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Transitivity Action of A_n on (n=4,5,6,7) on Unordered and Ordered Quadrupples

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ABSTRACT

In this paper, we study some transitivity action properties of the alternating group An(n=4,5,6,7 ,) acting on unordered and ordered pairs from the set $X = \{1,2,\ldots,n\}$ through determination of the number of disjoint equivalence classes called orbits.when n$\leq$ 7 ,the alternating group acts transitively on both $X^{(4)}$ and $X^{[4]}$.
key words : Orbits ,alternating group A_n , A_n on unordered and ordered quadruples from the set X.

1.Preliminaries

In 1964, Higman [2] introduced the rank of a group when he worked on finite permutation groups of rank 3.

In 1970, he calculated the rank and subdegrees of the symmetric group Sn acting on 2−$e$lements subsets from the set X = {1,2,...,n}. He showed that the rank is 3 and the subdegrees are 1,2(n−2), $\binom{n-2}{2}$.

In 1972, Cameron [1] worked on suborbits of multiply transitive permutation groups and later in 1974, he studied suborbits of primitive groups.

In 1999 Rosen [6] dealt with the properties arising from the action of a group on unordered and ordered pairs. Based on these results we investigate some properties of the action of An on $X^{(4)}$, the set of all unordered quadrupples from the set $X = \{1,2,\ldots,n\}$ and on $X^{[4]}$, the set of all ordered quadruples from $X = \{1,2,\ldots,n\}$. Let G = An act naturally on X, then G acts on $X^{(4)}$ by the rule $g\{a,b,c,d\}=\{ga,gb,gc,gd\}\forall g\epsilon G$ *and* $\{a,b,c,d\}\epsilon X^{(4)}$ and on $X^{[4]}$

1.1 NOTATION AND TERMINOLOGIES

In this paper, we shall represent the following notations as: $\sum_i$ −sum over i; $\binom{m}{n}$−m combination n; S_n– Symmetric group of degree n and order $n!$; A_n–an alternating group of degree n and order $\frac{n!}{2}$; $|G|$ − The order of a group G; $|G:H|$ − Index of H in G; $X^{(4)}$– The set of an unordered quadruples from set $X = \{1,2,\ldots,n\}$; $X^{[4]}$– The set of an ordered quadruples from set $X = \{1,2,\ldots,n\}$; {a,b,c,d} – Unordered quadruple; [a,b,c,d] – Ordered quadruple . We also define some basic terminologies on permutation group and give some results on group actions as:

Definition 1.1.1:
Let X be a non-empty set. A permutation of X is a one-to-one mapping of X onto itself.

Definition 1.1.2:
Let X be the set $\{1,2,\ldots,n\}$, the symmetric group of degree n is the group of all permutations of X under the binary operation of composition of maps. It is denoted by S_n and has order $n!$**.**

Definition 1.1.3:
A permutation of finite set is even or odd according to whether it can be expressed as the product of an even or odd number of 2−$c$ycles (transpositions).

Definition 1.1.4: The subgroup of S_n consisting of all even permutation in S_n is called the alternating group. It is denoted by A_n and $|A_n| = \frac{n!}{2}$.

Definition 1.1.5: Let X be a non-empty set. The group G acts on the left of X if for each $g \epsilon G$ and each $x \epsilon X$ there corresponds a unique element$gx\epsilon X$ such that;

i) (g_1g_2)=$g_1(g_2x)$ ∀ g_1,g_2 ϵX and $x \epsilon X$.

ii) For any $x\epsilon X$, Ix= x, where I is the identity in G

The action of G from the right on X can be defined in a similar way. In fact it is merely a matter of taste whether one writes the group element on the left or on the right.

Definition 1.1.6

Let G act on a set X and let x ∈ X. The stabilizer of x in G, denoted by $stab_G$ (x), is the set of all elements in G which fix x i.e. $stab_G$ (x)=$\{g \in G|gx = x\}$.

Note This set is also denoted by G_x. $Stab_G$ (x) is a subgroup of G, that is; $stab_G$ (x)≤G.

Definition 1.1.7

let G act on a set X. The set of elements of X fixed by g ∈ G is called the fixed point set of G, denoted by Fix(g). Thus, Fix(g)=$\{x \in X|gx = x\}$.

Definition 1.1.8

If a finite group G acts on a set X with n elements, each g ∈ G corresponds to a permutation σ of X, which can be written uniquely as a product of disjoint cycles. If σ has α_1 cycles of length1, α_2 cycles of length 2,…,α_n cycles of length n, we say that σ and hence g has cycle type $(\alpha_1,\alpha_2,\ldots,\alpha_n)$.

Definition 1.1.9

If the action of a group G on a set X has only one orbit, then G is said to act transitively on X. In other words, a group G acts transitively on X if for every pair of points x,y ∈ X, there exists g ∈ G such that gx=y.

Definition1.1.10

Let G act on a set X. Then G is said to act doubly transitively on X if for every two ordered pairs (x_1,x_2) and (y_1,y_2) of distinct elements in X, there exists g ∈ G such that $gx_1=y_1$ and $gx_2=y_2$.

