CERTAIN SUBCLASSES OF P-VALENT MEROMORPHIC CONVEX FUNCTIONS ASSOCIATED WITH MOSTAFA OPERATOR

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Abstract: In this paper we study certain subclasses of analytic p-valent meromorphic convex functions with positive coefficients in the puncture unit disk. The result presented coefficient estimate, growth and distortion properties for functions belonging to this subclasses. Further results of modified hadamard product, inclusion properties, radii of close-to-convexity; starlikeness and convexity for functions belonging to the subclasses are discussed.

Keywords: p-Valent meromorphic functions, convex function, Modified Hadamard product, inclusion properties and radii of starlikeness.

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1. Preliminaries and Definitions:

Let $\boldsymbol{\Sigma_p}$ be in the class of p-valent meromorphic functions of the form

$$\boldsymbol{f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p}, (p \in N = 1, 2, 3, \dots)}$$

(1.1)

(1.1)

which are analytic and p-valent meromorphic in puncture unit disk $(U^* = z \in C: 0 < |z| < 1)$.

A function $f \in \Sigma_p$ is said to be in the class $\Sigma C_p^k(\eta, \beta)$ of meromorphic p-valent convex of order η and type β if it satisfies

$$\left| \frac{1 + \frac{zf''(z)}{f'(z)} + p}{1 + \frac{zf''(z)}{f'(z)} - p + 2\eta} \right| < \beta, (z \in U^*, 0 \leq \eta < p, 0 < \beta \leq 1, p \in N).$$

Let $f \in \Sigma_p$ given by (1.1) and $g \in \Sigma_p$ defined by

$$\boldsymbol{g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p}, (p \in N = 1, 2, 3, \dots).}$$

Then hadamard (convolution) product of $f(z)$ and $g(z)$, is defined as

$$(f*g)(z)=z^{-p}+\sum_{k=1}^{\infty}a_{k-p}b_{k-p}z^{k-p},(p\in N=1,2,3,\dots)$$

Aqlan et. Al. [11] defined the operator of $Q_{\vartheta,p}^{\alpha}f(z):\Sigma_p\longrightarrow\Sigma_p$

$$Q_{v,p}^{\alpha}f(z)=\begin{cases} z^{-p}+\dfrac{\Gamma(\alpha+\vartheta)}{\Gamma(\vartheta)}\displaystyle\sum_{k=1}^{\infty}\frac{\Gamma(k+\vartheta)}{\Gamma(k+\vartheta+\alpha)}a_{k-p}z^{k-p},(\alpha>0,\vartheta>-1,p\in N)\\ f(z), \qquad (\alpha=0,\vartheta>-1,p\in N)\end{cases}$$

Now, Mostafa [12, 13], P. He and D. Zhang [14] defined the operator $H_{\vartheta,p,\mu}^{\alpha}f(z):\Sigma_p\longrightarrow\Sigma_p$ as follow,

First put

$$G_{\vartheta,p}^{\alpha}f(z):z^{-p}+\frac{\Gamma(\alpha+\vartheta)}{\Gamma(\vartheta)}\sum_{k=1}^{\infty}\frac{\Gamma(k+\vartheta)}{\Gamma(k+\vartheta+\alpha)}a_{k-p}z^{k-p},(p\in N)\tag{1.2}$$

And let $G_{\vartheta,p,\mu}^{\alpha*}f(z)$ be defined by

$$G_{\vartheta,p}^{\alpha}(z)*G_{\vartheta,p,\mu}^{\alpha*}(z)=\frac{1}{z^p(1-z)^{\mu}}, \qquad (\mu>0,p\in N)$$

$$H_{\vartheta,p,\mu}^{\alpha}=G_{\vartheta,p,\mu}^{\alpha*}(z)*f(z),\ \left(f\in\Sigma_p\right),\tag{1.3}$$

Using (1.2) and (1.3), we have,

$$H_{\vartheta,p,\mu}^{\alpha}f(z)=z^{-p}+\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\sum_{n=1}^{\infty}\frac{\Gamma(k+\vartheta+\alpha)(\mu)_n}{\Gamma(k+\vartheta)(1)_n}a_{n-p}z^{n-p}\tag{1.4}$$

Where $(\mu)_n$ denote the Pochhammer symbol given by

Where $(\mu)_n=\frac{\Gamma(\alpha+\vartheta)}{\Gamma(\vartheta)}=\begin{cases}1 & (n=0)\\ \vartheta\,(\vartheta+1)\dots(\vartheta+n-1), & (n\in N)\end{cases}$

It is notice that, putting $\mu=1$ in (1.4), we obtain the operator

$$H_{\vartheta,p,1}^{\alpha}f(z)=H_{\vartheta,p}^{\alpha}f(z)=z^{-p}+\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\sum_{n=1}^{\infty}\frac{\Gamma(k+\vartheta+\alpha)}{\Gamma(k+\vartheta)}a_{n-p}z^{n-p}$$

