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MIXED POISSON DISTRIBUTIONS ASSOCIATED WITH HAZARD FUNCTIONS OF EXPONENTIAL MIXTURES

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Abstract

The hazard function of an exponential mixture characterizes an infinitely divisible mixed Poisson distribution which is also a compound Poisson distribution.

Given the hazard function, the probability generating functions (pgf) of the compound Poisson distribution and its independent and identically distributed (iid) random variables are derived.The recursive forms of the distributions are also given.

Hofmann hazard function has been discussed and re-parameterized. The recursive form of the distribution of the iid random variables for the Hofmann distribution follows Panjer's model.

Key words: Mixed Poisson distributions, Laplace transform, exponential mixtures, complete monotocity, infinite divisibility, compound Poisson distribution, Hofmann distributions, Panjer's model.

1 INTRODUCTION

Motivated by Walhin and Paris (1999, 2002), this paper shows that there is a link between Poisson and exponential mixtures.
Specifically mixed Poisson distributions, in the form of Laplace transform, can be expressed in terms of hazard functions of exponential mixtures.

The hazard function of an exponential mixture is completely monotone, and thus the mixing distribution is infinitely divisible through Laplace transform.

A Poisson mixture with an infinitely divisible mixing distribution is infinitely divisible too. Further, an infinitely divisible mixed Poisson distribution is a compound Poisson distribution.

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To obtain the pgf of the iid random variables of a compound Poisson distribution, the pgf of a mixed Poisson distribution which is infinitely divisible and expressed in terms of the hazard function of the exponential mixture is equated to the pgf of the compound Poisson distribution.

By differentiation method, the pmf of the iid random variables is obtained. The compound Poisson distribution is also obtained recursively in terms of the pmf of the iid random variables and the hazard function of the exponential mixture.

The models developed have been applied to a class of mixed Poisson distribution known as Hofmann distributions. In this case both the differentiation method and the power series expansion have been used to obtain the distribution of the iid random variables.

The paper has the following sections: In section 2 mixed Poisson distribution is expressed in terms of Laplace transform.
In section 3 the mixed Poisson distribution is expressed in terms of hazard function of exponential mixture.
Section 4 discusses infinite divisibility in relation to complete monotocity and Laplace transform leading to infinite divisible mixed Poisson distribution.
Section 5 derives compound Poisson distribution in terms of pgfs and recursive form.
Section 6 deals with applications of the models derived to various cases of Hoffmann distributions.
Hoffman hazard function has been re-parameterized in section 7 and concluding remarks are in section 8.

2 Mixed Poisson Distribution in Terms of Laplace Transform

Let

$$p_n(t) = prob(Z(t) = n) \tag{2.1}$$

where Z(t) is a discrete random variable.
A mixed Poisson distribution is given as

$$p_n(t) = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} g(\lambda) d\lambda \qquad n = 0, 1, 2, ... \tag{2.2}$$

$$= E(e^{-\lambda t} \frac{(\lambda t)^n}{n!})$$

$$= \frac{(-1)^n t^n}{n!} E((-1)^n \lambda^n e^{-\lambda t})$$

When n = 0, we have

$$\begin{aligned} p_0(t) &= E(e^{-\lambda t}) \\ &= L_\Lambda(t) \end{aligned} \tag{2.3}$$

the Laplace transform of the mixing distribution g(λ)

Differentiating $p_0(t)$, n times we get

$$\begin{aligned} p_0^{(n)}(t) &= E[(-1)^n(\lambda)^n e^{-\lambda t}] \\ &= (-1)^n \frac{t^n}{n!}\, p_0^{(n)}(t) \end{aligned}$$

$$p_n(t) = (-1)^n \frac{t^n}{n!}\, L_\Lambda^{(n)}(t) \qquad n = 0, 1, 2, 3, ... \tag{2.4}$$

which is the mixed Poisson distribution in terms of the Laplace transform of the mixing distribution.

3 Mixed Poisson Distribution in Terms of the Hazard Function of the Exponential Mixture

The pdf of an exponential mixture is defined by

$$f(t) = \int_0^\infty \lambda\, e^{-\lambda\, t}\, g(\lambda)\, d\lambda \tag{3.1}$$

where g(λ) is the mixing distribution.
The corresponding survival function is

$$\begin{aligned} S(t) &= \int_0^\infty S(t|\lambda)\, g(\lambda)\, d\lambda \\ &= \int_0^\infty e^{-\lambda t}\, g(\lambda) d\lambda \\ &= L_\Lambda(t) \end{aligned}$$

Remark 1 *The survival function of an exponential mixture is the Laplace transform of the mixing distribution.*

The hazard function of the exponential mixture is given by

$$\begin{aligned} h(t) &= \frac{f(t)}{S(t)} \\ &= -\frac{1}{S(t)}\ \frac{dS}{dt} \\ &= -\frac{L'(t)}{L(t)} \end{aligned} \tag{3.2}$$

Let

$$\theta(t) = In(\frac{1}{L_{\Lambda}(t)}) \tag{3.3}$$

$\therefore$

$$\begin{aligned} \theta^{'}(t) &= -\frac{L'(t)}{L(t)} \\ &= h(t) \end{aligned} \tag{3.4}$$

$\therefore$

$$\begin{aligned} p_0(t) &= L_{\Lambda}(t)) \\ &= e^{In\ L_{\Lambda}(t)} \\ &= e^{-In(\frac{1}{L_{\Lambda}(t)})} \\ &= e^{-\theta(t)} \end{aligned} \tag{3.5}$$

where

$$p_0(t) = e^{\int\limits_0^t h(x)\ dx} \tag{3.6}$$

and

$$\theta(t) = -\int\limits_0^t h(x)\ dx \tag{3.7}$$

is the cumulative or integrated hazard function.

Using (3.6) and (3.7), $p_0(t)$ can be obtained and then use formula (3.5) to derive $p_n(t)$.

The pgf of the mixed Poisson distribution is given by

$$\begin{aligned}
H(s,t) &= \sum_{n=0}^{\infty} p_n(t)\ s^n \\
&= \sum_{n=0}^{\infty} [\int_0^{\infty} e^{-\lambda t}\ \frac{(\lambda t)^n}{n!}\ g(\lambda)\ d\lambda]\ S^n \\
&= \int_0^{\infty} e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda ts)^n}{n!}\ g(\lambda)\ d\lambda \\
&= \int_0^{\infty} e^{-\lambda t}\ e^{\lambda ts}\ g(\lambda)\ d\lambda \\
&= \int_0^{\infty} e^{-\lambda t(1-s)}\ g(\lambda)\ d\lambda \\
&= L_\Lambda(t-ts) \\
&= e^{-In\ \frac{1}{L_\Lambda(t-ts)}}
\end{aligned}$$

$$H(s,t) = e^{-\theta(t-ts)} \tag{3.8}$$

which is the survival function of the exponential mixture at time $\mathrm{t} - \mathrm{ts}$.

Remark 2 *Given a hazard function of an exponential mixture, the pgf of the mixed Poisson distribution is the survival function of the exponential mixture at time* $\mathrm{t} - \mathrm{ts}$.

The mean and variance of the mixed Poisson distribution can hence be obtained as follows:

$$\begin{aligned}
H'(s,t) &= \frac{d}{ds}\ H(s,t) \\
&= t\theta'(t-ts)\ H(s,t) \\
H''(s,t) &= t^2[(\theta'(t-ts))^2 - \theta''(t-ts)]\ H(s,t)]
\end{aligned}$$

then $\mathrm{H}(1,\mathrm{t}) = 1$, $\mathrm{H}'(1,\mathrm{t}) = \mathrm{t}\ \theta'(0)$ and $\mathrm{H}''(1,\mathrm{t}) = \mathrm{t}^2\ [\theta'(0)]^2 - \theta''(0)]$

$$\begin{aligned}
E[Z(t)] &= H'(1,t) \\
&= t\ \theta'(0) \\
Var[Z(t)] &= H''(1,t) + H'(1,t) - (H'(1,t))^2 \\
&= t\ \theta'(0) - t^2\ \theta''(0)
\end{aligned} \tag{3.9}$$

4 Infinite Divisibility

Definition 1 : *Infinite Divisibility (Karlis and Xekalaki, 2005)*
A random variable X is said to have an infinitely divisible distribution, if its characteristic function $\Psi(t)$ can be written in the form:

$$\Psi(\mathrm{t}) = [\Psi_{\mathrm{n}}(\mathrm{t})]^{\mathrm{n}}$$

where $\Psi_{\mathrm{n}}(\mathrm{t})$ *is a characteristic function for any* $\mathrm{n} = 1, 2, 3, ...$

In other words a distribution is infinitely divisible if it can be written as the distribution of the sum of an arbitrary number n of independently and identically distributed random variables.

Definition 2 : *Completely Monotonicity*
A function Ψ on $[0, \infty]$ is completely monotone if it possesses derivatives $\Psi^{(\mathrm{n})}$ of all orders and

$$(-1)^{\mathrm{n}}\Psi^{(\mathrm{n})}(\mathrm{t}) \geq 0, \qquad \mathrm{t} > 0$$

The link between infinite divisibility and complete monotonicity is given in the following propositions.

Proposition 1 : ***(Feller, 1971, Vol II, Chapter XIII)***
The function ω is the Laplace transform of an infinitely divisible probability distribution iff

$$\omega = \mathrm{e}^{-\Psi}$$

where Ψ has a completely monotone derivative and $\Psi(0) = 0$

Remark 3 : *In our situation*

$$\begin{aligned} p_0(t) &= L_\Lambda(t) \\ &= e^{-\theta(t)} \end{aligned} \tag{4.1}$$

and

$$\therefore \quad \begin{aligned} L_\lambda(0) &= 1 \\ L_\Lambda(t) &= e^{-\theta(t)} \qquad \textit{and} \\ \theta(0) &= 0 \end{aligned} \tag{4.2}$$

The derivative of $\theta(\mathrm{t})$ is $\theta'(\mathrm{t}) = \mathrm{h}(\mathrm{t})$ which is the hazard function of an exponential mixture.

