

Characterization of δ-small submodule

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Abstract
Pseudo Projectivity and M- Pseudo Projectivity is a generalization of Projevtevity.
[2], [8] studied M-Pseudo Projective module and small M-Pseudo Projective module. In this paper we consider some generalization of small M-Pseudo Projective module, that is δ-small M-Pseudo Projective module with the help of δ-small and δ- cover.
Key words: Singular module, S.F. small M-Pseudo Projective module, δ-small and projective δ- cover.

Introduction:

Throughout this paper R is an associative ring with unity module and all modules are unitary left R-modules. A sub module K of a module M. K ≤ M. Let M be a module, K ≤ M is said to be small in M if for every L ≤ M, the equality K + L = M implies L = M, (denoted by $K \ll M$). The concept of δ-small sub modules was introduced by Zhon [10]. A sub module K of M is said to be δ-small sub module of M (denoted by $K \ll_\delta M$) if whenever M = K+ L, with M/K is singular, then M = L. The sum of all δ-small sub module of M is denoted by δ(M). δ(M) is the reject in M of class of all singular simple modules. $\delta(M) = Rej_M(\wp) = \cap \left\{N \leq M: \frac{M}{N} \in \wp\right\}$, where $\wp$ be the class of all singular modules. ($Rej_M(\mathbb{Q})$ is the intersection of all K ≤ M, with M/K torsion free). An R-module M is said to be hollow (δ-hollow) if all proper sub modules of M are small (δ-small) in M. An R-module M is S.F. if zero is only small sub module in M.

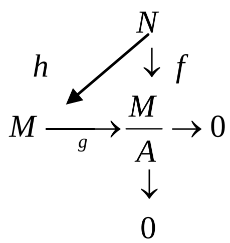
G. Azumay introduced projective cover. W.xue [12] generalized projective cover. A module epimorphism $f: P \to M$ is a cover in case $\ker f \leq Rad(P)$, $Kerf \ll M$. A cover $f: P \to M$ is called a protective cover in case P is projective module. An epimorphism $f: P \to M$ is called a δ-projective cover of module M in case $Kerf \ll_\delta \delta(P)$ and P is projective. A δ-cover $f: P \to M$ of a module M, is said to be a self projective δ-cover in case p is self projective module. Projective cover is denoted by P(M), if there is an epimorphism $f_M: P(M) \to M$ with P(M) is projective and $kerf_M \ll P(M)$.

In last section we introducing a new characterization of small M-pseudo projective module. We prove that N is hollow, then N is δ-small M-pseudo projective module if and only if N is M-pseudo projective module, and let M be a δ-small pseudo projective module then M is S.F. if and only if M/A is isomorphic to direct summand of M, A ≤ M.

1.δ-Small

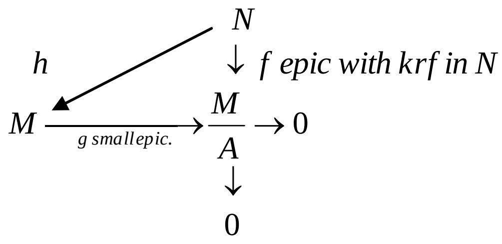
Definition:1.1. The sub module $Z(M) = \{x \in M : r_R(x) \text{ is essential in } R\}$ is called singular sub module of M. The module M is Called a singular module if Z(M) = M. The module M is Called a non- singular module if Z(M) = 0.

Definition:1.2. An R-module N is called small M- pseudo projective module if for every sub module A of M, any epimorphism $f: N \to \frac{M}{A}$ with $Kerf \ll N$, Can be lifted to a homomorphism $h: N \to M$



i.e. $g \cdot h = f$.

Definition:1.3 An R-module N is called δ-small M- pseudo projective module if for every sub module A of M, any small epimorphism $f : N \to \frac{M}{A}$ with $Ker f \ll_{\delta} N$, Can be lifted to a homomorphism $h : N \to M$



i.e. $g \cdot h = f$.

Examples:1

i) Every small sub module of M is δ-small in M.
ii) Every non singular semisimple sub module of M is δ-small in M.
iii) Every simple module M is hollow.
iv) $\mathbf{Z_6}$ as Z-module is not δ-small.

Example:2. Consider the Z-modules $\mathbf{Z_4}$ and $\mathbf{Z_2}$. An epimorphism $f : Z_4 \to Z_2$ define by $f(\bar{1}) = f(\bar{3}) = \bar{1}$ *and* $f(\bar{0}) = f(\bar{2}) = \bar{0}$ Then f is small epimorphism .

Every small sub module of M is δ-small in M.

