \beta_{32}C_{52} + \beta_{33}C_{53} = 0.5$$

$$\overline{C_{54}} = \beta_{41}C_{51} + \beta_{42}C_{52} + \beta_{43}C_{53} + \beta_{44}C_{53} = 0.5,\ \|\psi_5\|^2 = C_{55} = 1.75$$

$$\beta_{51} = \frac{-1 + 1 - 0 - 0}{0.25} = 0$$

$$\beta_{52} = \frac{-\overline{C_{52}}\beta_{22} - \overline{C_{53}}\beta_{32} - \overline{C_{54}}\beta_{42}}{\sqrt{\|\psi_5\|^2 - (\overline{C_{51}})^2 - (\overline{C_{52}})^2 - (\overline{C_{53}})^2 - (\overline{C_{54}})^2}} = \frac{-1 + 1 - 0}{0.5} = 0$$

$$\beta_{53} = \frac{-\overline{C_{53}}\beta_{33} - \overline{C_{54}}\beta_{43}}{\sqrt{\|\psi_5\|^2 - (\overline{C_{51}})^2 - (\overline{C_{52}})^2 - (\overline{C_{53}})^2 - (\overline{C_{54}})^2}} = \frac{-1 + 1}{0.5} = 0$$

$$\beta_{54} = \frac{-\overline{C_{54}}\beta_{44}}{\sqrt{\|\psi_5\|^2 - (\overline{C_{51}})^2 - (\overline{C_{52}})^2 - (\overline{C_{53}})^2 - (\overline{C_{54}})^2}} = \frac{-1}{0.5} = -2$$

$$\beta_{55} = \frac{1}{\sqrt{\|\psi_5\|^2 - (\overline{C_{51}})^2 - (\overline{C_{52}})^2 - (\overline{C_{53}})^2 - (\overline{C_{54}})^2}} = 2$$

$$\overline{\psi_5}(x) = -2\left(\frac{1}{2}x^2 + \frac{1}{6}x^3\right) + 2\left(\frac{1}{2}x^2 + \frac{1}{6}x^3\right) = 0$$

$u_n(x) = \sum_{i=1}^{n} A_i\overline{\psi_i}(x)$, Where $A_i = \sum_{k=1}^{i} \beta_{ik} f(x_k)$ where $f(x_k) = 6x_k + 2$

$A_1 = \beta_{11} f(x_1) = (1)(6x_1 + 2) = 2$

$A_2 = \beta_{21} f(x_1) + \beta_{22} f(x_2) = (-2)(6x_1 + 2) + (2)(6x_2 + 2) = 3$

$A_3 = \beta_{31} f(x_1) + \beta_{32} f(x_2) + \beta_{33} f(x_3) = 0 + (-2)(6x_2 + 2) + (2)(6x_3 + 2) = 3$

$A_4 = \beta_{41} f(x_1) + \beta_{42} f(x_2) + \beta_{43} f(x_3) + \beta_{44} f(x_4)$

$= 0 + 0 + (-2)(63 + 2) + (2)(6x_4 + 2) = 3$

$A_5 = \beta_{51} f(x_1) + \beta_{52} f(x_2) + \beta_{53} f(x_3) + \beta_{54} f(x_4) + \beta_{55} f(x_5)$

$= 0 + 0 + 0 + (-2)(6x_4 + 2) + (2)(6x_5 + 2)$

$$u_n(x) = \sum_{i=1}^{n} A_i \overline{\psi_i}(x)$$

$u_1(x) = A_1 \overline{\psi_1}(x) = x^2$

$u_2(x) = A_1 \overline{\psi_1}(x) + A_2 \overline{\psi_2}(x) = x^2 + x^3$

$u_3(x) = A_1 \overline{\psi_1}(x) + A_2 \overline{\psi_2}(x) + A_3 \overline{\psi_3}(x) = x^2 + x^3$

$u_4(x) = A_1 \overline{\psi_1}(x) + A_2 \overline{\psi_2}(x) + A_3 \overline{\psi_3}(x) + A_4 \overline{\psi_4}(x) = x^2 + x^3$

$u_5(x) = A_1 \overline{\psi_1}(x) + A_2 \overline{\psi_2}(x) + A_3 \overline{\psi_3}(x) + A_4 \overline{\psi_4}(x) + A_5 \overline{\psi_5}(x) = x^2 + x^3.$

$U_n(x_i) = u_n(x_i) - tru_0(x_i), i = 1, \dots, 5$, where $tru_0(x_i) = -2x_i - 1$ is the approximate solution

$x_1 = 0 \Rightarrow U_1(0) = u_1(0) - tru_0(0) = 1$

$x_2 = 0.25 \Rightarrow U_2(0.25) = u_2(0.25) - tru_0(0.25) = (0.25)^2 + (0.25)^3 + 2(0.25) + 1$

$= 1.578125$

$x_3 = 0.5 \Rightarrow U_3(0.5) = u_3(0.5) - tru_0(0.5) = (0.5)^2 + (0.5)^3 + 2(0.5) + 1$

$= 2.375$

$x_4 = 0.75\,, U_4(0.75) = u_4(0.75) - tru_0(0.75) = (0.75)^2 + (0.75)^3 + 2(0.75) + 1$

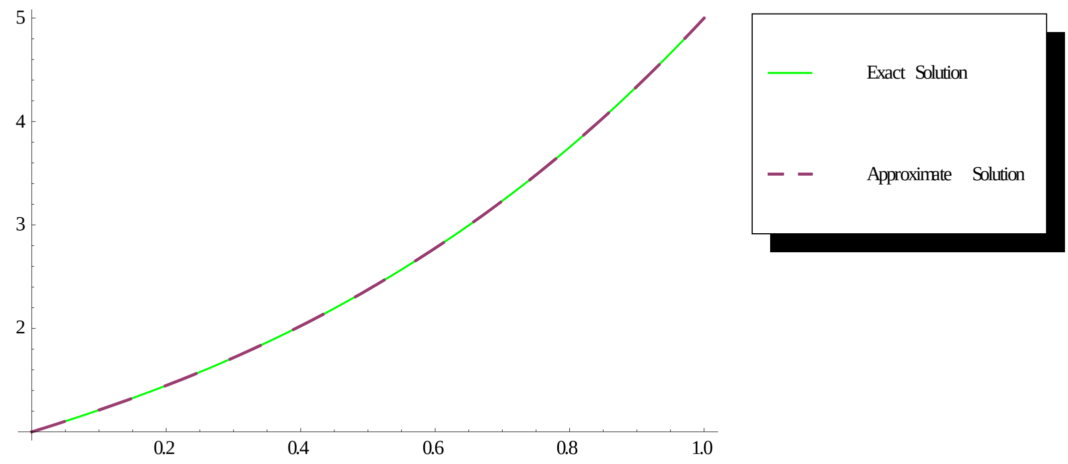
$= 3.484375$

$x_5 = 1\,, U_5(1) = u_5(1) - tru_0(1) = 5$

Table 1: Numerical Results for problem (1)

X	Exact solution	Approximate solution	Absolute error
0	1.578125	1.578125	0
0.25	1.578125	1.578125	0
0.5	2.375	2.375	0
0.75	3.484375	3.484375	0
1	5	5	0

Figure 1: Plots of exact solution and approximate solution.



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