Hyers- Ulam Rassias Stability of multiplicative Cauchy equation

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Abstract . The aim of this paper is to prove the stability of multiplicative equation in sprit of Hyers- Ulam- Rassias.

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1. INTRODUCTION

The problem of stability of homomorphism stemmed from the question posed by S.M.Ulam in 1940 in his lecture before the mathematical club of the University of Wisconsin. He demanded an answer to the following question of stability of homomorphism for metric groups.

Let G' be a group and let G" be a metric group with the metric d. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G' \to G''$ satisfies the following inequality

$d[h (x y) , h (x) h (y)] < \delta$ for all x and y in G',

then there exists a homomorphism $H : G' \to G''$ with

$d[h (x) , H (x)] < \varepsilon$ for all x in G' ?

In 1941 D.H.Hyers answered his question considering the case of Banach spaces. D.H.Hyers [18] proved the following result where E' and E" are Banach spaces.

Result

Let $f : E' \to E''$ be a mapping between Banach spaces. If f satisfies the following inequality

$\| f(x+y) - f(x) - f(y) \| \leq \delta$

for all x and y in E' and some $\delta > 0$ then the limit

$T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$

exists for all x in E' and T : E' $\to$ E'' is a unique additive mapping such that

$\| f(x) - T(x) \| \leq \delta$ for all x in E'.

Moreover, if f(t x) is continuous in t for each fixed x in E', then the mapping T is linear.

In 1978 Th.M.Rassias [2] generalized the result of Hyers by proving the following result.

Result

Let f : E' $\to$ E'' be a mapping between Banach spaces. If f satisfies the following inequality

$\| f(x+y) - f(x) - f(y) \| \leq \theta(\| x \|^p + \| y \|^p)$

for all x and y in E' and for some $\theta > 0$ and some p with $0 \leq p < 1$, then there exists a unique additive mapping T : E' $\to$ E'' such that

$$\| T(x) - f(x) \| \leq 2\theta \left(\frac{\| x \|^p}{2 - 2^p} \right)$$

for all x and y in E'. In addition , if f(t x) is continuous in t for each fixed x in E', then the mapping T is linear.

The method adopted by D.H.Hyers is designated as *Direct Method* and the stability of any functional inequality which had an independent bound is termed as *Hyers - Ulam Stability*.

Th.M.Rassias and thereafter others like George Isac, P.Gauvruta, G.L.Forti, John A.Baker , F.Skof, S.M.Jung etc. obtained several useful result on the stability of functional inequalities which had bounds dependant in some way on the elements in the domain of the function under consideration. Such type of stability is termed as *Hyers - Ulam - Rassias Stability*

In section 3 the stability of the multiplicative Cauchy equation

$$f\left(\frac{3x+3y}{2}\right) . \ f\left(\frac{y-x}{2}\right) = 2f(x)\,f(y)$$

is obtained , where f : R $\rightarrow$ Y and R is a set of real numbers and Y is a Banach algebra.

2. PRELIMINARIES

2.1 Multiplicative Cauchy Equation

Let (G, .) be a semigroup and E is a normed algebra with the property that the norm is multiplicative. A mapping f : G $\rightarrow$ E is called *multiplicative Cauchy equation* , if function f satisfies

f (x . y) = f (x) . f (y) for all x and y in G.

2.2 Banach Space

A *Banach space* is a complete normed vector space.

2.3. Banach algebra

Banach algebra is an algebra (a vector space with multiplication, satisfying the usual algebraic rules) and also a Banach space.

3. Stability of Multiplicative Cauchy equation

3.1 Theorem

Let f: R $\rightarrow$ Y where Y is a Banach algebra which satisfies the following inequality

$$\left\| f\left(\frac{3x+3y}{2}\right) f\left(\frac{y-x}{2}\right) - 2f(x)f(y) \right\| \leq \theta[\, \| x \|^p + \| y \|^p \,] \qquad (3.2)$$

for all $\theta > 0$, x and y in R and $p < 1$.

Further let $\| f(x) \| > 1$ for all x in R.