Theorem 1.1.13 [Krishnamurthy 1985, p.68]

Two permutations in A_n are conjugate if and only if they have the same cycle type; and if g ∈ A_n has cycle type $(\alpha_1, \alpha_2,\ldots, \alpha_n)$, then the number of permutations in A_n conjugate to g is $\frac{n!}{\prod_{i=1}^{n} \alpha_i! i^{\alpha_i}}$.

Theorem 1.1.14 [Orbit- Stabilizer Theorem –Rose 1978, p.72]

Let G be a group acting on a finite set X and x ∈ X. Then $|Orb_G(x)|$ = $[G:Stab_G(x)]$.

Theorem 1.1.15 [Cauchy- Frobenius Lemma-Rotman 1973, p.45]

Let G be a group acting on finite set X. Then the number of G-orbits in X is $\frac{1}{|G|}\sum_{g\in G}|Fix(g)|$.

This theorem is usually but erroneously attributed to Burnside (1911) cf. Neumann (1977).

1.2 INTRODUCTION

2.ACTION OF THE ALTERNATING GROUP An ON UNORDERED QUADRUPPLES

2.1 some general results of permutation groups ascting on $X^{(4)}$

We first give two the proofs of two lemmas which will be useful in the investigation of transitivity of the action of An on $X^{(4)}$

Lemma 2.1.1

Let the cycle type of $g \in A_n$ be $(\alpha_1, \alpha_2,\ldots,\alpha_n)$. Then the number of elements in $X^{(4)}$ fixed by g is given by the formula $|Fix(g)| = \binom{\alpha_1}{4}+\binom{\alpha_1}{2}\binom{\alpha_2}{1}+\binom{\alpha_2}{2}+\alpha_1\alpha_3+\alpha_4$.

Proof

Let $\{a,b,c,d\} \in X^{(4)}$ and $g \in A_n$. Then g fixes {a,b,c,d} if and only if g permutes the elements in the set {a,b,c,d} as in the following cases;

Case 1:

Each of the elements a, b, c and d comes from a single-cycle in g. In this case the number of unordered quadruples fixed by g is $\binom{\alpha_1}{4}$, for $\alpha_1 \geq 4$.

Case 2:

Two of the elements a, b, c and d come from single-cycles and the other two elements come from a 2-cycle, say (ab)(c)(d)… In this case the number of unordered quadruples fixed by g is $\binom{\alpha_1}{2}\binom{\alpha_2}{1}$, for $\alpha_1 \geq 2$, and $\alpha_2 \geq 1$.

Case 3:

Each of the elements a, b, c and d come from a 2-cycle in g, say (ab)(cd)… In this case the number of unordered quadruples fixed by g is $\binom{\alpha_2}{2}$, $\alpha_2 \geq 2$.

Case 4:

Three of the elements a, b, c and d come from a 3-cycle and one element comes from a single-cycle say (abc)(d)…. In this case the number of unordered quadruples fixed by g is $\alpha_1\alpha_3$.

Case 5:

The elements a,b,c and d come from a 4-cycle in g say (abcd)…. In this case the number of unordered quadruples fixed by g is α_4. Thus the total number of unordered quadruples fixed by g is

$\binom{\alpha_1}{4}+\binom{\alpha_1}{2}\binom{\alpha_2}{1}+\binom{\alpha_2}{2}+\alpha_1\alpha_3+\alpha_4$. ■

Lemma 2.1.2

Let $g \in A_n$ have cycle type $(\alpha_1, \alpha_2, \ldots,\alpha_n)$. Then the number of permutations in A_n that fix $\{a,b,c,d\} \in X^{(4)}$ and having the same cycle type as g is given by

$$\frac{(n-4)!}{1^{\alpha_1-4}(\alpha_1-4)!\prod_{i=2}^{n}\alpha_i!\,i^{\alpha_i}}+\frac{6(n-4)!}{1^{\alpha_1-2}(\alpha_1-2)!2^{\alpha_2-1}(\alpha_2-1)!\prod_{i=3}^{n}\alpha_i!\,i^{\alpha_i}}+\frac{3(n-4)!}{\alpha_1!1^{\alpha_1}2^{\alpha_2-2}(\alpha_2-2)!\prod_{i=3}^{n}\alpha_i!\,i^{\alpha_i}}+$$
$$\frac{8(n-4)!}{1^{\alpha_1-1}(\alpha_1-1)!\alpha_2!2^{\alpha_2}3^{\alpha_3-1}(\alpha_3-1)!\prod_{i=4}^{n}\alpha_i!\,i^{\alpha_i}}+\frac{6(n-4)!}{\alpha_1!1^{\alpha_1}\alpha_2!2^{\alpha_2}\alpha_3!3^{\alpha_3}4^{\alpha_4-1}(\alpha_4-1)!\prod_{i=5}^{n}\alpha_i!\,i^{\alpha_i}}.$$

Proof

Let $\{a, b, c, d\} \in X^{(4)}$ and $g \in A_n$. Then g fixes {a, b, c, d} if and only if it permutes the elements in the set {a,b,c, d} as in the following cases;

Case 1:

Each of the elements a, b, c and d comes from a single cycle in g. In this case the number of permutations in A_n fixing {a, b, c, d} and with the same cycle type as g is equal to the number of permutations of A_{n-4} with cycle type $(\alpha_1\text{-}4, \alpha_2,\ldots\alpha_n)$. By Theorem 1.1.13, this number is $\frac{(n-4)!}{(\alpha_1-4)!\prod_{i=2}^{n}\alpha_i!\,i^{\alpha_i}}$, for $\alpha_1 \geq 4$.