Hesselager et. al. (1998) have stated the following theorem: "A distribution with a completely monotone hazard function is a mixed exponential distribution".

Remark 4 *The theorem implies that a hazard function of an exponential mixture is completely monotone.*

According to proposition 1, $\theta'(\mathrm{t}) = \mathrm{h}(\mathrm{t})$ is completely monotone and $\theta(0) = 0$ Therefore

$$\mathrm{p}_0(\mathrm{t}) = \mathrm{L}_\Lambda(\mathrm{t})$$

is the Laplace transform of an infinitely divisible distribution.

Thus the mixing distribution, $\mathrm{g}(\lambda)$ is infinitely divisible.

Proposition 2 *(Maceda, 1948)*
If in a Poisson mixture the mixing distribution is infinitely divisible, the resulting mixture is also infinitely divisible.

Proposition 3 *(Feller, 1968; Ospina and Gerbes, 1987)*
Any discrete infinitely divisible distribution can arise as a compound Poisson distribution.

From the above discussion we have the following:

Theorem 1 *Mixed Poisson distributions expressed in terms of Laplace transforms can also be expressed in terms of hazard functions of exponential mixtures.*
Such Poisson mixtures are infinitely divisible and hence are compound Poisson distributions.

5 Compound Poisson Distribution

Let

$$\begin{aligned} Z(t) &= Z_{N(t)} \\ &= X_1 + X_2 + ... + X_{N(t)} \end{aligned} \tag{5.1}$$

where $\mathrm{X}_\mathrm{i}'\mathrm{s}$ are iid random variables and N(t), is also a random variable independent of $\mathrm{X}_\mathrm{i}\mathrm{s}$

Then $\mathrm{Z}_{\mathrm{N(t)}}$ is said to have a compound Poisson distribution

5.1 Compound Poisson Distribution in terms of pgf

Let

$$\begin{aligned} H(s,t) &= E[s^{Z_{N(t)}}] \\ &= \sum_{n=0}^{\infty} p_n(t)\ s^n \\ &= the\ pgf\ of\ Z_{N(t))} \end{aligned}$$

$$\begin{aligned} F(s,t) &= E(s^{N(t)}) \\ &= \sum_{j=0}^{\infty} f_j \; s^j \\ &= the\; pgf\; of\; N(t) \end{aligned}$$

and

$$\begin{aligned} G(s,t) &= E(s^{X_i}) \\ &= \sum_{x=0}^{\infty} g_x(t) \; s^x \\ &= the\; pgf\; of\; X_i \end{aligned}$$

It can be proved that

$$\mathrm{H(s,t) = F[G(s,t)]} \tag{5.2}$$

If N(t) is Poisson with parameter θ(t), then

$$\mathrm{H(s,t) = e^{-\theta(t)[1-G(s,t)]}} \tag{5.3}$$

5.2 The Distribution of iid Random Variables of the Compound Poisson Distribution

Since an infinitely divisible mixed Poisson distribution is also a compound Poisson distribution, we equate their pgfs given in (3.9) and (5.3) i.e

$$\begin{aligned} e^{-\theta(t-t\;s)} &= e^{-\theta(t)[1-G(s,t)]} \\ G(s,t) &= 1 - \frac{\theta(t-t\;s)}{\theta(t)} \end{aligned} \tag{5.4}$$

which is the probability generating function of iid random variables expressed in terms of the cumulative hazard function of the exponential mixture.
The corresponding probability mass function obtained by differentiation method is given by:

$$\begin{aligned} g_x(t) &= \frac{1}{x!}\,\frac{d^x}{ds^x}\; G(s,t)|_{s=0} \qquad for\; x = 1,2,3,... \\ g_0(t) &= 0 \end{aligned} \tag{5.5}$$

5.3 Compound Poisson Distribution in Recursive Form

By differentiating equation (5.3),

$$\mathrm{H^{'}(s,t) = \theta(t)\; G^{'}(s,t)\; H(s,t)} \tag{5.6}$$

where

$$\mathrm{G}'(\mathrm{s},\mathrm{t}) = \frac{\mathrm{d}}{\mathrm{ds}}\,\mathrm{G}(\mathrm{s},\mathrm{t})$$

By definition

$$\mathrm{H}(\mathrm{s},\mathrm{t}) = \sum_{\mathrm{n}=0}^{\infty} \mathrm{p_n}(\mathrm{t})\ \mathrm{s^n} \qquad \therefore \qquad \mathrm{H}'(\mathrm{s},\mathrm{t}) = \sum_{\mathrm{n}=1}^{\infty}\ \mathrm{n}\ \mathrm{p_n}(\mathrm{t})\ \mathrm{s^{n-1}} \tag{5.7}$$

$$\mathrm{G}(\mathrm{s},\mathrm{t}) = \sum_{\mathrm{x}=0}^{\infty}\ \mathrm{g_x}(\mathrm{t})\ \mathrm{s^x} \qquad \therefore \qquad \mathrm{G}'(\mathrm{s},\mathrm{t}) = \sum_{\mathrm{x}=1}^{\infty}\ \mathrm{x}\ \mathrm{g_x}(\mathrm{t})\ \mathrm{s^{x-1}} \tag{5.8}$$

By comparing (5.6) to (5.7) we get

$$\begin{aligned}
\sum_{n=1}^{\infty} n\ p_n(t)\ s^{n-1} &= \theta(t)\ G'(s,t)\ H(s,t) \\
&= \theta(t)\ [\sum_{x=1}^{\infty} x\ g_x(t)\ s^{x-1}]\ \sum_{n=0}^{\infty} p_n(t)\ s^n \sum_{n=1}^{\infty} n\ p_n(t)\ s^n \\
&= \sum_{n=1}^{\infty} [\theta(t)\ \sum_{x=1}^{\infty} x\ g_x(t)\ \ p_{n-x}(t)]\ s^n \\
n\ p_n(t) = \theta(t)\ \sum_{x=1}^{n} x\ g_x(t)\ \ p_{n-x}(t) &\qquad for\ n = 1, 2, 3, \ldots
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
&Put \\
&\quad x = i + 1 \\
(n+1)\ p_{n+1}(t) &= \theta(t)\ \sum_{i=0}^{n} (i+1)\ g_{i+1}(t)\ \ p_{n-i}(t) \qquad for\ n = 0, 1, 2.....
\end{aligned} \tag{5.10}$$

Equation (5.10) can be used to obtain $\mathrm{p_n}(\mathrm{t})$ iteratively and can also be used to obtain the mean and the variance of the mixed Poisson distribution.

5.4 Panjer's Recursive Model

Let

$$\mathrm{p_n} = (\mathrm{a} + \frac{\mathrm{b}}{\mathrm{n}}\ \mathrm{p_{n-1}}); \qquad \mathrm{n} = 1, 2, 3, ...$$

where a and b are real numbers.
This recursive model is known as Panjer's recursive model of class zero denoted by $(a, b, 0)$ class of distributions.
We can extend it to

$$\mathrm{p_n} = (\mathrm{a} + \frac{\mathrm{b}}{\mathrm{n}}\ \mathrm{p_{n-1}}); \qquad \mathrm{n} = 2, 3, ...$$

which is Panjer's $(a, b, 1)$ class.
In general, we have

$$\mathrm{p_n = (a + \frac{b}{n}\, p_{n-1}; \qquad n = k+1, k+2, ...}$$

which is Panjer's (a, b, k) class for $k = 0, 1, 2, ...$
Some probability mass functions in this paper take Panjer's recursive form.

Remark 5 *In actuarial literature, this recursive relation is due to the work of Panjer (1981). In statistical literature, however, this relation had been published by Katz (1965) based on his PhD dissertation of 1945.*

6 Hofmann Hazard Function

A class of mixed Poisson distributions known as Hofmann distribution has been described by Walhin and Paris (1999) as:

$$p_0(t) = e^{-\theta(t)} \tag{6.1}$$

and

$$p_n(t) = (-1)^n \frac{t^n}{n!} p_0^n(t) \qquad for \qquad n = 1, 2, ... \tag{6.2}$$

where

$$\theta'(t) = \frac{p}{(1+ct)^a} \qquad for \qquad p > 0,\ c > 0 \ and \ a \geq 0 \tag{6.3}$$

and

$$\theta(0) = 0 \tag{6.4}$$

Remark 6 *Clearly* $\theta'(\mathrm{t})$ *is a hazard function of an exponential mixture.*

We shall refer to it as Hofmann hazard function of an exponential mixture; i.e

$$\mathrm{\theta'(t) = h(t) = \frac{p}{(1+ct)^a} \qquad for \qquad p > 0,\ c > 0 \text{ and } a \geq 0} \tag{6.5}$$

We will now consider various cases of a

6.1 When $a = 0$

$$\begin{aligned}\theta'(t) &= h(t) \\ &= p, \qquad a\ constant\end{aligned} \tag{6.6}$$

which is the hazard function of the exponential distribution with parameter p
Therefore

$$h^{(n)}(t) = 0$$
$$(-1)^n \frac{d^n}{dt^n} h(t) = (-1)^n h^n(t) \geq 0$$

Therefore h(t) is completely monotone.
The cumulative hazard function is therefore

$$\begin{aligned}\theta(t) &= p \int_0^t dx \\ &= pt\end{aligned}$$

implying that

$$\begin{aligned}\theta\,(t - ts) &= p\,t\,(1-s) \\ \theta(0) &= 0, \qquad \theta'(0) = p \qquad and \qquad \theta''(0) = 0 \\ \therefore \qquad p_0(t) &= e^{-\theta(t)} \\ &= e^{-pt} \\ p_0^{(n)}(t) &= (-1)^n\, p^n\, e^{-pt}\end{aligned}$$

Therefore,

$$p_n(t) = (-1)^n \frac{t^n}{n!}\, p_0^{(n)}(t)$$
$$p_n(t) = \frac{e^{-pt}\,(pt)^n}{n!}$$

which is a Poisson distribution with parameter pt.

$$\begin{aligned}H(s,t) &= e^{-\theta\,(t-ts)} \\ &= e^{-p\,t\,(1-s)}\end{aligned} \tag{6.7}$$

which is the pgf of Poisson distribution with parameter pt
Therefore

$$\mathrm{E[Z(t)] = pt} \qquad \text{and} \qquad \mathrm{Var(Z_{N(t)}) = pt}$$

Since $\theta(0) = 0$ and $\mathrm{h(t)} = \theta'(\mathrm{t})$ is completely monotone, then $\mathrm{p_0(t)}$ is a Laplace transform of an infinitely divisible mixing distribution.

