Bilateral generating relations for modified Konhauser polynomial with discrete variable

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Abstract

In [10], we introduced a modified Konhauser polynomial with discrete variable. In this paper an attempt has been made to obtain some bilateral generating relations for modified Konhauser polynomial with discrete variable. Each result is followed by its applications to the classical orthogonal polynomials.

Key words: Konhauser polynomial, modified Jacobi polynomial, Laguerre polynomial.

Introduction

In the previous paper [10], we introduced modified Konhauser polynomial with discrete variable $Z_n^{\alpha}(x;w,k)$ defined as follows:

$$(1.1)\qquad Z_n^{\alpha}(x;w,k) = \frac{\Gamma(kn+\alpha+1)}{n!}\sum_{j=0}^{n}\frac{(-1)^j \binom{n}{j}^{[jw]}\left\{x^{[kw]}\right\}}{\Gamma(kj+\alpha+1)}$$

where Milne-Thomson [6]

$$x^{[kw]} = x(x-w)(x-2w)....(x-kw+w)$$

and the following notations have been adopted

$$^{[kw]}x = x(x+w)(x+2w)...(x+kw-w)$$

So that

$$^{[kw]}x = \binom{x}{w}_j w^j \text{ and } ^{[kw]}\left\{x^{[kw]}\right\} = {}^{[jw]}x.{}^{[jw]}(x-w).{}^{[jw]}(x-2w)...{}^{[jw]}(x-kw+w)$$

We also derived the following relation

(1.2) $$Z_n^{\alpha}(x;w,k) = \frac{(1+\alpha)_{kn}}{n!}\, {}_{k+1}F_k\left(\begin{matrix} -n, \frac{x}{w}, \frac{x}{w}-1, ..., \frac{x}{w}-k+1; \\ \Delta(k;1+\alpha); \end{matrix} \left(\frac{w}{k}\right)^k\right),$$

where, well known notation Satyanarayana [9, pp.30(vii)]

(1.3) $$\Delta(k,1+\alpha) = \frac{1+\alpha}{k}\frac{2+\alpha}{k}....\frac{k-1+\alpha}{k}$$

and well-known lemma Rainville [8, 22(2)],

(1.4) $$(\alpha)_{kn} = k^{kn}\left(\frac{\alpha}{k}\right)_n\left(\frac{\alpha+1}{k}\right)_n....\left(\frac{\alpha+k-1}{k}\right)_n$$

And ${}_pF_q$ is generalized hypergeometric function Rainville [8].

Taking the limit $w \to 0$ in (1.1), we obtain

(1.5) $$\lim_{w\to 0} Z_n^{\alpha}(x,w,k) = Z_n^{\alpha}(x;k),$$

where $Z_n^{\alpha}(x;k)$ is Konhauser polynomial of two variables [2].

In [3], Lahiri and Satyanarayana defined and studied a generalized hypergeometric function $I_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)$ and also proved that

(1.6) $$\sum_{n=0}^{\infty}\binom{n+r}{r}\frac{(\rho+r)_n}{(1+\alpha+r)_n} I_{n+r}^{\alpha}(x,w)t^n$$

$$=(1-t)^{-\rho-r}\binom{\alpha+r}{r}F^{(3)}\left[\begin{matrix}(a_p)::\frac{x}{w}-\mu+1;—;—;\rho+r;-r;\frac{-x}{w}+\lambda \\ (b_q)::1+\alpha;—;—;—;—;—;\end{matrix} -\frac{wt}{1-t}, w, w\right]$$

Where $F^{(3)}$ is generalized hypergeometric function of three variables (see Srivastava and Karlsson [11]).

In particular for $\lambda = 0$, $\mu = 1$, $p = q$ and $a_j = b_j (j = 1, 2, …, p)$ we have

(1.7) $$I_{n;0;(a_p)}^{\alpha;1;(a_p)}(x,w) = (1-w)^{\frac{x}{w}} J_n^{\alpha}(x,w),$$

Where $J_n^{\alpha}(x,w)$ is modified Jacobi polynomial (see Parihar and Patel [7]) and

(1.8) $$\lim_{w\to 0} I_{n;\lambda;(a_p)}^{\alpha;\mu;(a_p)}(x,w) = e^{-x} L_n^{\alpha}(x),$$

where $L_n^{\alpha}(x)$ is Laguerre polynomial [8].