Case 2:

Two of the elements a, b, c, and d come from single- cycles and the other two elements come from a 2-cycle, say, (ab)(c)(d)…. In this case the number of permutations in A_n fixing {a,b,c,d} and with the same cycle type as g is equal to the number of permutations of A_{n-4} with cycle type $(\alpha_1\text{-}2, \alpha_2\text{-}1,\alpha_3, \ldots, \alpha_n)$. By Theorem 1.1.13 this number is

$$\frac{(n-4)!}{1^{\alpha_1-2}(\alpha_1-2)!2^{\alpha_2-1}(\alpha_2-1)!\prod_{i=3}^{n}\alpha_i!\,i^{\alpha_i}}\text{ , for }\alpha_1 \geq 2\text{ and }\alpha_2 \geq 1.$$

But the number of ways of filling the blanks (- -) (-) (-) with a, b, c, and d is 6 giving a permutation of the same cycle type as g and fixing {a, b, c, d}. Therefore the number of permutations in A_n fixing {a, b, c, d} and with the same cycle type with g is

$$\frac{6(n-4)!}{1^{\alpha_1-2}(\alpha_1-2)!2^{\alpha_2-1}(\alpha_2-1)!\prod_{i=3}^{n}\alpha_i!\,i^{\alpha_i}}.$$

Case 3:
Each of the elements a, b, c, and d come from a 2-cycle in g say (ab)(cd)…. In this case the number of permutations in A_n fixing {a, b, c, d} and with the same cycle type as g is equal to the number of permutations of A_{n-4} with cycle type (α_1, α_2-2,α_3,… ,α_n). By Theorem 1.1.13 this number is $\frac{(n-4)!}{\alpha_1!1^{\alpha_1}2^{\alpha_2-2}(\alpha_2-2)!\prod_{i=3}^{n}\alpha_i!\,i^{\alpha_i}}$, for $\alpha_2\geq 2$.
But the number of ways of filing the blanks (- -) (- -) with a, b, c and d is 3, giving a permutation of the same cycle type as g and fixing {a, b, c, d}. Therefore the number of permutations in A_n fixing {a, b, c, d} and with the same cycle type as g is $\frac{3(n-4)!}{\alpha_1!1^{\alpha_1}2^{\alpha_2-2}(\alpha_2-2)!\prod_{i=3}^{n}\alpha_i!\,i^{\alpha_i}}$.

Case 4:
Three of the elements a, b, c and d come from a 3-cycle and one element comes from a single-cycle say (abc)(d)…. In this case the number of permutations in A_n fixing {a, b, c, d} and with the same cycle type as g is equal to the number of permutations of A_{n-4} with cycle type (α_1-1, α_2, α_3-1, α_4,…, α_n). By Theorem 1.1.13 this number is

$$\frac{(n-4)!}{1^{\alpha_1-1}(\alpha_1-1)!\alpha_2!2^{\alpha_2}3^{\alpha_3-1}(\alpha_3-1)!\prod_{i=4}^{n}\alpha_i!\,i^{\alpha_i}},\ \text{for } \alpha_1\geq 1,\ \alpha_3\geq 1.$$

However the number of ways of filling the blanks (- - -)(-) with a, b, c,and d is 8, giving a permutation of the same cycle type as g and fixing {a, b, c, d}. Therefore the number of permutations in A_n fixing {a, b, c, d} and having the same cycle type as g is

$$\frac{8(n-4)!}{1^{\alpha_1-1}(\alpha_1-1)!\alpha_2!2^{\alpha_2}3^{\alpha_3-1}(\alpha_3-1)!\prod_{i=4}^{n}\alpha_i!\,i^{\alpha_i}}.$$

Case 5:
The elements a, b, c and d come from a 4-cycle of g, say (abcd)…. In this case the number of permutations in A_n fixing {a, b, c, d} and with the same cycle type as g is equal to the number of permutations of A_{n-4} with cycle type (α_1,α_2,α_3,α_4-1,α_5,…,α_n). By Theorem 1.1.13 this number is $\frac{(n-4)!}{\alpha_1!1^{\alpha_1}\alpha_2!2^{\alpha_2}\alpha_3!3^{\alpha_3}4^{\alpha_4-1}(\alpha_4-1)!\prod_{i=5}^{n}\alpha_i!\,i^{\alpha_i}}$, for $\alpha_4\geq 1$.
But the number of ways of filling the blanks (- - - -) with a,b,c and d is 6, giving a permutation of the same cycle type as g and fixing {a, b, c, d}. Therefore the number of permutations in A_n fixing {a,b,c,d} and having the same cycle type as g is

$$\frac{6(n-4)!}{\alpha_1!1^{\alpha_1}\alpha_2!2^{\alpha_2}\alpha_3!3^{\alpha_3}4^{\alpha_4-1}(\alpha_4-1)!\prod_{i=5}^{n}\alpha_i!\,i^{\alpha_i}}.$$

Therefore the total number of permutations in A_n that fix {a, b, c, d} $\in X^{(4)}$ and with the same cycle type as g is the sum of the formulas in the five cases which yield the given formula. ■

2.2 Some properties of the alternating group $A_n(n\leq 7)$ acting on unordered quadruples

Theorem 2.2.1

$G=A_4$ acts transitively on $X^{(4)}$.