The corresponding infinitely divisible mixed Poisson distribution is a compound Poisson distribution whose iid random variables have pgf given by

$$\begin{aligned} G(s,t) &= 1 - \frac{pt\,(1-s)}{pt} \\ &= s \end{aligned}$$

implying that the pmf is

$$\begin{aligned} \mathrm{g_x(t) = 1} \qquad &\mathrm{for} \qquad \mathrm{x = 1} \\ \mathrm{g_x(t) = 0} \qquad &\mathrm{for} \qquad \mathrm{x \neq 1} \end{aligned} \tag{6.8}$$

The compound Poisson distribution in recursive form is

$$\begin{aligned} n\,p_n(t) &= \theta(t) \sum_{x=1}^{n} x\,g_x(t)\;\; p_{n-x}(t) \\ \therefore \qquad n\,p_n(t) &= \theta(t)\,p_{n-1}(t) \qquad for \qquad n = 1,2,3..... \end{aligned} \tag{6.9}$$

By iteration

$$\begin{aligned} For \quad n = 1 \qquad\qquad & p_1(t) = \theta(t)\,p_0(t) \\ For \quad n = 2 \qquad\qquad & p_2(t) = \frac{(\theta(t))^2}{2!}\,p_0(t) \\ For \quad n = 3 \qquad\qquad & p_3(t) = \frac{(\theta(t))^3}{3!}\,p_0(t) \end{aligned}$$

$$\begin{aligned} &By\ induction, \\ \therefore \qquad & p_n(t) = \frac{e^{-pt}\,(pt)^n}{n!} \qquad for \qquad n = 0,1,2..... \end{aligned}$$

Moments

$$\mathrm{E(Z(t)} = \sum_{\mathrm{n=1}}^{\infty} \mathrm{n\ p_n(t)}$$

Sum equation (6.9) over n; i.e.

$$\begin{aligned} \sum_{\mathrm{n=1}}^{\infty} \mathrm{n\ p_n(t)} &= \mathrm{E(Z(t)} \\ \mathrm{E(Z(t)} &= \theta\mathrm{(t) = pt} \end{aligned}$$

Multiply (6.9) by n and then sum the result over n; i.e.

$$\begin{aligned} E(Z(t))^2 &= \sum_{n=1}^{\infty} n^2\,p_n(t) \\ E(Z(t))^2 &= (pt)^2 + pt \\ Var(Z(t)) &= E(Z(t))^2 - (E(Z(t))^2 \\ &= pt \end{aligned}$$

6.2 When $a = 1$ and $c = p$

$$\mathrm{h(t)} = \theta'(\mathrm{t}) = \frac{\mathrm{p}}{1 + \mathrm{pt}}$$

which is a hazard function of Pareto (exponential-exponential) distribution, with parameters p and p.
Therefore

$$h^{(n)}(t) = (-1)^n \, p^{n+1} \, (1 + pt)^{-(n+1)}$$

$$(-1)^n \, \frac{d^n}{dt^n} \, h(t) = (-1)^n \, h^n(t) \geq 0$$

Therefore h(t) is completely monotone.
The cumulative hazard function is

$$\theta(t) = p \int_0^t \frac{p}{1 + px} \, dx$$

$$= In(1 + pt)$$

Implying that,

$$\theta(t - ts) = In(1 + pt - pts)$$

Therefore,

$$\theta(0) = 0, \qquad \theta'(0) = p \qquad and \qquad \theta''(0) = -p^2$$

$$p_0(t) = e^{-In(1+pt)} = \frac{1}{1 + pt}$$

$$p_0^{(n)}(t) = (-1)^n \, n! \, p^n \, (1 + pt)^{-n-1}$$

Therefore,

$$p_n(t) = (\frac{pt}{1 + pt})^n \, \frac{1}{1 + pt} \qquad n = 0, 1, 2, 3, ...$$

which is a geometric(Poisson-exponential) distribution with parameter $\frac{pt}{1+pt}$
The pgf is given by

$$\mathrm{H(s,t)} = (\frac{\frac{1}{1+\mathrm{pt}}}{1 - \frac{\mathrm{pt}}{1+\mathrm{pt}}\,\mathrm{s}})$$

which is the pgf of a geometric distribution with parameter $\frac{1}{1+pt}$
Therefore,

$$\mathrm{E[Z(t)} = \mathrm{pt} \qquad \text{and} \qquad \mathrm{Var(Z(t))} = \mathrm{pt} + (\mathrm{pt})^2$$

Since $\theta(0) = 0$ and $\mathrm{h(t)} = \theta'(\mathrm{t})$ is completely monotone, then $\mathrm{p_0(t)}$ is a Laplace transform of an infinitely divisible mixing distribution.

The corresponding infinitely divisible mixed Poisson distribution is a compound Poisson distribution whose iid random variables have pgf given by

$$G(s,t) = 1 - \frac{In(1+pt-pts))}{In\ (1+pt)}$$

By power series expansion,

$$G(s,t) = \sum_{x=1}^{\infty} \frac{(\frac{pt}{1+pt})^x}{-x\ In\ (1-\frac{pt}{1+pt})}\ s^x$$

$$g_x(t) = \frac{(\frac{pt}{1+pt})^x}{-x\ In\ (1-\frac{pt}{1+pt})}$$

which is logarithmic series distribution with parameter $\frac{1}{1+pt}$.

By the differentiation method, we have

$$G^x(s,t) = \frac{(x-1)!}{In(1+pt)}\ (\frac{pt}{1+pt})^x\ (1-\frac{pt}{1+pt}\ s)^{-x}$$

$$g_x(t) = \frac{(\frac{pt}{1+pt})^x}{-x\ In(1-\frac{pt}{1+pt})} \qquad x = 1, 2...$$

$$\frac{g_x(t)}{g_{x-1}(t)} = \frac{x-1}{x}\ \frac{pt}{1+pt} = \frac{pt}{1+pt}\ (1-\frac{1}{x})$$

$$g_x(t) = (a+\frac{b}{x})\ g_{x-1}(t) \qquad for \quad x = 2, 3, 4...$$

which is Panjer's recursive model with

$$a = \frac{pt}{1+pt} \qquad and \qquad b = -\frac{pt}{1+pt}$$

The compound Poisson distribution in recursive form is:

$$n\ p_n(t) = In(1+pt) \sum_{x=1}^{n} \frac{x\ (\frac{pt}{1+pt})^x}{x\ In(1+pt)}\ p_{n-x}(t) \qquad for \quad x = 2, 3, 4, ...$$

$$n\ p_n(t) = \sum_{x=1}^{n} (\frac{pt}{1+pt})^x\ p_{n-x}(t) \tag{6.10}$$

$$For \quad n = 1, \qquad p_1(t) = p\ \frac{pt}{1+pt}\ p_0(t)$$

$$For \quad n = 2, \qquad p_2(t) = (\frac{t}{1+t})^2\ p_0(t)$$

In general,

$$p_n(t) = (\frac{pt}{1+pt})^n \frac{1}{1+pt} \qquad n = 0, 1, 2, ...$$

which is the geometric distribution with parameter $\frac{1}{1+\text{pt}}$

Moments

Sum the recursive relation (6.10) over n; thus

$$\sum_{n=1}^{\infty} n\, p_n(t) = E(Z(t))$$
$$= pt$$

Next, multiply the recursive relation (6.10) by n + 1 and then sum the result over n; thus

$$\text{E(Z(t))}^2 = (\text{pt})^2 + \text{pt} + (\text{pt})^2 \qquad \text{and} \qquad \text{Var(Z(t))} = \text{pt} + (\text{pt})^2$$

6.3 When $a = 1 \quad c = 1$

$$\theta'(t) = h(t)$$
$$= \frac{p}{1+t}$$

which is a hazard function of Pareto (exponential-gamma) distribution, with parameters p and 1.

Therefore,

$$h^{(n)}(t) = (-1)^n\, p^n\, (1+pt)^{-(n+1)}$$
$$(-1)^n \frac{d^n}{dt^n} h(t) = (-1)^n\, h^n(t) \geq 0$$

Therefore h(t) is completely monotone.

The cumulative hazard function is

$$\theta(t) = p \int_0^t \frac{1}{1+x}\, dx$$
$$= p\, In(1+t)$$

implying that,

$$\theta(t - ts) = p\, In(1 + t - ts)$$

Therefore

$$\theta(0) = 0, \qquad \theta'(0) = p \qquad and \qquad \theta''(0) = -p$$

$$\begin{aligned}
p_0(t) &= e^{-\theta(t)} \\
&= e^{-p\ In(1+t)} \\
&= (1+t)^{-p} \\
p_0^{(n)}(t) &= (-1)^n\ \ n!\ \frac{(p+n-1)!)}{n!\ (p-1)!}\ (1+t)^{-p-n} \\
p_n(t) &= (-1)^n \frac{t^n}{n!}\ (-1)^n\ \ n!\ \frac{(p+n-1)!)}{n!\ (p-1)!}\ (1+t)^{-p-n} \\
p_n(t) &= \binom{p+n-1}{n}\ (\frac{t}{1+t})^n\ (\frac{1}{1+t})^p \qquad n = 1, 2, 3, ...
\end{aligned}$$

which is a negative binomial (Poisson-gamma) distribution with parameters p and $\frac{1}{1+t}$

The pgf is given by

$$\begin{aligned}
H(s,t) &= e^{-\theta(t-ts)} \\
&= e^{-p\ In\ (1+t-ts)} \\
&= (\frac{1}{1+t-ts})^p \\
&= (\frac{\frac{1}{1+t}}{1-\frac{t}{1+t}\ s})^p
\end{aligned}$$

which is the pgf of a negative binomial distribution with parameters p and $\frac{1}{1+t}$

$$\begin{aligned}
E[Z(t)] &= t\ \theta'(0) \\
&= pt \\
Var(Z(t)) &= t\ \theta'(0) - t^2\ \theta''(0) \\
&= tp + t^2\ p \\
&= pt + pt^2
\end{aligned}$$

Since $\theta(0) = 0$ and $\mathrm{h(t)} = \theta'(\mathrm{t})$ is completely monotone, then $\mathrm{p_0(t)}$ is a Laplace transform of an infinitely divisible mixing distribution.