The following definitions and results given by Rainville [8 , p.302] Gottlieb polynomial

(1.9) $$\phi_n(x;\lambda) = e^{-n\lambda}\, {}_2F_1(-n,-x;1;1-e^{\lambda}),$$

Generalized Sylvester polynomial

(1.10) $$f_n(x;a) = \frac{(ax)^n}{n!}\, {}_2F_0(-n,x;-;-\frac{1}{ax})$$

Agarwal and Manocha [1, p.1372(2.2), p.1374(5.5)]

(1.11) $$\sum_{n=0}^{\infty}\binom{n+k}{k}\phi_{n+k}(x;\lambda)t^n = (1-t)^{x-k}(1-te^{-\lambda})^{-x-1}\phi_k(x;\log_e\left(\frac{e^{\lambda}-t}{1-t}\right))$$

and

(1.12) $$\sum_{n=0}^{\infty}\binom{n+k}{k} f_{n+k}(x;a)t^n = (1-t)^{-x-k} e^{axt} f_k(x;a(1-t)).$$

2. Bilateral generating relations

We prove the following main bilateral generating relations for the modified Konhauser polynomial with discrete variable [10].

(2.1) $$\sum_{n=0}^{\infty} \frac{n!(\eta)_n}{(1+\alpha)_n(1+\beta)_{nk}} Z_n^{\beta}(y;\nu,k) I_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)t^n$$

$$= (1-t)^{-\eta} F_{q+k+1:0;0;0;0}^{p+k+2:0;0;0;1}\left[\begin{matrix}[(a_p):0,1,1,1],[\eta:1,1,1,0]: \\ [(b_q):0,1,1,1]: \quad - \quad :\end{matrix}\right.$$

$$\begin{matrix}\left[\frac{y}{\nu}:1,0,1,0\right],\left[\frac{y}{\nu}-1:1,0,1,0\right],---,\left[\frac{y}{\nu}-k+1:1,0,1,0\right]; \\ [\Delta(k;1+\beta):1,0,1,0] \qquad ;\end{matrix}$$

$$\left.\begin{matrix}[\frac{x}{w}-\mu+1:0,1,1,0];-;-;-;-\frac{x}{w}+\lambda; \\ [1+\alpha:0,1,1,0];-;-;\ -\ ;\ -\ ;\end{matrix}\quad E,F,G,H\right]$$

where $E = \frac{t}{t-1}\left(\frac{v}{k}\right)^k$, $F = \frac{-wt}{1-t}$, $G = \frac{wt}{t-1}\left(\frac{v}{k}\right)^k$ *and* $H = w$

and F is a generalized Lauricella hypergeometric function of 4 variables.

$$(2.2)\ \sum_{n=0}^{\infty} \frac{n!}{(1+\beta)_{nk}} Z_n^{\beta}(y;v,k)\,\phi_n(x;\lambda)t^n$$

$$= (1-t)^x \left(1-te^{-\lambda}\right)^{-x-1} F^{k+1:1;0}_{k:1;0}\left[\begin{matrix}\left[\frac{y}{v}:1,1\right],\left[\frac{y}{v}-1:1,1\right],---,\left[\frac{y}{v}-k+1:1,1\right],\\ [\Delta(k;1+\beta):1,1]\end{matrix}\right.$$

$$\left.\begin{matrix}[1:1,1]:-x;-;\\ :1\,;-\,;\end{matrix}\ -\frac{t}{1-t}\left(\frac{e^{\lambda}-1}{e^{\lambda}-t}\right)\left(\frac{v}{k}\right)^k, \left(\frac{-t}{e^{\lambda}-t}\right)\left(\frac{v}{k}\right)^k\right]$$

Where $F^{p:u;v}_{q:w;l}(x,y)$ is hypergeometric functions of two variables (see Srivastava and Karlsson [11])..

and

$$(2.3)\quad \sum_{n=0}^{\infty} \frac{n!}{(1+\beta)_{nk}} Z_n^{\beta}(y;v,k)\, f_n(x;a)t^n$$

$$= (1-t)^{-x} e^{axt} F^{k:1;0}_{k:0;0}\left[\begin{matrix}\left[\frac{y}{v}:1,1\right],\left[\frac{y}{v}-1:1,1\right],---,\left[\frac{y}{v}-k+1:1,1\right],[1:1,1]:\\ [\Delta(k;1+\beta):1,1]\qquad :\end{matrix}\right.$$

$$\left.\begin{matrix}x;-;\\ -;-;\end{matrix}\quad -\frac{t}{1-t}\left(\frac{v}{k}\right)^k,\ -axt\left(\frac{v}{k}\right)^k\right]$$

Where $F^{p:u;v}_{q:w;l}(x,y)$ is hypergeometric functions of two variables (see Srivastava and Karlsson [11])..