Proof

We can prove this by using the Cauchy-Frobenius Lemma (Theorem 1.1.15). Let $g\in A_4$ have cycle type (α_1, α_2, α_3, α_4), then the number of permutations in A_4 having the same cycle type as g is given by Theorem 1.1.13 and the number of elements in $X^{(4)}$ fixed by each $g\in A_4$ is given by Lemma 2.1.1. We now have the following Table

Table 2.2.1: Permutations in A_4 and number of fixed points

Permutations in A_4	**Cycle type**	**Number of permutations**	**$\lvert Fix(g)\rvert$ in $X^{(4)}$**
1	(4,0,0,0)	1	1
(abc)	(1,0,1,0)	8	1
(ab)(cd)	(0,2,0,0)	3	1

By Cauchy-Frobenius Lemma, we get the number of the orbits of A_4 acting on $X^{(4)}$,

$$\frac{1}{|A_4|}\sum_{g\in A_4}|Fix(g)| = \frac{1}{12}[(1\times 1)+(8\times 1)+(3\times 1)] = \frac{12}{12} = 1$$

This implies that A_4 acts transitively on $X^{(4)}$.

Alternatively we can use the Orbit-Stabilizer Theorem (Theorem 1.1.14). In this case we have to show that the length of the orbit of a point say {1,2,3,4} in $X^{(4)}$ is 1, the same as the number of points in $X^{(4)}$. Now let $g \in A_4$ have cycle type $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, then the number of permutations in A_4 fixing {1,2,3,4} and having the same cycle type as g is given by Lemma 2.1.2.

We now have the following Table;

Table 2.2.2: Number of permutations in G=A_4 fixing {1,2,3,4}

Permutation in A_4	**Cycle type**	**Number fixing {1,2,3,4}**
1	(4,0,0,0)	1
(abc)	(1,0,1,0)	8
(ab)(cd)	(0,2,0,0)	3
Total		12

From the table $\left|G_{\{1,2,3,4\}}\right|$=12.

Therefore by Orbit-Stabilizer Theorem,

$$|Orb_G\{1,2,3,4\}| = |G: Stab_G\{1,2,3,4\}| = \frac{|G|}{|Stab_G\{1,2,3,4\}|} = \frac{12}{12} = 1 = \left|X^{(4)}\right|.$$

Hence the orbit of {1,2,3,4} is the whole of $X^{(4)}$ and therefore A_4 acts transitively on $X^{(4)}$. ■

Theorem 2.2.2

G=A_5 acts transitively on $X^{(4)}$.

Proof

We can prove this by Cauchy-Frobenius Lemma (Theorem 1.1.15). Let $g \in A_5$ have cycle type

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, then the number of permutations in A_5 having the same cycle type as g is given by Theorem 1.1.13 and the number of elements in $X^{(4)}$ fixed by each $g \in A_5$ is given by Lemma 2.1.1. We now have the following Table;

Table 2.2.3: Permutations in A_5 and number of fixed points

Permutations in A_5	Cycle type	Number of permutations	$\lvert Fix(g)\rvert$ in $X^{(4)}$
1	(5,0,0,0,0)	1	5
(abc)	(2,0,1,0,0)	20	2
(ab)(cd)	(1,2,0,0,0)	15	1
(abcde)	(0,0,0,0,1)	24	0

By Cauchy Frobenius Lemma, we get the number of the orbits of A_5 acting on $X^{(4)}$,

$$\frac{1}{|A_5|}\sum_{g\in A_5}|Fix(g)| = \frac{1}{60}[(1\times 5)+(20\times 2)+(15\times 1)+(24\times 0)]$$
$$=\frac{60}{60}=1.$$

This implies that A_5 acts transitively on $X^{(4)}$.

Alternatively we can use the Orbit-Stabilizer Theorem (Theorem 1.1.12). In this case we have to show that the length of the orbit of a point say {1,2,3,4} in $X^{(4)}$ is 5, the same as the number of points in $X^{(4)}$. Let $g \in A_5$ have cycle type $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, then the number of permutations in A_5 fixing {1,2,3,4} and having the same cycle type as g is given by Lemma 2.1.2. We now have the following Table;

Table 2.2.4: Number of permutations in G=A_5 fixing {1,2,3,4}

Permutation in A_5	Cycle type	Number fixing {1,2,3,4}
1	(5,0,0,0,0)	1
(abc)	(2,0,1,0,0)	8
(ab)(cd)	(1,2,0,0,0)	3
(abcd)	(0,0,0,0,1)	0
Total		12

From the table $\left|G_{\{1,2,3,4\}}\right|$=12.