The corresponding infinitely divisible mixed Poisson distribution is a compound Poisson distribution whose iid random variables have pgf given by

$$G(s,t) = 1 - \frac{In(1+t-ts))}{In\ (1+t)}$$

By power series expansion,

$$
\begin{aligned}
G(s,t) &= 1 - \frac{In[(1+t)(1-\frac{t}{1+t}\,s)]}{In(1+t)} \\
&= \sum_{x=1}^{\infty} \frac{1}{In\,(\frac{1+t}{1})}\;\frac{(\frac{t}{1+t})^x}{x}\;s^x \\
&= \sum_{x=1}^{\infty} \frac{(\frac{t}{1+t})^x}{-x\;In\;(1-\frac{t}{1+t})}\;s^x \\
g_x(t) &= \frac{(\frac{t}{1+t})^x}{-x\;In\;(1-\frac{t}{1+t})}, \qquad x = 1, 2, 3...
\end{aligned}
$$

which is logarithmic series distribution with parameter $\frac{1}{1+t}$.
By the differentiation method, we have

$$
\begin{aligned}
G^x(s,t) &= \frac{(x-1)!}{In(1+t)}\;(\frac{t}{1+t})^x\;(1-\frac{t}{1+t}\;s)^{-x} \\
g_x(t) &= \frac{1}{x!}\;\frac{d^x\;G(s,t)}{ds^x}|_{s=0} = \frac{(\frac{t}{1+t})^x}{x\;In(1+t)} \\
&= \frac{(\frac{t}{1+t})^x}{-x\;In\frac{1}{1+t}} \qquad x = 1, 2... \\
\frac{g_x(t)}{g_{x-1}(t)} &= \frac{x-1}{x}\;\frac{t}{1+t} \\
&= \frac{t}{1+t}\;(1-\frac{1}{x}) \\
g_x(t) &= (a+\frac{b}{x})\;g_{x-1}(t) \qquad for \quad x = 2, 3, 4..
\end{aligned}
$$

which is Panjer's recursive model with

$$
\mathrm{a} = \frac{\mathrm{t}}{1+\mathrm{t}} \qquad \text{and} \qquad \mathrm{b} = -\frac{\mathrm{t}}{1+\mathrm{t}}
$$

The compound Poisson distribution in recursive form is:

$$
n\;p_n(t) = p\sum_{x=1}^{n}(\frac{t}{1+t})^x\;p_{n-x}(t); n = 1, 2, 3, ... \tag{6.11}
$$

$$
\begin{aligned}
&For \quad n = 1, \qquad && p_1(t) = p\;\frac{t}{1+t}\;p_0(t) \\
&For \quad n = 2, \qquad && p_2(t) = \binom{p+1}{2}\;(\frac{t}{1+t})^2\;p_0(t) \\
&For \quad n = 3, \qquad && p_3(t) = \binom{p+2}{3}\;(\frac{t}{1+t})^3\;p_0(t)
\end{aligned}
$$

By induction,

$$p_n(t) = \binom{p+n-1}{n} \; (\frac{t}{1+t})^n \; (\frac{1}{1+t})^p, \quad n = 0.1, 2, ... \tag{6.12}$$

which is the negative binomial distribution with parameters p and $\frac{1}{1+t}$

Moments
Sum equation (6.11) over n

$$\sum_{n=1}^{\infty} n \; p_n(t) = p \sum_{n=1}^{\infty} \sum_{x=1}^{n} (\frac{t}{1+t})^x \; p_{n-x}(t)$$

$$E(Z(t)) = pt$$

Next, multiply the recursive relation (6.11) by n and then sum the result over n; thus

$$\begin{aligned} E(Z(t))^2 &= \sum_{n=1}^{\infty} n^2 \; p_n(t)] \\ &= pt + 2pt^2 \\ Var(Z(t)) &= pt + pt^2 \end{aligned}$$

6.4 When $a = 1$

$$\theta'(\mathrm{t}) = \mathrm{h}(\mathrm{t}) = \frac{\mathrm{p}}{1 + \mathrm{ct}}$$

which is a hazard function of Pareto (exponential-gamma) distribution, with parameters p and c.
Therefore

$$h^{(n)}(t) = (-1)^n \; p \; (c)^n \; (1+ct)^{-(n+1)})$$

$$(-1)^n \; \frac{d^n}{dt^n} \; h(t) = (-1)^n \; h^n(t) \geq 0$$

Therefore h(t) is completely monotone.
The cumulative hazard function is

$$\begin{aligned} \theta(t) &= p \int_0^t \frac{1}{1+ct} \, dx \\ &= \frac{p}{c} \; In(1+ct) \end{aligned}$$

Implying that,

$$\theta(t - ts) = \frac{p}{c} \; In(1 + ct - cts)$$

Therefore,

$$\theta(0) = 0, \qquad \theta'(0) = p \qquad and \qquad \theta''(0) = -pc$$
$$p_0(t) = (1+ct)^{-\frac{p}{c}}$$
$$p_0^{(n)}(t) = (-1)^n \, c^n \, \frac{\Gamma(\frac{p}{c}+n)}{\Gamma(\frac{p}{c})} \, (1+ct)^{-\frac{p}{c}-n}$$
$$= (-1)^n \, c^n \, n! \, \frac{(\frac{p}{c}+n-1)!)}{n! \, (\frac{p}{c}-1)!} \, (1+ct)^{-\frac{p}{c}-n}$$
$$p_n(t) = (-1)^n \frac{t^n}{n!} \, (-1)^n \, c^n \, n! \, \frac{(\frac{p}{c}+n-1)!)}{n! \, (\frac{p}{c}-1)!} \, (1+ct)^{-\frac{p}{c}-n}$$
$$p_n(t) = \binom{\frac{p}{c}+n-1}{n} \, (\frac{ct}{1+ct})^n \, (\frac{1}{1+ct})^{\frac{p}{c}} \qquad for \quad n = 0, 1, 2, ...$$

which is a negative binomial (Poisson-gamma) distribution with parameters $\frac{p}{c}$ and $\frac{1}{1+ct}$

The pgf is given by

$$H(s,t) = e^{-\theta(t-ts)}$$
$$= e^{-\frac{p}{c} \, In \, (1+ct-cts)}$$
$$= (\frac{1}{1+ct-cts})^{\frac{p}{c}} = (\frac{\frac{1}{1+ct}}{1-\frac{ct}{1+ct} \, s})^{\frac{p}{c}}$$

which is the pgf of a negative binomial distribution with parameters $\frac{p}{c}$ and $\frac{1}{1+ct}$

$$E[Z(t) = t \, \theta'(0) = pt \quad Var(Z(t)) = t \, \theta'(0) - t^2 \, \theta''(0) = pt + pct^2$$

Since $\theta(0) = 0$ and $\mathrm{h(t)} = \theta'(\mathrm{t})$ is completely monotone, then $\mathrm{p_0(t)}$ is a Laplace transform of an infinitely divisible mixing distribution.

The corresponding infinitely divisible mixed Poisson distribution is a compound Poisson distribution whose iid random variables have pgf given by

$$\mathrm{G(s,t) = 1 - \frac{In(1+ct-cts))}{In\,(1+ct)}}$$

By power series expansion

$$\begin{aligned}
G(s,t) &= 1 - \frac{In[(1+ct)(1-\frac{ct}{1+ct}\ s)]}{In(1+ct)} \\
&= \sum_{x=1}^{\infty} \frac{1}{In\ (\frac{1+ct}{1})}\ \frac{(\frac{ct}{1+ct})^x}{x}\ s^x \\
&= \sum_{x=1}^{\infty} \frac{(\frac{ct}{1+ct})^x}{-x\ In\ (1-\frac{ct}{1+ct})}\ s^x \\
g_x(t) &= \frac{(\frac{ct}{1+ct})^x}{-x\ In\ (1-\frac{ct}{1+ct})} \qquad for \quad x = 0,1,2,...
\end{aligned}$$

which is logarithmic series distribution with parameter $\frac{1}{1+ct}$.