Example:3. Let $R = M = Z_6$. Then two non-trivial sub modules of M, $M_1 = \{\bar{0}, \bar{3}\}$ *and* $M_2 = \{\bar{0}, \bar{2}, \bar{4}\}$ are δ-small in M, but neither M_1 and M_2 is small in M, Moreover $M \ll_{\delta} M$

Proposition:1.1 Let M, L and N be R-Modules. If $\alpha : M \to N$ and $\beta : N \to L$ are two epimorphisms. Then $\beta \circ \alpha$ is small if and only if both α, β are small.

Proof: [8]

Lemma:1.1. Let N be a submodule of M. The following are equivalent:

i) $N \ll_{\delta} M$
ii) If X + N = M, then M = X ⊕ Y for a projective semi simple sub module Y with Y ⊆ N.
iii) If X + N = M, with M/X Goldie torsion, then X = M.

Proof: [10].

Lemma:1.2. Let M be a module, then

i) For sub module N, K, L with K ≤ N, We have
   a) $N \ll_{\delta} M$ if and only if $K \ll_{\delta} M$ and $N/K \ll_{\delta} M/K$
   b) $N + L \ll_{\delta} M$ if and only if $N \ll_{\delta} M$ and $L \ll_{\delta} M$.
ii) If $K \ll_{\delta} M$ and $f : M \to N$ is an homomorphism, then $f(K) \ll_{\delta} N$,
   In particular, if $K \ll_{\delta} M \le N$ and $K \ll_{\delta} N$.
iii) Let $K_1 \le M_1 \le M$, $K_2 \le M_2 \le M$ and $M = M_1 \oplus M_2$,
   then $K_1 \oplus K_2 \ll_{\delta} M_1 \oplus M_1$ *if and only if* $K_1 \ll_{\delta} M_1$ *and* $K_2 \ll_{\delta} M_2$.

Proof: [4].

Lemma:1.3. Let M be a module. Then

i) $\delta(M) = \sum\{L \le M : L \ll_{\delta} M\} = \cap \{K \le M : \frac{M}{K}$ *is singular module*$\}$.
ii) If f: M→N is an R-homomorphism, then $f(\delta(M)) \le \delta(N)$. Therefore δ(M) is fully invariant sub module of M. In particular if K ≤ M, then δ(K) ≤ δ(M).
iii) *If* $M = \oplus_{i \in I} (M_i)$, *then* $\delta(M) \le \oplus_{i \in I} \delta(M_i)$.
iv) If every proper sub module of M is contained in a maximal sub module of m, then δ(M) is the unique largest δ-small sub module of M. In particular if M is finitely generated, then δ(M) is δ- small in M.

Proof: [4]

Lemma:1.4. If K ≤ N ≤ M, $K \ll_{\delta} M$ and N is a direct summand of M, Then $K \ll_{\delta} N$.

Proof: [10]

Proposition: Given a module M, each of the following sets is equal δ(M).

i) $\delta(M) = \sum\{A : A \ll \delta M\}$.
ii) $\delta(M) = \cap \{B : B \le M$ *with* M/B *is singular*$\}$.
iii) $\delta(M) = \cap \{ker\Phi : \Phi \in Hom(M, N)$ *such that* N *is singular simple*$\}$

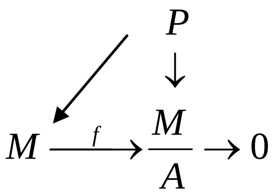
iv) $\delta(M) = \cap\{ker\Phi : \Phi \in Hom(M, N)$ such that N is singular semi singular$\}$

Proposition:1.2. If f: M→N is an epimorphism with $kerf \leq \delta(M)$, then $\delta(N) = f(\delta(M))$.

Proof: [4].

Lemma:1.5. Let P be a small projective module, then $\delta(M) \ll_\delta$ P.

Proof: Let P be a small-projective module and $P = \delta(P) + Y$, where P/Y is singular, by hypothesis P= A ⊕ B such that



A ≤ Y and B∩Y ≤ δ(P), Then Y = A ⊕ (B∩Y) and so P = δ(P) ⊕A. since A is summand of P, there exists a sub module X ≤ δ(P) such that P = = X ⊕ A. Since $\delta(X) = X \cap \delta(P) = X$,
X is semi simple projective and P/Y is epimorphic image of $P/A \cong X \Rightarrow P/Y$ is projective and singular, we have P = Y. Hence δ(P) $\ll_\delta$ P.

Proposition:1.3. Let M and N be any R-modules. Then Following are equivalent:

i) $N \ll_\delta M$.
ii) If X + N =M, then M = X⊕Y for projective semi simple sub module Y, with Y ≤ N.