Therefore by Orbit-Stabilizer Theorem,

$$|Orb_G\{1,2,3,4\}|=|G:Stab_G\{1,2,3,4\}|$$
$$=\frac{|G|}{|Stab_G\{1,2,3,4\}|}$$
$$=\frac{60}{12}=5=\left|X^{(4)}\right|.$$

Hence the orbit of {1,2,3,4} is the whole of $X^{(4)}$ and therefore A_5 acts transitively on $X^{(4)}$. ■

Theorem 2.2.3

A_5 does not act doubly transitively on $X^{(4)}$

Proof

Given any two pair of points say {1, 2, 3, 4}, {1, 2, 3, 5} $\in X^{(4)}$ and {1, 2,4,5}, {1, 5, 3, 4} $\in X^{(4)}$ and suppose that there exists a permutation $g \in A_5$ such that g[{1, 2, 3, 4}, {1, 2, 3,5}]=[{1, 2, 4,5}, {1,5, 3, 4}]; then g {1, 2, 3, 4}= {g(1),g(2), g(3), g(4)}= {1,2,4,5} and g {1, 2, 3, 5}= {g(1),g(2), g(3), g(5)}={1, 5, 3, 4}. Implying that g(2)=2 and g(2)=5, this is impossible. Thus A_5 does not act doubly transitively on $X^{(4)}$. ■

Theorem 2.2.4 G=A_6 acts transitively on $X^{(4)}$.

Proof

We will prove this by Cauchy-Frobenius Lemma (Theorem 1.1.15). Let g ∈ A_6 have cycle type (α_1, α_2, α_3, α_4, α_5, α_6), then the number of permutations in A_6 having the same cycle type as g is given by Theorem 1.1.13 and the number of elements in $X^{(4)}$ fixed by each g ∈ A_6 is given by Lemma 2.1.1. We now have the following Table;**Table 2.2.5: Permutations in A_6 and number of fixed points**

Permutations in A_6	**Cycle type**	**Number of permutations**	**$\|Fix(g)\|$ in $X^{(4)}$**
1	(6,0,0,0,0,0)	1	360
(abc)	(3,0,1,0,0,0)	40	0
(ab)(cd)	(2,2,0,0,0,0)	45	0
(abcde)	(1,0,0,0,1,0)	144	0
(ab)(cdef)	(0,1,0,1,0,0)	90	0
(abc)(def)	(0,0,2,0,0,0)	40	0

By Cauchy-Frobenius Lemma, we get the number of the orbits of A_6 acting on $X^{(4)}$,

$$\frac{1}{|A_6|}\sum_{g\in A_6}|\text{Fix}(g)| = \frac{1}{360}[(1\times 360)+(0\times 40)+(0\times 144)+(0\times 45)+(0\times 90)+(0\times 40)]$$

$$=\frac{360}{360}=1.$$

This implies that A_6 acts transitively on $X^{(4)}$.

Alternatively we can use the Orbit-Stabilizer Theorem (Theorem 1.1.14). In this case we have to show that the length of the orbit of a point say {1,2,3,4} in $X^{(4)}$ is 15, the same as the number of points in $X^{(4)}$. Let g ∈ A have cycle type (α_1, α_2, α_3, α_4, α_5, α_6), then the number of permutations in A_6 fixing {1,2,3,4} and having the same cycle type as g is given by Lemma 2.1.2.

We now have the following Table;

Table 2.2.6: Number of permutations in G=A_6 fixing {1,2,3,4}

Permutation in A_6	**Cycle type**	**Number fixing {1,2,3,4}**
1	(6,0,0,0,0,0)	1
(abc)	(3,0,1,0,0,0)	8
(ab)(cd)	(2,2,0,0,0,0)	9
(abcde)	(1,0,0,0,1,0)	0
(ab)(cdef)	(0,1,0,1,0,0)	6
(abc)(def)	(0,0,2,0,0,0)	0
Total		24

From the table $|G_{\{1,2,3,4\}}|$=24.

Therefore by Orbit-Stabilizer Theorem,

$$|\mathrm{Orb}_G\{1,2,3,4\}|=|G:\mathrm{Stab}_G\{1,2,3,4\}| = \frac{|G|}{|\mathrm{Stab}_G\{1,2,3,4\}|} = \frac{360}{24} = 15 = |X^{(4)}| .$$

Hence the orbit of {1,2,3,4} is the whole of $X^{(4)}$ and therefore A_6 acts transitively on $X^{(4)}$. ■

Theorem 2.2.5

A_6 does not act doubly transitively on $X^{(4)}$.

Proof

Given any two pair of points say {1, 2, 3, 4}, {1, 2, 3, 5} $\in X^{(4)}$ and {1, 2,4,6}, {1, 6, 3, 4} $\in X^{(4)}$ and suppose that there exists a permutation $g \in A_6$ such that g[{1, 2, 3, 4}, {1, 2, 3,5}]=[{1, 2, 4,6}, {1,6, 3, 4}]; then g {1, 2, 3, 4}= {g(1),g(2), g(3), g(4)}= {1,2,4,6} and g {1, 2, 3, 5}= {g(1),g(2), g(3), g(5)}={1, 6, 3, 4}. Implying that g(2)=2 and g(2)=6, this is impossible. Thus A_6 does not act doubly transitively on $X^{(4)}$. ■

Theorem 2.2.6

$G=A_7$ acts transitively on $X^{(4)}$.