By the differentiation method, we have

$$\begin{aligned}
G^x(s,t) &= \frac{(x-1)!}{In(1+ct)}\ (\frac{ct}{1+ct})^x\ (1-\frac{ct}{1+ct}\ s)^{-x} \\
g_x(t) &= \frac{1}{x!}\ \frac{d^x\ G(s,t)}{ds^x}|_{s=0} \\
&= \frac{(\frac{ct}{1+ct})^x}{-x\ In\frac{1}{1+ct}} \qquad x = 1,2... \\
\frac{g_x(t)}{g_{x-1}(t)} &= \frac{x-1}{x}\ \frac{ct}{1+ct} \\
&= \frac{ct}{1+ct}\ (1-\frac{1}{x}) \\
g_x(t) &= (a+\frac{b}{x})\ g_{x-1}(t) \qquad for \quad x = 2,3,4..
\end{aligned}$$

which is Panjer's recursive model with

$$\mathrm{a} = \frac{\mathrm{ct}}{1+\mathrm{ct}} \qquad \text{and} \qquad \mathrm{b} = -\frac{\mathrm{ct}}{1+\mathrm{ct}}$$

The compound Poisson distribution in recursive form is:

$$n\ p_n(t) = \frac{p}{c} \sum_{x=1}^{n} (\frac{ct}{1+ct})^x\ p_{n-x}(t) \qquad for \quad n = 1,2,3,... \tag{6.13}$$

$$\begin{aligned}
For \quad n = 1,\ \ p_1(t) &= \frac{p}{c}\ \frac{ct}{1+ct}\ p_0(t) \\
For \quad n = 2,\ \ p_2(t) &= \frac{1}{2}(\frac{ct}{1+ct})^2\ [\frac{p}{c}\ (\frac{p}{c}+1)p_0(t) = \binom{\frac{p}{c}+1}{2}\ (\frac{ct}{1+ct})^2\ p_0(t) \\
For \quad n = 3,\ \ p_3(t) &= \binom{\frac{p}{c}+2}{3}\ (\frac{ct}{1+ct})^3\ p_0(t)
\end{aligned}$$

By induction,

$$\begin{aligned} p_n(t) &= \binom{\frac{p}{c}+n-1}{n} \; (\frac{ct}{1+ct})^n \; p_0(t) \\ &= \binom{\frac{p}{c}+n-1}{n} \; (\frac{ct}{1+ct})^n \; (\frac{1}{1+ct})^{\frac{p}{c}} \qquad for \quad n = 1, 2, 3, ... \end{aligned}$$

which is the negative binomial distribution with parameters $\frac{p}{c}$ and $\frac{1}{1+ct}$

Moments
Sum equation (6.13) over n

$$\begin{aligned} \sum_{n=1}^{\infty} n \; p_n(t) &= \frac{p}{c} \sum_{n=1}^{\infty} \sum_{x=1}^{n} (\frac{ct}{1+ct})^x \; p_{n-x}(t) \\ E(Z(t)) &= pt \end{aligned}$$

Next, multiply the recursive relation (6.13) by $n+1$ and then sum the result over n; thus

$$\begin{aligned} E(Z(t))^2 &= \sum_{n=1}^{\infty} n^2 \; p_n(t) \\ &= (pt)^2 + pt + pct^2 \\ Var(Z(t)) &= pt + pct^2 \end{aligned}$$

6.5 When $a = \frac{1}{2}$

$$\begin{aligned} h(t) &= \theta'(t) \\ &= \frac{p}{(1+ct)^{\frac{1}{2}}} \quad for \quad p > 0, \quad and \quad c > 0 \end{aligned}$$

which is a hazard function of the exponential-inverse Gaussian distribution. Therefore

$$h^{(n)}(t) = (-1)^n \; \frac{(2n-1)}{2} \; \frac{(2n-3)}{2} \; \frac{(2n-5)}{2} \; ......\frac{5}{2} \; \frac{3}{2} \; \frac{1}{2} \; (1+ct)^{-\frac{(2n+1)}{2}}$$

$$(-1)^n \; \frac{d^n}{dt^n} \; h(t) = (-1)^n \; h^n(t) \geq 0$$

Therefore h(t) is completely monotone.

$$\begin{aligned} \theta(t) &= p \int_0^t (1+cx)^{-\frac{1}{2}} \; dx \\ &= \frac{2p}{c} \; [(1+ct)^{\frac{1}{2}} - 1] \end{aligned}$$

implying that

$$\theta(t-ts)=\frac{2p}{c}\ [(1+ct-cts)^{\frac{1}{2}}-1]$$

Therefore,

$$\theta(0)=0, \qquad \theta'(0)=p \qquad and \qquad \theta''(0)=-\frac{1}{2}pc$$

$$p_0(t)=e^{-\frac{2p}{c}\ [(1+ct)^{\frac{1}{2}}-1]}$$

$$p_0^n(t)=(-1)^n\ p^n\sum_{k=0}^{n-1}\frac{(n-1+k)!}{(n-1-k)!k!}(\frac{c}{4p})^k\ (1+ct)^{-\frac{(n+k)}{2}}]\ p_0(t)$$

Therefore,

$$\begin{aligned}p_n(t)&=(-1)^n\frac{t^n}{n!}\ (-1)^n\ p^n\sum_{k=0}^{n-1}\frac{(n-1+k)!}{(n-1-k)!k!}(\frac{c}{4p})^k\ (1+ct)^{-\frac{(n+k)}{2}}]\ p_0(t)\\&=\frac{(pt)^n}{n!}\sum_{k=0}^{n-1}\frac{(n-1+k)!}{(n-1-k)!k!}(\frac{c}{4p})^k\ (1+ct)^{-\frac{(n+k)}{2}}]\ p_0(t) \qquad for \quad n=1,2,3,...\end{aligned}$$

where

$$p_0(t)=e^{-\frac{2p}{c}\ [(1+ct)^{\frac{1}{2}}-1]}$$

The pgf is

$$\begin{aligned}H(s,t)&=e^{-\theta(t-ts)}\\&=e^{-\frac{2p}{c}\ [(1+ct-cts)^{\frac{1}{2}}-1]}\end{aligned}$$

Therefore

$$\mathrm{E(Z(t))}=\mathrm{t}\ \theta'(0)=\mathrm{pt} \qquad \text{and} \qquad \mathrm{Var(Z(t))}=\mathrm{t}\ \theta'(0)-\mathrm{t}^2\ \theta''(0)=\mathrm{pt}+\frac{1}{2}\ \mathrm{pc}\ \mathrm{t}^2$$

Since $\theta(0)=0$ and $\mathrm{h(t)}=\theta'(\mathrm{t})$ is completely monotone, then $\mathrm{p_0(t)}$ is a Laplace transform of an infinitely divisible mixing distribution.

The corresponding infinitely divisible mixed Poisson distribution is a compound Poisson distribution whose iid random variables have pgf given by

$$\begin{aligned}G(s,t)&=1-\frac{(1+ct-cts)^{\frac{1}{2}}-1}{(1+ct)^{\frac{1}{2}}-1}\\&=\frac{(1+ct)^{\frac{1}{2}}-(1+ct-cts)^{\frac{1}{2}}}{(1+ct)^{\frac{1}{2}}-1}\end{aligned}$$

By power series expansion

$$G(s,t) = \frac{(1+ct)^{\frac{1}{2}}}{(1+ct)^{\frac{1}{2}-1}\,[1-(1-\frac{ct}{1+ct}\,s)^{\frac{1}{2}}]}$$

$$G(s,t) = \frac{(1+ct)^{\frac{1}{2}}}{(1+ct)^{\frac{1}{2}}-1}\sum_{x=1}^{\infty}\frac{\frac{1}{2}}{x!\,\Gamma(\frac{1}{2})}\,\Gamma(x-\frac{1}{2})\,(\frac{ct}{1+ct})^{x}\,s^{x}$$

$$g_x(t) = \frac{(1+ct)^{\frac{1}{2}}}{(1+ct)^{\frac{1}{2}}-1}\,\frac{\frac{1}{2}\,\Gamma(x-\frac{1}{2})}{x!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{x} \qquad x=1,2,3,...$$

$$= \frac{pt\,(1+ct)^{-\frac{1}{2}}}{\theta(t)}\,\frac{\Gamma(x-\frac{1}{2})}{x!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{x-1} \qquad x=1,2,3,...$$

By differentiation method

$$G^{(x)}(s,t) = \frac{\Gamma[x-\frac{1}{2}]\,(\frac{1}{2})\,(ct)^{x}\,(1+ct-cts)^{-\frac{(2x-1)}{2}}}{\Gamma(\frac{1}{2})\,[(1+ct)^{\frac{1}{2}}-1]}$$

$$g_x(t) = \frac{pt\,(1+ct)^{-\frac{1}{2}}}{\theta(t)}\,\frac{\Gamma[x-\frac{1}{2}]}{x!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{x-1} \qquad for \quad x=1,2,3...$$

$$\frac{g_x(t)}{g_{x-1}(t)} = \frac{\Gamma(x-\frac{1}{2})}{x!}\,(\frac{ct}{1+ct})^{x-1}\,\frac{(x-1)!}{\Gamma[x-1-\frac{1}{2}]\,(\frac{ct}{1+ct})^{x-2}}$$

$$= (1-\frac{3}{2x})\,\frac{ct}{1+ct} \qquad x=2,3,4...$$

which is in Panjer's recursive form, where $\mathrm{a} = \frac{\mathrm{ct}}{1+\mathrm{ct}}$ and $\mathrm{b} = -\frac{3}{2}\,\frac{\mathrm{ct}}{1+\mathrm{ct}}$

$$n\,p_n(t) = pt\,(1+ct)^{-\frac{1}{2}}\sum_{x=1}^{n}\frac{\Gamma[x-\frac{1}{2}]}{(x-1)!!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{x-1}\,p_{n-x}(t) \qquad for\; n=1,2,3,\ldots$$

Replace n by n + 1 to get

$$(n+1)p_{n+1}(t) = pt\,(1+ct)^{-\frac{1}{2}}\sum_{x=1}^{n+1}\frac{\Gamma[x-\frac{1}{2}]}{(x-1)!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{x-1}\,p_{n+1-x}(t)$$

$$Let \qquad x = i+1$$

$$(n+1)p_{n+1}(t) = pt\,(1+ct)^{-\frac{1}{2}}\sum_{i=0}^{n}\frac{\Gamma[i+\frac{1}{2}]}{i!\,\Gamma(\frac{1}{2})}\,(\frac{ct}{1+ct})^{i}\,p_{n-i}(t) \qquad n=1,2,... \tag{6.14}$$

By iteration,

$$For \quad n=0, \qquad p_1(t) = pt\,(1+ct)^{-\frac{1}{2}}\,p_0(t$$

$$For \quad n=1, \qquad p_2(t) = \frac{(pt)^2}{2!}\,[(1+ct)^{-\frac{2}{2}} + 2!\,\frac{c}{4p}\,(1+ct)^{-\frac{3}{2}}]\,p_0(t)$$