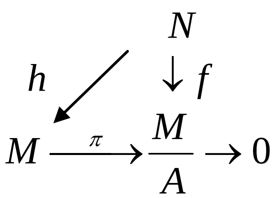
Proof: [4]

Proposition:1.4. For Hollow module N the following conditions are equivalent:

i) N is δ-small M-pseudo projective module.
ii) N is M-pseudo projective module.

Proof: (i) => (ii) is obvious.

(ii)=> (i) Let R-module N be a M- pseudo projective module. A ≤ M, any small epimorphism $f : N \to \frac{M}{A}$ and natural epimorphism $\pi : M \to \frac{M}{A}$. For a sub module A of M, $A \ll_\delta M$, then A is direct summand of M, there exists a decomposition $M = A \oplus B$ such that $A \leq N + F$ and $B \cap N \cap F \ll_\delta M$, there exists a homomorphism $h : N \to M$ such that the diagram



i.e. $g.h = f$. Hence N is M-pseudo projective module.//

Proposition:1.5. Let M be a δ-small M-Pseudo projective module, then following conditions are equivalent:

i) M is S.F. If M/A is isomorphic to direct summand of M, A ≤ M.
ii) M is direct sum of sub modules of A, B with $A \leq A \cap F$ and $B \cap N \cap F \ll_\delta M$.

Proof: (i) => (ii) by S1

(ii)=> (i) Since M = A ⊕ B, i.e. M = A + B and A ∩ B = 0. Now M = A + B, for some B ≤ M and $\frac{M}{B}$ is singular, then $S = A + (S \cap B)$. Suppose that $A \cap B \neq A$, then $\frac{\delta(M)}{A \cap B}$ is finitely cogenerated by A. But $\frac{S}{A} = \frac{A + (S + B)}{A \cap B} \leq Soc\left(\frac{\delta(M)}{A \cap B}\right)$. Hence $\frac{S}{A}$ is f.g. this is contradiction. Thus $A = A \cap B \leq B$. We have $M = A + B = B$. So $A \ll_\delta M$. //

2. δ-Cover

Definition;2.1. Let P and M be modules. δ-small f: P →M is a called a δ-cover of M in case kerf $\ll_\delta$ P.

Definition 2.2. A δ-cover f: P →M is called a projective δ-cover in case P is a projective module.

Some module may not have projective δ-cover and some module have projective δ-cover but not projective cover.

Definition:2.3. A module M is called a semi perfect module if any homomorphic image of M has a projective δ-cover.

Lemma:2.1 If f: P→M and g: M→N are δ-covers then g○f : P→N is a δ-cover.

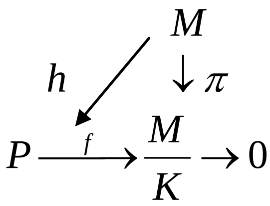
M= Kerf.g + L, with P/L is singular. Then M = kerg + f(L). Since P/f(L) is singular, M = F(L).This implies that P = L, P/L is singular and kerf $\ll_\delta$ P as desired.

Lemma:2.2 If $f_i: P_i \to M_i$ is a δ-cover for i = 1,2,3,…n, then $\oplus_{i=1}{}^{n} f_i : \oplus_{i=1}{}^{n} P_i \to M_i$ is δ- cover.

Lemma:2.3. If N is a direct summand of M and $A \ll_\delta M$, then $A \cap N \ll_\delta N$.

Lemma:2.4. Let K be a sub module of a projective module M. if M/K has a δ-cover, then, it has a δ-cover of the form $f: \frac{M}{L} \to \frac{M}{K}$, with Kerf = K/L, where L≤ K.

Proof: Let small epimorphism $f: P \to M/K$ be a δ-cover of M/K and $\pi: M \to M/K$ is a natural epimorphism. Since M is projective module, there exists $h: M \to P$ such that following diagram commute.



i.e. $f.h = \pi$. Then $P = \ker f + \operatorname{Im} h$ by lemma 1. $P = Y + \operatorname{Im} h$ for a semi simple Y with $Y \subseteq \ker f$ also by lemma 2. $\ker(fI_{\operatorname{Im} h}) \ll_\delta Imh$. So f$|_{\text{Imh}}$ is also δ- cover of M/K. But $\frac{M}{\ker h} \cong \operatorname{Im} h$ (by Isomorphic the.) and $f.h = \pi,\ \ker h \subset K$ .If we consider the isomorphism $h': \frac{M}{\ker h} \to \operatorname{Im} h$, then we obtain $\ker(\mathrm{f}\ \mathrm{I}_{\mathrm{Imh}} h') \ll_\delta$ M/Imh by lemma 2.//

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