Proof

We can prove this by Cauchy-Frobenius Lemma (Theorem 1.1.15). Let $g \in A_7$ have cycle type ($\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\ \alpha_7$), then the number of permutations in A_7 having the same cycle type as g is given by Theorem 1.1.13 and the number of elements in $X^{(4)}$ fixed by each $g \in A_7$ is given by Lemma 2.1.1. We now have the following Table;

Table 2.2.7: Permutations in A_7 and the number of fixed points

Permutation g in A_7	Cycle type	Number of permutations	$\|Fix(g)\|$ in $X^{(4)}$
1	(7,0,0,0,0,0,0,)	1	35
(abc)	(4,0,1,0,0,0,0)	70	5
(abcde)	(2,0,0,0,1,0,0)	504	0
(abcdefg)	(0,0,0,0,0,0,1)	720	0
(ab)(cdef)	(1,1,0,1,0,0,0)	630	1
(ab)(cd)	(3,2,0,0,0,0,0)	105	7
(ab)(cd)(efg)	(0,2,1,0,0,0,0)	210	1
(abc)(def)	(1,0,2,0,0,0,0)	280	2

By Cauchy-Frobenius Lemma we get the number of the orbits of A_7 acting on $X^{(4)}$,

$$\frac{1}{|A_7|}\sum_{g\in A_7}|\mathrm{Fix}(g)| = \frac{1}{2520}[(35x1) + (5x70) + (0x504) + (0x720) + (1x630) + (7x105) + (1x210) + (2x280)]$$

$$= \frac{1}{2520}[35+350+630+735+210+560]$$

$$= \frac{2520}{2520} = 1 .$$

This implies that A_7 acts transitively on $X^{(4)}$.

Alternatively we can use the Orbit-Stabilizer Theorem (Theorem 1.1.14). In this case we have to show that the length of the orbit of a point say {1, 2, 3, 4} in $X^{(4)}$ is 35, the same as the number of points in $X^{(4)}$. Let $g \in A_7$ have a cycle type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, the number of permutations in A_7 fixing {1, 2, 3, 4} and having the same cycle type as g is given by Lemma 2.1.2. We now have the following Table;

Table 2.2.8: Number of permutations in G=A_7 fixing {1, 2, 3, 4}

Permutation g in A_7	Cycle type	Number fixing {1,2,3,4}
1	(7,0,0,0,0,0,0,)	1
(abc)	(4,0,1,0,0,0,0)	10
(abcde)	(2,0,0,0,1,0,0)	0
(abcdefg)	(0,0,0,0,0,0,1)	0
(ab)(cdef)	(1,1,0,1,0,0,0)	18
(ab)(cd)	(3,2,0,0,0,0,0)	21
(ab)(cd)(efg)	(0,2,1,0,0,0,0)	6
(abc)(def)	(1,0,2,0,0,0,0)	16
Total		72

From the table $\left|G_{\{1,2,3,4\}}\right|$=72.

Therefore by Orbit-Stabilizer Theorem,

$$|\mathrm{Orb}_G\,\{1,2,3,4\}| = |G:\mathrm{Stab}_G\,\{1,2,3,4\}|$$
$$= \frac{|G|}{|\mathrm{Stab}_G\,\{1,2,3,4\}|}$$
$$= \frac{2520}{72} = 35 = \left|X^{(4)}\right| .$$

Hence the orbit of {1, 2, 3, 4} is the whole of $X^{(4)}$ and therefore A_7 acts transitively on $X^{(4)}$. ■

Does not act doubly transitively on $X^{(4)}$.

Proof

Given any two pair of points say {1, 2, 3, 4},{1, 2, 3, 5}∈ $X^{(4)}$ and {1, 2, 3, 6},{1, 7, 3, 4}∈ $X^{(4)}$ and suppose that there exists a permutation $g \in A_7$ such that $g[\{1,2,3,4\},\{1,2,3,5\}] = [\{1,2,3,6\},\{1,7,3,4\}]$; then g {1, 2, 3, 4}= {g(1),g(2), g(3), g(4)}= {1,2,3, 6} and g {1, 2, 3, 5}= {g(1),g(2), g(3), g(5)} = {1, 7, 3, 4}. Implying that g(2)=2 and g(2)=7, this is impossible. Thus A_7 does not act doubly transitively on $X^{(4)}$

. ■

3.ACTIONS OF THE ALTERNATING GROUP An ON ORDERED QUADRUPPLES

3.1 some general results of permutation groups acting on $X[4]$

Similarly like in section 2.1 we give the proofs of two lemmas which will be very useful in the investigation of transitivity of the action of An on $X[4]$

Lemma 2.1.3

Let $g \in A_n$ be a permutation with cycle type $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Then |Fix (g)| in $X^{[4]}$ is given by the formula

$4!\binom{\alpha_1}{4}$.