For $\mathrm{n} = 2$,

$$\mathrm{p_3(t) = pt\ (1+ct)^{-\frac{1}{2}}\ [\frac{(pt)^3}{3!} \sum_{k=0}^{2} \frac{(2+k)!}{(2-k)!k!}\ (\frac{c}{4p})^k\ (1+ct)^{-\frac{(3+k)}{2}} +}$$

$$\mathrm{\frac{(pt)^3}{3!}\ \frac{3}{pt}\ \frac{ct}{2\ (1+ct)} \sum_{k=0}^{1} \frac{(1+k)!}{(1-k)!k!}\ (\frac{c}{4p})^k\ (1+ct)^{-\frac{(2+k)}{2}} +}$$

$$\mathrm{\frac{(pt)^3}{3!}\ \frac{3}{4}\ \frac{3}{(pt)^2}\ (\frac{ct}{1+ct})^2\ (1+ct)^{-\frac{1}{2}} + \frac{(pt)^3}{3!}\ \frac{15}{8}\ \frac{1}{(pt)^3}\ (\frac{ct}{1+ct})^3]\ p_0(t)]}$$

$$\mathrm{= \frac{(pt)^4}{4!}\ [\frac{(3+0)!}{(3-0)!0!}\ (\frac{c}{4p})^0\ (1+ct)^{-\frac{(4+0)}{2}} + \frac{(3+1)!}{(3-1)!1!}\ (\frac{c}{4p})^1\ (1+ct)^{-\frac{(4+1)}{2}} +}$$

$$\mathrm{\frac{(3+2)!}{(3-2)!2!}\ (\frac{c}{p})^2\ (1+ct)^{-\frac{(4+2)}{2}} + \frac{(3+3)!}{(3-3)!3!}\ (\frac{c}{p})^3\ (1+ct)^{-\frac{(4+3)}{2}}]\ p_0(t)}$$

$$\mathrm{p_4(t) = \frac{(pt)^4}{4!} \sum_{k=0}^{3} \frac{(3+k)!}{(3-k)!k!}\ (\frac{c}{4p})^k\ (1+ct)^{-\frac{(4+k)}{2}}\ p_0(t)}$$

In general therefore, we have

$$p_n(t) = \frac{(pt)^n}{n!} \sum_{k=0}^{n-1} \frac{(n-1+k)!}{(n-1-k)!k!}\ (\frac{c}{4p})^k\ (1+ct)^{-\frac{(n+k)}{2}}\ p_0(t) \qquad for \quad n = 1, 2, 3...$$

where,

$$p_0(t) = e^{-\frac{2p}{c}\ [(1+ct)^{\frac{1}{2}}-1]}$$

Moments

Sum the recursive relation (6.14) over n; thus

$$\sum_{n=0}^{\infty} (n+1)\ p_{n+1}(t) = E(Z(t))$$

$$= pt$$

Next, multiply the recursive relation (6.14) by $\mathrm{n}+1$ and then sum the result over n; thus

$$\sum_{n=0}^{\infty} (n+1)^2\ p_{n+1}(t) = E[Z(t)]^2$$

$$= (pt)^2 + pt + \frac{pc\, t^2}{2}$$

$$Var(Z(t)) = pt + \frac{pc\, t^2}{2}$$

Using notations by Willmot (1986), let

$$\mathrm{t} = 1, \qquad \mathrm{c} = 2\beta \qquad \mathrm{p} = \mu \qquad \text{and} \qquad \mathrm{s} = \mathrm{z}$$

Then we get

$$p_n(t) = \frac{p_0\ (\mu)^n}{n!} \sum_{k=0}^{n-1} \frac{(n-1+k)!}{(n-1-k)!k!} \left(\frac{\beta}{2\mu}\right)^k (1+2\beta)^{-\frac{(n+k)}{2}}\ p_0(t) \qquad for \quad n = 1, 2, 3...$$

where,

$$p_0(t) = e^{-\frac{\mu}{\beta}\ [(1+2\beta)^{\frac{1}{2}}-1]}$$

The pgf $H(s,t)$ is given by

$$\mathrm{P(z)} = \mathrm{e}^{-\frac{\mu}{\beta}\ [(1-2\beta(\mathrm{z}-1))^{\frac{1}{2}}-1]}$$

The mean and variance are

$$\mathrm{E(N)} = \mathrm{E(Z(t))} = \mu; \qquad \mathrm{Var(N)} = \mathrm{Var(Z(t))} = \mu(1+\beta)$$

The pgf, $\mathrm{G(s,t)}$ of the iid random variables is given by

$$\mathrm{Q(z)} = \frac{[(1-2\beta(\mathrm{z}-1))]^{\frac{1}{2}} - (1+2\beta)^{\frac{1}{2}}}{1-(1+2\beta)^{\frac{1}{2}}}$$

The pmf $g_x(t)$ of the iid random variables is given by

$$\mathrm{q_n} = \frac{\frac{1}{2}\Gamma(\mathrm{n}-\frac{1}{2})\ (1+2\beta)^{\frac{1}{2}}\ (\frac{2\beta}{1+2\beta})^{\mathrm{n}}}{\mathrm{n}!\Gamma(\frac{1}{2})\ [(1+2\beta)^{\frac{1}{2}-1]}} \qquad \text{for} \quad \mathrm{n} = 1, 2, 3, ...$$

which satisfies Panjer's recursive model with

$$\mathrm{a} = \frac{2\beta}{1+2\beta} \qquad \text{and} \qquad \mathrm{b} = \frac{-3\beta}{1+2\beta}$$

Using notations of Sankara (1968), let

$$\mathrm{t} = 1, \qquad \mathrm{n} = \mathrm{r} \qquad \frac{\mathrm{p}}{(1+\mathrm{c})^{\frac{1}{2}}} \qquad \mathrm{b} = -\frac{\mathrm{c}}{1+\mathrm{c}} \qquad \text{and} \qquad \mathrm{i} = \mathrm{k}$$

The recursive formula for the Poisson-inverse Gaussian distribution becomes

$$(r+1)p_{r+1}(t) = a \sum_{k=0}^{r} \frac{\Gamma(k+\frac{1}{2})}{k!\Gamma(\frac{1}{2})}\ (-b)^k\ p_{r-k} = a \sum_{k=0}^{r} \binom{\frac{1}{2}+k-1}{k}\ (-b)^k\ p_{r-k}$$
$$= a \sum_{k=0}^{r} \frac{(2k-1)(2k-3)....(5.3.1)\ (\frac{-b}{k})^k\ \ p_{r-k}}{k!} \qquad for \qquad r = 0, 1, 2, ...$$

6.6 When $\mathrm{a} = 2$

$$\mathrm{h(t)} = \theta'(\mathrm{t}) = \frac{\mathrm{p}}{(1+\mathrm{ct})^2} \quad \text{for} \quad \mathrm{p} > 0, \quad \text{and} \quad \mathrm{c} > 0$$

which shall be referred to as Polya-Aeppli hazard function.
Therefore

$$h^{(n)}(t) = (-1)^n \, (n+1)! pc^n (1+ct)^{-(n+2)}$$

$$(-1)^n \frac{d^n}{dt^n} h(t) = (-1)^n \, h^n(t)$$
$$= (-1)^n \, (n+1)! pc^n (1+ct)^{-(n+2)} \geq 0$$

Therefore h(t) is completely monotone.

$$\theta(t) = p \int_0^t (1+cx)^{-2} \, dx$$

$$\theta(t) = \frac{pt}{1+ct}$$

implying that

$$\theta(\mathrm{t}-\mathrm{ts}) = \frac{\mathrm{pt}-\mathrm{pts}}{1-\mathrm{ct}-\mathrm{cts}}$$

Therefore

$$\theta(0) = 0, \qquad \theta'(0) = \mathrm{p} \qquad \text{and} \qquad \theta''(0) = -2\mathrm{pc}$$

$$\mathrm{p}_0(\mathrm{t}) = \mathrm{e}^{-\frac{\mathrm{pt}}{1+\mathrm{ct}}}$$

The pgf is

$$H(s,t) = e^{-\theta(t-ts)}$$
$$= e^{-\frac{pt-pts}{1-ct-cts}}$$

Therefore

$$\mathrm{E(Z(t))} = \mathrm{t}\,\theta'(0) = \mathrm{pt} \qquad \text{and} \qquad \mathrm{Var(Z(t))} = \mathrm{t}\,\theta'(0) - \mathrm{t}^2\,\theta''(0) = \mathrm{pt} + 2\,\mathrm{pc}\,\mathrm{t}^2$$

Since $\theta(0) = 0$ and $\mathrm{h(t)} = \theta'(\mathrm{t})$ is completely monotone, then $\mathrm{p}_0(\mathrm{t})$ is a Laplace transform of an infinitely divisible mixing distribution.
The corresponding infinitely divisible mixed Poisson distribution is a compound Poisson distribution whose iid random variables have pgf given by

$$G(s,t) = 1 - \frac{pt-pts}{1-ct-cts}\,\frac{1+ct}{pt}$$
$$= \frac{s}{(1+ct)(1-\frac{ct}{1+ct}\,s)}$$
$$= \frac{\frac{1}{1+ct}\,s}{1-\frac{ct}{1+ct}\,s}$$

which is the pgf of zero-truncated (shifted) geometric distribution.