Then the limit $T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ exists for all x in R such that

$$\| T(x) - f(x) \| \le \left(\frac{\theta(\| x \|^p)}{2 - 2^p} \right) (1 + 3^{-p}) \qquad (3.3)$$

for all x in R. Also the function T : R → Y is the unique and a solution of (3.2)

Proof

Replace y by 3x in (3.2) to obtain

$$\| f[3(2x)] f(x) - 2 f(x) f(3x) \| \le [\theta(1 + 3^p)(\| x \|^p)]$$

for all x in R.

or

$$\| f(x) \| . \| f[3(2x)] - 2 f(3x) \| \le [\theta(1 + 3^p)(\| x \|^p)]$$

for all x in R.

or

Replace x by x/3 in last inequality to obtain

$$\| f(x/3) \| . \| f(2x) - 2f(x) \| \le [\theta . 3^{-p}(1 + 3^p)(\| x \|^p)]$$

for all x in R.

or

$$\| f(2x) - 2f(x) \| \le \left(\frac{\theta . 3^{-p}(1 + 3^p) . \| x \|^p}{\| f(x/3) \|} \right) \text{for all } x \text{ in R.} \qquad (3.4)$$

If $\| f(x) \| > 1$ for all x in R .
Then

$1 / \| f(x) \| < 1$ for all x in R . Therefore from (3.4) it follows that

$$\| f(2x) - 2f(x) \| \leq [\theta(1 + 3^{-p})(\| x \|^p)] \text{ for all } x \text{ in R.} \quad (3.5)$$

From a direct method there is sequence $\{2^{-n} f(2^n x)\}$ which is Cauchy and convergent in Y.

$$\text{Fix } T(x) = \lim_{n \to \infty} 2^{-n} f(2^n x) \quad \text{for all } x \text{ in R.} \quad (3.6)$$

Therefore from (3.5) it follows that

$$\| T(x) - f(x) \| \leq \sum_{n=0}^{\infty} \left(\frac{\theta(1 + 3^{-p})(\| 2^n x \|^p)}{2^{n+1}} \right)$$

for all x in R and n in N.
or

$$\| T(x) - f(x) \| \leq \left(\frac{\theta(1 + 3^{-p}) . \| x \|^p}{2 - 2^p} \right)$$

for all x in R and for all n in N, which is exactly (3.3).

T is a solution

Replace x by $2^n x$ and y by $2^n y$ in (3.2) then divide it by 2^{2n} to obtain

$$\left\| f\left(\frac{3[\,2^n(x+y)/2\,]}{2^n}\right) . f\left(\frac{[2^n(y-x)/2\,]}{2^n}\right) - 2f\left(\frac{2^n x}{2^n}\right) f\left(\frac{2^n y}{2^n}\right)\right\|$$

$$\leq \quad 2^{n(p-2)}.\theta\,[\ \|\ x\ \|^p + \|\ y\ \|^p\ ]$$

for all x in R and n in N and $p < 1$.

or

$$\lim_{n\to\infty}\left\| f\left(\frac{3[2^n(x+y)/2]}{2^n}\right) f\left(\frac{[2^n(y-x)/2]}{2^n}\right) - 2f\left(\frac{2^n x}{2^n}\right) f\left(\frac{2^n y}{2^n}\right)\right\| = 0$$

Or

$$T\left(\frac{3x+3y}{2}\right) T\left(\frac{y-x}{2}\right) = 2\,T(x)\,.\,T(y) \quad \text{for all } x \text{ and } y \text{ in R.}$$

Uniqueness

Let $T'(x) : R \rightarrow Y$ be another map which satisfies (3.2)and (3.6). Obviously

$T(2^n x) = 2^n T(x)$ and $T'(2^n x) = 2^n T'(x)$

For all x in R and n in N. Therefore

$$\|\,T(x) - T'(x)\,\| \leq \quad 2^{-n}\,\|\,T(2^n x) - f(2^n x)\,\| + 2^{-n}\,\|\,T'(2^n x) - f(2^n x)\,\|$$

$$\leq \left(\frac{2^{-(n+1)}\,[\theta(1+3^{-p})\,(\,\|\,x\,\|^p\,)]}{2 - 2^p}\right)$$

for all x in R and n in N . On taking the limit as n tend to infinity there is a uniqueness of T . Hence $T(x) = T'(x)$ for all x in R.

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EXPECTED OUTCOME OF THE PRESENT WORK

1. Generalization of these result to topological vector space can be further taken up .

2. In addition to these, some open problems posed by TH.M.RASSIAS[2] in his Survey paper on the topic shall be taken for further investigation.