Proof

Let [a,b,c,d] ∈ $X^{[4]}$ and g ∈ A_n. Then g fixes [a,b,c,d] if and only if each of the elements a,b,c,d are mapped onto themselves, that is, g [a,b,c,d]=[g(a),g(b),g(c),g(d)]=[a,b,c,d] implying ga=a, gb=b, gc=c and gd=d. Thus each of a,b,c and d comes from single cycles. Therefore the number of unordered quadruples fixed by g ∈ A_n is

$$\binom{\alpha_1}{4}.$$

But unordered quadruple, can be rearranged to give 24=4! distinct ordered quadruples. Thus the number of ordered quadruples fixed by g ∈ A_n is

$$4!\binom{\alpha_1}{4}.$$ ■

Lemma 2.1.4

Let g ∈ A_n be a permutation with cycle type ($\alpha_1, \alpha_2,\ldots,\alpha_n$). Then the number of permutations in A_n fixing [a,b,c,d] ∈ $X^{[4]}$ and having the same cycle type as g is given by

$$\frac{(n-4)!}{(\alpha_1-4)!1^{\alpha_1-4}\prod_{i=2}^{n}\alpha_i!i^{\alpha_i}}, \text{ for } \alpha_1 \geq 4.$$

Proof

Let g ∈ A_n have cycle type ($\alpha_1, \alpha_2,\ldots, \alpha_n$) and let g fix [a,b,c,d] .Then each of a,b,c and d must come from a single cycle in g. Thus to count the number of permutations in A_n having the the same cycle types as g and fixing a,b,c and d is the same as counting the number of permutations in A_n-4 having cycle type (α_1-4, $\alpha_2,\ldots, \alpha_n$). By Theorem 1.1.13 this number is

$$\frac{(n-4)!}{(\alpha_1-4)!1^{\alpha_1-4}\prod_{i=2}^{n}\alpha_i!i^{\alpha_i}}, \quad \text{for } \alpha_1 \geq 4.$$ ■

.3 Some properties of the alternating groups A_n (n≤7) acting on $X^{[4]}$

Theorem 2.3.1 G=A_6 acts transitively on $X^{[4]}$.

Proof

We can prove this by the use of Cauchy-Frobenius Lemma (Theorem 1.1.15). Let g ∈ A_6 have cycle type (α_1, $\alpha_2,\ldots,\alpha_6$), then the number of permutations in A_6 having the same cycle type as g is given by Theorem 1.1.13 and the number of elements in $X^{[4]}$ fixed by g is given by Lemma 2.1.3. We now have the following Table;

Table 2.3.1: Permutations in A_6 and the number of fixed points

Permutation in A_6	**Cycle type**	**Number of permutations**	**|Fix(g)| in $X^{[4]}$**
1	(6,0,0,0,0,0)	1	360
(abc)	(3,0,1,0,0,0)	40	0
(abcde)	(1,0,0,0,1,0)	144	0
(ab)(cd)	(2,2,0,0,0,0)	45	0
(ab)(cdef)	(0,1,0,1,0,0)	90	0
(abc)(def)	(0,0,2,0,0,0)	40	0

By Cauchy-Frobenius Theorem we get the number of the orbits of A_6 acting on $X^{[4]}$,

$$\frac{1}{|A_6|}\sum_{g\in A_6}|\text{Fix}(g)| = \frac{1}{360}[(360\text{x}1) + (0\text{x}40) + (0\text{x}144) + (0\text{x}45) + (0\text{x}90) + (0\text{x}40)]$$

$$= \frac{360}{360} = 1.$$

This implies that A_6 acts transitively on $X^{[4]}$. ■

Alternatively we can use the Orbit-Stabilizer Theorem (Theorem 1.1.14). In this case we have to show that the length of the orbit of a point say [1, 2, 3, 4] in $X^{[4]}$ is 360, the same as the number of points in $X^{[4]}$. Let $g \in A_6$ have cycle type $(\alpha_1\alpha_{2,}\ldots,\alpha_6)$, then the number of permutations in A_6 fixing [1, 2, 3, 4] and having the same cycle type as g is given by Lemma 2.1.4. We now have the following Table;

Table 2.3.2: Number of permutations in G=A_6 fixing [1, 2, 3, 4]

Permutation in A_6	Cycle type	Number fixing [12,3,4]
1	(6,0,0,0,0,0)	1
(abc)	(3,0,1,0,0,0)	0
(abcde)	(1,0,0,0,1,0)	0
(ab)(cd)	(2,2,0,0,0,0)	0
(ab)(cdef)	(0,1,0,1,0,0)	0
(abc)(def)	(0,0,2,0,0,0)	0

By Orbit-Stabilizer Theorem,

$$|\mathrm{Orb}_G\,[1,2,3,4]| = |G:\mathrm{Stab}_G\,[1,2,3,4]| = \frac{|G|}{|\mathrm{Stab}_G\,[1,2,3,4]|} = \frac{360}{1} = 360 = \left|X^{[4]}\right|.$$

Hence the orbit of [1, 2, 3, 4] is the whole of $X^{[4]}$ and therefore A_6 acts transitively on $X^{[4]}$. ■

Theorem 2.3.2

G=A_6 does not act doubly transitively on $X^{[4]}$.