By power series expansion

$$\begin{aligned}
G(s,t) &= \frac{1}{1+ct}\; s \sum_{x=0}^{\infty} (\frac{ct}{1+ct})^{x}\; s^{x} \\
g_x(t) &= \frac{1}{1+ct}\; (\frac{ct}{1+ct})^{x-1} \qquad x = 1,2,3,... \qquad and \qquad g_0(t) = 0 \\
\frac{g_x(t)}{g_{x-1}(t)} &= (\frac{ct}{1+ct})^{x-1}\; (\frac{(1+ct)}{(ct)})^{x-2} \\
&= \frac{ct}{1+ct} \qquad x = 2,3,... \\
&= (\frac{ct}{1+ct} + \frac{0}{x}) \qquad for \qquad x = 2,3,...
\end{aligned}$$

which is Panjer's form with $a = \frac{ct}{1+ct}$ and $b = 0$
The compound Poisson distribution in the recursive form is given by

$$(n+1)p_{(n+1)}(t) = \frac{pt}{(1+ct)^2} \sum_{i=0}^{n} (i+1)\; (\frac{ct}{1+ct})^{i}\; p_{n-i}(t) \qquad (6.15)$$

By iteration,

$$n = 0, \qquad p_1(t) = \frac{pt}{(1+ct)^2}\; p_0(t)$$

$$n = 1, \qquad p_2(t) = [\frac{1}{2}\; \frac{(pt)^2}{(1+ct)^4} + \frac{pt\; ct}{(1+ct)^3}]\; p_0(t)$$

$$n = 2, \qquad 3\; p_3(t) = \frac{pt}{(1+ct)^2} \sum_{i=0}^{2} (i+1)\; (\frac{ct}{1+ct})^{i}\; p_{2-i}(t)$$

$$= \frac{pt}{(1+ct)^2}\; [p_2(t) + \frac{ct}{1+ct}\; p_1(t) + (\frac{ct}{1+ct})^2\; p_0(t)]$$

$$= \frac{pt}{(1+ct)^2}\; [\frac{1}{2}\; \frac{(pt)^2}{(1+ct)^4}\; p_0(t) + \frac{pt\; ct}{(1+ct)^3}\; p_0(t) + 2\frac{pt\; ct}{(1+ct)^3}\; p_0(t) + (\frac{ct}{1+ct})^2\; p_0(t)]$$

$$p_3(t) = [\frac{1}{6}\; \frac{(pt)^3}{(1+ct)^6} + \frac{(pt)^2\; ct}{(1+ct)^5} + \frac{pt\; (ct)^2}{(1+ct)^4}]\; p_0(t)$$

$$n = 3, \qquad 4\; p_4(t) = \frac{pt}{(1+ct)^2} \sum_{i=0}^{3} (i+1)\; (\frac{ct}{1+ct})^{i}\; p_{3-i}(t)$$

$$p_4(t) = \frac{pt}{(1+ct)^5} \sum_{j=1}^{4} \binom{4-1}{j-1}\; [\frac{1}{j!}\; (\frac{pt}{1+ct})^{j-1}\; (ct)^{4-j}]\; p_0(t)$$

$$\mathrm{p_4(t)} = (\frac{\mathrm{ct}}{1+\mathrm{ct}})^4\ \mathrm{p_0(t)} \sum_{\mathrm{j=1}}^{4} \binom{4-1}{j-1}\ (\frac{\mathrm{pt}}{\mathrm{ct}\ (1+\mathrm{ct})})^{\mathrm{j}}\ \frac{1}{\mathrm{j!}}$$

In general, the pattern is

$$\mathrm{p_n(t)} = (\frac{\mathrm{ct}}{1+\mathrm{ct}})^{\mathrm{n}}\ \mathrm{p_0(t)} \sum_{\mathrm{j=1}}^{\mathrm{n}} \binom{n-1}{j-1}\ (\frac{\mathrm{pt}}{\mathrm{ct}\ (1+\mathrm{ct})})^{\mathrm{j}}\ \frac{1}{\mathrm{j!}}$$

where

$$\mathrm{p_0(t)} = \mathrm{e}^{-\frac{\mathrm{pt}}{1+\mathrm{ct}}}$$

This is the Polya-Aeppli distribution.

6.7 When a $\to \infty$

$$\begin{aligned} h(t) &= \theta'(t) \\ &= lim_{\ a \to \infty}\ p_2\ (1+c_2 t)^{-a} \quad for \quad p>0, \quad and \quad c>0 \\ &= lim_{\ a\to\infty}\ p_2 \sum_{k=0}^{\infty} \frac{-a\ (-a-1)\ (-a-2)...[-a-(k-1)]\ (c_2 t)^k}{k!} \\ &= lim_{\ a\to\infty}\ p_2 \sum_{k=0}^{\infty} \frac{(-1)^k\ a^k\ (1+\frac{1}{a})\ (1+\frac{2}{a})...[(1+\frac{(k-1)}{a})]\ (c_2 t)^k}{k!} \\ &= p_2 \sum_{k=0}^{\infty} lim_{\ a\to\infty} \frac{(-ac)^k\ (t)^k}{k!} \end{aligned}$$

Let b = ac as a $\to$ ∞

$$\begin{aligned} h(t) &= \theta'(t) \\ &= p_2 \sum_{k=0}^{\infty} \frac{(-bt)^k}{k!} \\ &= p_2\ e^{-bt} \end{aligned}$$

which is the hazard function for Gompertz distribution.
Therefore,

$$h^{(n)}(t) = (-1)^n\ b^n\ p_2\ e^{-bt}$$

$$(-1)^n\ \frac{d^n}{dt^n}\ h(t) = (-1)^n\ h^n(t) \geq 0$$

Therefore h(t) is completely monotone.

The cumulative hazard function is

$$\begin{aligned} \theta(t) &= p_2 \int_0^t e^{-bx}\ dx \\ &= \frac{p}{b}\ [1-e^{-bt}] \end{aligned}$$

implying that

$$\theta(\mathrm{t-ts}) = \frac{\mathrm{p}}{\mathrm{b}}\,[1-\mathrm{e}^{-\mathrm{bt}(1-\mathrm{s})}]$$

Therefore

$$\theta(0) = 0, \qquad \theta'(0) = \mathrm{p} \qquad \text{and} \qquad \theta''(0) = -\mathrm{b\,p}$$

Since Gompertz distribution is an exponential mixture, the survival function is the Laplace transform of the mixing distribution and hence,

$$\begin{aligned}
p_0(t) &= e^{\frac{p}{b}\,[e^{-bt}-1]} \\
&= e^{-\frac{p}{b}} \sum_{j=0}^{\infty} [\frac{p}{b}\, e^{-bt}]^j\, \frac{1}{j!} \\
&= \sum_{j=0}^{\infty} e^{-btj}\, [e^{-\frac{p}{b}}\, \frac{[\frac{p}{b}]^j}{j!}] \\
p_0^{(n)}(t) &= \sum_{j=0}^{\infty} (-bj)^n\, e^{-btj}\, p_j \\
p_n(t) &= (-1)^n \frac{t^n}{n!} \sum_{j=0}^{\infty} (-bj)^n\, e^{-btj}\, p_j \\
p_n(t) &= \sum_{j=0}^{\infty} (-1)^n \frac{t^n}{n!}\, (-bj)^n\, e^{-btj}\, p_j \\
&= \sum_{j=0}^{\infty} \frac{(btj)^n}{n!}\, e^{btj}\, e^{-\frac{p}{b}}\, \frac{[\frac{p}{b}]^j}{j!}; \qquad j = 0, 1, 2....
\end{aligned}$$

which is Poisson mixture of Poisson distribution also known as Neyman Type A distribution.

The pgf is

$$\begin{aligned}
H(s,t) &= e^{-\theta(t-ts)} \\
&= e^{-\frac{p}{b}\,[1-e^{-bt(1-s)}]}
\end{aligned}$$

Therefore

$$\mathrm{E(Z(t))} = \mathrm{t}\,\theta'(0) = \mathrm{pt} \qquad \text{and} \qquad \mathrm{Var(Z(t))} = \mathrm{t}\,\theta'(0) - \mathrm{t}^2\,\theta''(0) = \mathrm{pt} + \mathrm{pb\,t}^2$$

Since $\theta(0) = 0$ and $\mathrm{h(t)} = \theta'(\mathrm{t})$ is completely monotone, then $\mathrm{p_0(t)}$ is a Laplace transform of an infinitely divisible mixing distribution.

The corresponding infinitely divisible mixed Poisson distribution is a compound Poisson distribution whose iid random variables have pgf given by

$$
\begin{aligned}
G(s,t) &= 1-\frac{1-e^{-bt\,(1-s)}}{1-e^{-bt}} = \frac{e^{-bt\,(1-s)}-e^{-bt}}{1-e^{-bt}}\\
&= \frac{e^{-bt}}{1-e^{-bt}}\,[e^{bts}-1]\\
&= \frac{e^{-bt}}{1-e^{-bt}}\,[\sum_{x=0}^{\infty}\frac{(bt)^x\,s^x}{x!}-1]\\
g_x(t) &= \frac{1}{e^{bt}-1}\,\frac{(bt)^x}{x!} \qquad x=1,2...
\end{aligned}
$$

which is zero-truncated Poisson distribution with parameter bt

$$
\frac{\mathrm{g_x(t)}}{\mathrm{g_{x-1}(t)}} = 0+\frac{\mathrm{bt}}{\mathrm{x}} \qquad \mathrm{x}=1,2,...
$$

and it is in Panjer's recursive form, where $\mathrm{a}=0$ and $\mathrm{b}=\mathrm{bt}$

Remark 7 .
Whereas Klugman et. al. (2008) have stated that the distribution of the iid random variables is Poisson, in this study, the distribution is zero-truncated Poisson distribution.
The difference is due to the assumption that N is Poisson with a constant parameter λ instead of $\theta(t)$, which is a cumulative hazard function.