Proof of (2.1). The proof of (2.1) can be developed easily from (1.2)) as follows

$$\sum_{n=0}^{\infty} \frac{n!(\eta)_n}{(1+\alpha)_n(1+\beta)_{nk}} Z_n^{\beta}(y;v,k) I^{\alpha;\mu;(a_p)}_{n;\lambda;(b_q)}(x,w)t^n$$

$$= \sum_{n=0}^{\infty} \frac{(\eta)_n}{(1+\alpha)_n}\ {}_{k+1}F_k\left[\begin{matrix}-n,\frac{y}{v},\frac{y}{v}-1,---,\frac{y}{v}-k+1;\\ \Delta(k,1+\beta)\qquad ;\end{matrix}\left(\frac{v}{k}\right)^k\right] I^{\alpha;\mu;(a_p)}_{n;\lambda;(b_q)}(x,w)t^n$$

$$= \sum_{l=0}^{\infty} \frac{(\eta)_l \left(\frac{y}{v}\right)_l \left(\frac{y}{v}-1\right)_l ---\left(\frac{y}{v}-k+1\right)_l \left(\frac{v}{k}\right)^{lk} t^l}{(1+\alpha)_l \ \Delta_l(k;1+\beta)} \sum_{n=0}^{\infty} \binom{n+l}{l} \frac{(\eta+l)_n}{(1+\alpha+l)_n} I_{n+l;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w)t^n$$

By using (1. 6) and simplifying, we arrive (2.1).

Applications

(i) By writing $p=q,\ a_j=b_j,\ j=1,2,....,p;\ \mu=1 \ and \ \lambda=0$ in (2.1), we obtain

$$(2.4) \sum_{n=0}^{\infty} \frac{n!(\eta)_n}{(1+\alpha)_n(1+\beta)_{nk}} Z_n^{\beta}(y;v,k) J_n^{\alpha}(x,w)t^n$$

$$= (1-t)^{-\eta} F_{k+1:0;0;0}^{k+2:0;0;0}\left[\begin{matrix} [\eta:1,1,1],\left[\frac{y}{v}:1,0,1\right],\left[\frac{y}{v}-1:1,0,1\right],---, \left[\frac{y}{v}-k+1:1,0,1\right],[\frac{x}{w}:0,1,1]:-;-;-; \\ [\Delta(k;1+\beta):1,0,1],[1+\alpha:0,1,1] :-;-;-; \end{matrix} E,F,G\right]$$

where

$$E=\frac{t}{t-1}\left(\frac{v}{k}\right)^k,\ F=\frac{-wt}{1-t},\ G=\frac{wt}{t-1}\left(\frac{v}{k}\right)^k,$$

Where $J_n^{\alpha}(x,w)$ is modified Jacobi polynomial (see Parihar and Patel [7]).

(ii) By writing $p=q,\ a_j=b_j,\ j=1,2,....,p;\ \mu=1 \ and \ \lambda=0$ and taking $w\to 0$ in (2.1), we obtain

$$(2.5) \sum_{n=0}^{\infty} \frac{n!(\eta)_n}{(1+\alpha)_n(1+\beta)_{nk}} Z_n^{\beta}(y,v,k) L_n^{\alpha}(x)t^n$$

$$= (1-t)^{-\eta} F_{k+1:0;0;0}^{k+1:0;0;0}\left[\begin{matrix} [\eta:1,1,1],\left[\frac{y}{v}:1,0,1\right],\left[\frac{y}{v}-1:1,0,1\right],---, \\ [\Delta(k;1+\beta):1,0,1],[1+\alpha:1,0,1] \end{matrix}\right.$$

$$\left.\begin{matrix} \left[\frac{y}{v}-k+1:1,0,1\right]:-;-;-; \\ \qquad\qquad \frac{t}{t-1}\left(\frac{v}{k}\right)^k, \frac{-xt}{1-t}, \frac{xt}{t-1}\left(\frac{v}{k}\right)^k \\ :-;-;-; \end{matrix}\right]$$

where $L_n^{\alpha}(x)$ is Laguerre polynomial [8].