Proof

Given any two pair of points say [1, 2, 3, 4], [1, 2, 4, 5] $\in X^{[4]}$ and [1, 2, 3, 6], [1, 6, 3, 4] $\in X^{[4]}$ and suppose that there exists a permutation $g \in A_6$ such that $g\big[[1,2,3,4],[1,2,4,5]\big] = \big[[1,2,3,6],[1,6,3,4]\big]$; then g [1, 2, 3, 4]= [g(1),g(2), g(3), g(4)]= [1,2,3,6] and g [1, 2, 3, 5]= [g(1),g(2), g(3), g(5)] = [1, 6, 3, 4]. Implying that g(2)=2 and g(2)=6, this is impossible. Thus A_6 does not act doubly transitively on $X^{[4]}$. ■

Theorem 2.3.3

G=A_7 acts transitively on $X^{[4]}$.

Proof

We can prove this by the use of Cauchy–Frobenius Lemma (Theorem 1.1.15). Let $g \in A_7$ have cycle type $(\alpha_1, \alpha_2,\ldots,\alpha_7)$, then the number of permutations in A_7 having the same cycle type as g is given by Theorem 1.1.13 and the number of elements in $X^{[4]}$ fixed by g is given by Lemma 2.1.3. We now have the following Table;

Table 2.3.3: Permutations in A_7 and the number of fixed points

Permutation g in A7	Cycle type	Number of permutations	$\|Fix(g)\|$ in $X^{[4]}$
1	(7,0,0,0,0,0,0,)	1	840
(abc)	(4,0,1,0,0,0,0)	70	24
(abcde)	(2,0,0,0,1,0,0)	504	0
(abcdefg)	(0,0,0,0,0,0,1)	720	0
(ab)(cdef)	(1,1,0,1,0,0,0)	630	0
(ab)(cd)	(3,2,0,0,0,0,0)	105	0
(ab)(cd)(efg)	(0,2,1,0,0,0,0)	210	0
(abc)(def)	(1,0,2,0,0,0,0)	280	0

By Cauchy-Frobenius Lemma we get the number of the orbits of A_7 acting on $X^{[4]}$,

$\frac{1}{|A_7|}\sum_{g\in A_7}|Fix(g)|$

$=\frac{1}{2520}[(840x1) + (24x70) + (0x504) + (0x720) + (0x630) + (0x105) + (0x210) + (0x280)]$

$= \frac{1}{2520}[840+1680+0+0+0+0+0+0+0]$

$=\frac{2520}{2520}=1.$

This implies that A_7 acts transitively on $X^{[4]}$.

Alternatively we can use the Orbit-Stabilizer Theorem (Theorem 1.1.14). In this case we have to show that the length of the orbit of a point say [1, 2, 3, 4] in $X^{[4]}$ is 840, the same as the number of points in $X^{[4]}$. Let $g \in A_n$ have cycle type $(\alpha_1\alpha_{2,}\ldots,\alpha_7)$, then number of permutations in A_7 fixing [1, 2, 3, 4] and having the same cycle type as g is given by Lemma 2.1.4. We now have the following Table;

Table 2.3.4: Number of permutations in G=A_7 fixing [1, 2, 3, 4]

Permutation g in A_7	**Cycle type**	**Number fixing [12,3,4]**
1	(7,0,0,0,0,0,0,)	1
(abc)	(4,0,1,0,0,0,0)	2
(abcde)	(2,0,0,0,1,0,0)	0
(abcdefg)	(0,0,0,0,0,0,1)	0
(ab)(cdef)	(1,1,0,1,0,0,0)	0
(ab)(cd)	(3,2,0,0,0,0,0)	0
(ab)(cd)(efg)	(0,2,1,0,0,0,0)	0
(abc)(def)	(1,0,2,0,0,0,0)	0
Total		3

By Orbit-Stabilizer Theorem,

$|Orb_G\,[1,2,3,4]|=|G: Stab_G\,[1,2,3,4]|$

$=\frac{|G|}{|Stab_G\,[1,2,3,4]|}=\frac{2520}{3}$

$=840=|X^{[4]}|$.

Hence the orbit of [1, 2, 3, 4] is the whole of $X^{[4]}$ and therefore A_7 acts transitively on $X^{[4]}$. ∎

Theorem 2.3.4

G=A_7 does not act doubly transitively on $X^{[4]}$.

Proof

Given any two pair of points say [1, 2, 3, 4], [1, 2, 4, 7] ∈ $X^{[4]}$ and [1, 2, 3, 5], [1, 7, 3, 4] ∈ $X^{[4]}$ and suppose that there exists a permutation $g \in A_7$ such that $g[[1,2,3,4],[1,2,4,7]]$

$= [[1,2,3,5],[1,7,3,4]]$; then g [1, 2, 3, 4]= [g(1),g(2), g(3), g(4)]= [1,2,3,5] and g [1, 2, 4, 7]= [g(1),g(2), g(3), g(7)] = [1, 7, 3, 4]. Implying that g(2)=2 and g(2)=7, this is impossible. Thus A_7 does not act doubly transitively on $X^{[4]}$. ■

Therefore G acts transitively on $X^{[4]}$

Conclusion: This implies that An (for n$\leq$ 7) acts transitively on $X^{(4)}$ a nd $X^{[4]}$

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