The recursive formula for the compound Poisson distribution is given by:

$$
\begin{aligned}
(n+1)\,p_{n+1}(t) &= \frac{p}{b}\,[1-e^{-bt}]\sum_{i=0}^{n}(i+1)\,\frac{1}{e^{bt}-1}\,\frac{(bt)^{i+1}}{(i+1)!}\,p_{n-i}(t)\\
&= pt\,e^{-bt}\sum_{i=0}^{n}\frac{(bt)^i}{i!}\,p_{n-i}(t) \qquad n=0,1,2,...
\end{aligned}
$$

6.8 When $a \neq 0$ *and* $a \neq 1$

$$
\begin{aligned}
h(t) &= \theta'(t)\\
&= \frac{p}{(1+ct)^a} \quad for \quad p>0, \quad c>0 \quad and \quad a>0 \quad but \quad a\neq 1\\
h^{(n)}(t) &= \theta^{(n)}(t)\\
&= (-1)^n\,a(a+1)a+3)...(a+n-1)c^n p\,(1+ct)^{-a-n+1}
\end{aligned}
$$

Therefore $h(t)$ is completely monotone.

$$\theta(t) = p \int_0^t (1+cx)^{-a}\, dx$$
$$= \frac{p}{c(1-a)}\, [(1+ct)^{1-a} - 1]$$

implying that

$$\theta(\mathrm{t}-\mathrm{ts}) = \frac{\mathrm{p}}{\mathrm{c}(1-\mathrm{a})}\, [(1+\mathrm{ct}-\mathrm{cts})^{1-\mathrm{a}} - 1]$$

and

$$\theta(0) = 0 \quad \theta'(0) = \mathrm{p} \quad \text{and} \quad \theta''(0) = -\mathrm{acp}$$

also

$$p_0(t) = e^{-\frac{p}{c(1-a)}\,[(1+ct)^{1-a}-1]}$$
$$and \qquad H(s,t) = e^{-\theta(t-ts)}$$
$$= e^{-\frac{p}{c(1-a)}\,[(1+ct-cts)^{1-a}-1]}$$

Therefore

$$\mathrm{E(Z(t))} = \mathrm{t}\,\theta'(0) = \mathrm{pt} \qquad \text{and} \qquad \mathrm{Var(Z(t))} = \mathrm{t}\,\theta'(0) - \mathrm{t}^2\,\theta''(0) = \mathrm{pt} + \mathrm{acpt}^2$$

Since $\theta(0) = 0$ and $\mathrm{h(t)} = \theta'(\mathrm{t})$ is completely monotone, then $\mathrm{p_0(t)}$ is a Laplace transform of an infinitely divisible mixing distribution.
The corresponding infinitely divisible mixed Poisson distribution is a compound Poisson distribution whose iid random variables have pgf given by

$$G(s,t) = 1 - \frac{1+ct-cts)]^{1-a} - 1}{(1+ct)^{1-a} - 1}$$
$$= \frac{(1+ct)^{1-a} - (1+ct-cts)^{1-a}}{(1+ct)^{1-a} - 1}$$
$$= \frac{(1+ct)^{1-a}}{(1+ct)^{1-a} - 1}\,(-1)\sum_{x=1}^{\infty}\binom{1-a}{x}\,(\frac{-ct}{1+ct})^x\, s^x$$
$$g_x(t) = \frac{1-a}{x!}\,\frac{\Gamma(a+x-1)}{\Gamma(a)}\,(\frac{ct}{1+ct})^x\,\frac{(1+ct)^{1-a}}{(1+ct)^{1-a} - 1} \qquad for\ x = 1,2,3,...$$
$$and \qquad g_0(t) = 0$$

The recursive formula of the compound Poisson distribution is therefore

$$= \frac{\mathrm{pt}}{(1+\mathrm{ct})^{\mathrm{a}}}\sum_{\mathrm{x}=1}^{\mathrm{n}}\frac{\Gamma(\mathrm{a}+\mathrm{x}-1)}{(\mathrm{x}-1)!\,\Gamma(\mathrm{a})}\,(\frac{\mathrm{ct}}{1+\mathrm{ct}})^{\mathrm{x}-1}\,\mathrm{p_{n-x}(t)} \quad \text{for} \quad \mathrm{x} = 1,2,3,...$$

Therefore by replacing n by $n+1$, we have

$$(\mathrm{n}+1)\ \mathrm{p}_{(\mathrm{n}+1)}(\mathrm{t}) = \frac{\mathrm{pt}}{(1+\mathrm{ct})^{\mathrm{a}}} \sum_{\mathrm{x}=1}^{\mathrm{n}+1} \frac{\Gamma(\mathrm{a}+\mathrm{x}-1)}{(\mathrm{x}-1)!\,\Gamma(\mathrm{a})} \left(\frac{\mathrm{ct}}{1+\mathrm{ct}}\right)^{\mathrm{x}-1} \mathrm{p}_{\mathrm{n}+1-\mathrm{x}}(\mathrm{t}) \quad \text{for} \quad \mathrm{x} = 0,1,2,3,...$$

Put $x = i+1$ we get

$$(\mathrm{n}+1)\ \mathrm{p}_{(\mathrm{n}+1)}(\mathrm{t}) = \frac{\mathrm{pt}}{1+\mathrm{ct}} (1+\mathrm{ct})^{1-\mathrm{a})} \sum_{\mathrm{i}=0}^{\mathrm{n}} \frac{\Gamma(\mathrm{a}+\mathrm{i})}{\mathrm{i}!\,\Gamma(\mathrm{a})} \left(\frac{\mathrm{ct}}{1+\mathrm{ct}}\right)^{\mathrm{i}} \mathrm{p}_{\mathrm{n}-\mathrm{i}}(\mathrm{t}) \quad \text{for} \quad \mathrm{n} = 0,1,2,3,... \tag{6.16}$$

as given by Walhin and Paris (2002)

Moments

To obtain the first moment, sum (6.16) over n to get

$$\begin{aligned}\sum_{n=0}^{\infty} (n+1)\ p_{n+1}(t) &= E[Z(t)] \\ &= \frac{pt}{(1+ct)^{a}} \left(\frac{1}{1+ct}\right)^{-a} \\ &= pt\end{aligned}$$

Next, multiply (6.16) by $\mathrm{n}+1$ and then sum the result over n, to get

$$\begin{aligned}E[Z(t)]^2 &= \sum_{n=0}^{\infty} (n+1)^2\ p_{n+1}(t) \\ &= pt\ [(pt+1)+act] \\ Var(Z(t)) &= pt + acpt^2\end{aligned}$$

7 Parameterization of Hofmann Hazard Function

This section identifies the hazard function of an exponential mixture which accommodates the extended truncated negative binomial (ETNB) distribution as the distribution of the iid random variables for the compound Poisson distribution.

7.1 Constructing ETNB Distribution

$$(1-q(t))^{-r} = \sum_{k=0}^{\infty} \binom{-r}{k} (-q(t))^k$$

$$= 1 + \sum_{k=1}^{\infty} \binom{-r}{k} (-q(t))^k$$

therefore

$$Prob(X = k) = \frac{\binom{r+k-1}{k} (q(t))^k}{(1-q(t))^{-r} - 1} \qquad k = 1, 2, 3, ...$$

is a probability mass function called zero-truncated negative binomial distribution with parameter $q(t)$. It is also called the extended truncated negative binomial (ETNB) distribution according to Klugman et. al. (2008) because the parameter r can extend below zero.

Let

$$\mathrm{q(t)} = \frac{\mathrm{ct}}{1+\mathrm{ct}}, \qquad \mathrm{c} > 0, \mathrm{t} > 0.$$

Then

$$\mathrm{Prob(X = k)} = \frac{\binom{r+k-1}{k} (\frac{\mathrm{ct}}{1+\mathrm{ct}})^{\mathrm{k}}}{(1+\mathrm{ct})^{\mathrm{r}} - 1} \qquad \mathrm{k} = 1, 2, 3, ...$$

The probability generating function is given by:

$$G(s) = \sum_{k=1}^{\infty} p_k \, s^k$$

$$= \sum_{k=1}^{\infty} \frac{\binom{r+k-1}{k} (\frac{ct}{1+ct} \, s)^k}{(1+ct)^r - 1}$$

$$= \frac{\sum_{k=1}^{\infty} \binom{-r}{k} (-\frac{ct}{1+ct} s)^k}{(1+ct)^r - 1}$$

$$G(s) = \frac{(1 - \frac{ct}{1+ct} \, s)^{-r}) - 1}{(1+ct)^r - 1}$$

7.2 Identifying the Hazard Function

Let us parameterize Hofmann hazard function by putting a = r + 1. Thus

$$\begin{aligned}
h(t) &= \frac{p}{(1+ct)^{r+1}} \qquad p>0, \quad c>0 \quad and \quad r \geqslant -1\\
\theta(t) &= \frac{p}{rc}\,[1-(1+ct)^{-r}],\\
\theta(0) &= 0\\
&\textit{and hence}\\
\theta(t-t\ s) &= \frac{p}{rc}\,[1-(1+ct-cts)^{-r}]
\end{aligned}$$

For an infinitely mixed Poisson distribution

$$\begin{aligned}
G(s,t) &= 1-\frac{[1-(1+ct-cts)^{-r}}{1-(1+ct)^{-r}}\\
&= \frac{(1-\frac{ct}{1+ct}\ s)^{-r}-1}{(1+ct)^{r}-1}
\end{aligned}$$

which is the pgf of ETNB distribution.

$$\begin{aligned}
&= \frac{\sum_{x=1}^{\infty}\binom{-r}{x}\ (-\frac{ct}{1+ct}\ s)^{x}]}{(1+ct)^{r}-1}\\
g(x) &= \frac{\binom{r+x-1}{x}\ (\frac{ct}{1+ct})^{x}}{[(1+ct)^{r}-1]} \qquad x=1,2,3,...
\end{aligned}$$

which is the pmf of a zero-truncated negative binomial distribution or extended truncated negative binomial distribution with parameters $\frac{\text{ct}}{1+\text{ct}}$.

$$\begin{aligned}
\frac{g_x(t)}{g_{x-1}(t)} &= \frac{\binom{r+x-1}{x}\ (\frac{ct}{1+ct})^{x}}{\binom{r+x-2}{x-1}\ (\frac{ct}{1+ct})^{x-1}}\\
&= \frac{ct}{1+ct}\ \frac{r+x-1}{x}\\
&= \frac{ct}{1+ct}\ \frac{r+x-1}{x} \qquad x=2,3,...\\
\frac{g_x(t)}{g_{x-1}(t)} &\equiv \frac{ct}{1+ct}\ [1+\frac{r-1}{x}]\\
&= a+\frac{b}{x}
\end{aligned}$$

and this is Panjer's model

8 Concluding Remarks

The hazard function of an exponential mixture characterizes an infinitely divisible mixed Poisson distribution which is also a compound Poisson distribution.

Hofmann hazard function is a good illustration of the theory. For further research other classes of hazard functions should be considered, in particular those based on compound Poisson distributions with continuous iid random variables.

Hazard functions expressed in terms of the modified Bessel function of the third kind would be of interest.

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