(iii) By taking the limit as $v \to 0$ in (2.1), we arrive at the following result.

$$(2.6)\ \sum_{n=0}^{\infty} \frac{n!\,(\eta)_n}{(1+\alpha)_n (1+\beta)_{nk}} Z_n^{\beta}(y,k)\, I_{n;\lambda;(b_q)}^{\alpha;\mu;(a_p)}(x,w) t^n$$

$$= (1-t)^{-\eta} F_{q+k+1:0;0;0;0}^{p+2:0;0;0;1}\left[\begin{matrix} [(a_p):0,1,1,1],[\eta:1,1,1,0],[\frac{x}{w}-\mu+1:0,1,1,0] : \\ [(b_q):0,1,1,1],[\Delta(k;1+\beta):1,0,1,0],[1+\alpha:0,1,1,0]: \end{matrix}\right.$$

$$\left.\begin{matrix} -;-;-;-\frac{x}{w}+\lambda; \\ \qquad \frac{t}{t-1}\left(\frac{y}{k}\right)^k, \frac{-wt}{1-t}, \frac{wt}{t-1}\left(\frac{y}{k}\right)^k, w \\ -;-;-;\quad - \quad ; \end{matrix}\right]$$

where $Z_n^{\alpha}(x;k)$ is Konhauser polynomial [2].

(iv) By writing $p = q$, $a_j = b_j$, $j = 1, 2, ---, p$; $\mu = 1$, $\lambda = 0$ *and* $k = 1$ and taking the limit as $w \to 0$, $v \to 0$ in (2.1), we have

$$(2.7)\ \sum_{n=0}^{\infty} \frac{n!\,(\eta)_n}{(1+\alpha)_n (1+\beta)_n} L_n^{\beta}(y)\ L_n^{\alpha}(x) t^n$$

$$= (1-t)^{-\eta} F_{2:0;0;0;0}^{1:0;0;0;0}\left[\begin{matrix} [\eta:1,1,1,0]:-;-;-;-; \\ [1+\beta):1,0,1,0],[1+\alpha:0,1,1,0]:-;-;-;-; \end{matrix}\right.$$

$$\left.\frac{yt}{t-1},\frac{-xt}{1-t},\frac{xyt}{t-1},x\right]$$

Proof of (2.2). By using (1.2), (1.9) and (1.11) as in the proof (2.1), we can easily derive the proof of (2.2).

Applications :

(i) By applying $v \to 0$ in (2.2), it becomes

$$(2.8)\ \sum_{n=0}^{\infty} \frac{n!}{(1+\beta)_{nk}} Z_n^{\beta}(y;k)\phi_n(x;\lambda)t^n$$

$$=(1-t)^x (1-te^{-\lambda})^{-x-1} F_{1:1;0}^{1:1;0}\left[\begin{matrix}[1:1,1] & :-x;-; \\ [\Delta(k;1+\beta):1,1]:1;-; & \end{matrix} -\frac{(e^{\lambda}-1)y^k t}{(e^{\lambda}-t)(1-t)}, -\frac{y^k t}{(e^{\lambda}-t)}\right]$$

(ii) Applying $v \to 0$ & $k = 1$ in(2.2), we have

$$(2.9)\ \sum_{n=0}^{\infty} \frac{n!}{(1+\beta)_{nk}} L_n^{\beta}(y)\phi_n(x;\lambda)t^n$$

$$=(1-t)^x (1-te^{-\lambda})^{-x-1} F_{1:1;0}^{1:1;0}\left[\begin{matrix}[1:1,1]:-x;-; \\ [(1;1+\beta):1,1]:1\,;-;\end{matrix} -\frac{y(e^{\lambda}-1)t}{(e^{\lambda}-t)(1-t)}, -\frac{yt}{(e^{\lambda}-t)}\right]$$

Proof of (2.3). (2.3) can be proved in a manner similar to that of (2.1) and (2.2).

Applications

(i) By applying $v \to 0$ in (2.3), we get

$$(2.10)\ \sum_{n=0}^{\infty} \frac{n!}{(1+\beta)_{nk}} Z_n^{\beta}(y,k)\, f_n(x;a)t^n = (1-t)^{-x} e^{axt}\ \phi_3\left[\begin{matrix} x & ; \\ \Delta(k;1+\beta); & \end{matrix} \frac{-y^k t}{1-t}, -a\,x\,y^k t\right]$$

where $\phi_3[\beta;\gamma;x,y]$ is the confluent hypergeometric function of two variables(See Srivastava and Manocha [12, pp. 59 (40)] .

(ii) By applying $v \to 0$ and $k = 1$ in (2.3), we obtain

$$(2.11)\ \sum_{n=0}^{\infty} \frac{n!}{(1+\beta)_{nk}} L_n^{\beta}(y)\, f_n(x;a)t^n = (1-t)^{-x} e^{axt}\ \phi_3\left[\begin{matrix} x & ; \\ \Delta(1;1+\beta); & \end{matrix} \frac{-yt}{1-t}, -a\,x\,y\,t\right]$$

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