Aouf et al. [10] introduced a Mostafa operator. Waggas Galib Atshan and Assra Abdul Jaleel Husien [15] introduced a subclass class $\Sigma_{p,\eta}^{\alpha}(\mu;a;h)$ of meromorphically p-valent analytic functions. Using the operator $H_{\vartheta,p,\mu}^{\alpha}f(z)$ defined in (1.4). We introduced classes $\Sigma C_{\vartheta,p,\mu}^{\alpha}(\eta,\beta)$ of p-valent functions as follows

Definition: The function $f\in\Sigma_p$ is said to be in the class $\Sigma C_{\vartheta,p,\mu}^{\alpha}(\eta,\beta)$ if and only if

$$\left|\frac{1+\dfrac{z\left(H_{\vartheta,p,\mu}^{\alpha}f(z)\right)''}{\left(H_{\vartheta,p,\mu}^{\alpha}f(z)\right)'}+p}{1+\dfrac{z\left(H_{\vartheta,p,\mu}^{\alpha}f(z)\right)''}{\left(H_{\vartheta,p,\mu}^{\alpha}f(z)\right)'}-p+2\eta}\right|<\beta,\tag{1.5}$$

$(z\in U^*,0\le\eta<p,0<\beta\le 1,\alpha>0,\vartheta>-1,\mu>0,p\in N)$, where $H_{\vartheta,p,\mu}^{\alpha}f(z)$ defined as (1.4).

The class $\Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$ are called meromorphic p-valent convex functions of order η and type β with positive coefficient. M. K. Aouf et al.[5, 10], G. Murugusundaramoorthy and Aouf [2], F. Ganim and M. Darus [1], S. M. Khairnar and S. M. Rajas [4] studied the subclasses of p-valent functions of order η and type β.
In this paper we obtain coefficient estimates for the class $\Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$, Hadamard product, growth and distortion theorems, radii of close to convexity, starlikeness and convexity.

2. Coefficient Estimate:

Theorem2.1: Assume that $f \in \Sigma_p$ and

$$\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\sum_{n=1}^{\infty}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]a_{n-p} < 2\beta p(p-\eta). \tag{2.1}$$

(2.1)

Then $f \in \Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$.

Proof: Let us assume that inequality (2.1) is true. Further suppose that

$$\Omega(f) = \left|z\left(H^{\alpha}_{\vartheta,p,\mu}f(z)\right)'' + (1-p)\left(H^{\alpha}_{\vartheta,p,\mu}f(z)\right)'\right| - \beta\left|z\left(H^{\alpha}_{\vartheta,p,\mu}f(z)\right)'' + (2\eta+1-p)\left(H^{\alpha}_{\vartheta,p,\mu}f(z)\right)'\right|.$$

Using (1.4) and for $0 < |z| = r < 1$, we have

$$\Omega(f) \le \frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\sum_{n=1}^{\infty}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]a_{n-p}r^{n-p} - 2\beta p(p-\eta)r^{-p}, \tag{2.2}$$

Since above inequality holds for all r, $0 < r < 1$. Letting $r \to 1$ in (2.2), we easily get that $\Omega(f) \le 0$, hence $f \in \Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$.

Theorem2.1: Let $\boldsymbol{f \in \Sigma_p}$. Then $\boldsymbol{f \in \Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)}$ if and only if

$$\boldsymbol{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\sum_{n=1}^{\infty}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]a_{n-p} < 2\beta p(p-\eta)}, \tag{2.3}$$

(2.3)

Proof: In view of Theorem 2.1, it is sufficient to prove that the 'only if' part. Let us assume that $f \in \Sigma_p$.
Then

$$\boldsymbol{\left|\frac{1+\dfrac{z\left(H^{\alpha}_{\vartheta,p,\mu}f(z)\right)''}{\left(H^{\alpha}_{\vartheta,p,\mu}f(z)\right)'}+p}{1+\dfrac{z\left(H^{\alpha}_{\vartheta,p,\mu}f(z)\right)''}{\left(H^{\alpha}_{\vartheta,p,\mu}f(z)\right)'}-p+2\eta}\right|}$$

Since $Re(z) \le |z|$ for all z, it follows that

$$Re\left\{\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\sum_{n=1}^{\infty}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n)(n-p)a_{n-p}z^{n-p},}{2p(\eta-p)z^p-\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\sum_{n=1}^{\infty}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(2\eta-2p+n)a_{n-p}z^{n-p},}\right\}<\beta$$

Choosing values of z on the real axis so that $\frac{z\left(H_{\vartheta,p,\mu}^{\alpha}f(z)\right)''}{\left(H_{\vartheta,p,\mu}^{\alpha}f(z)\right)'}$ is real and letting $z\to 1^-$ through real axis, we get desired conclusion.

Corollary2.3: If the function $f(z)$is in the class $\Sigma C_{\vartheta,p,\mu}^{\alpha}(\eta,\beta)$, then

$$a_{n-p}<\frac{2\beta p(p-\eta)}{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]},p,n\in N. \tag{2.4}$$

The result (2.8) is sharp for the function $f(z)$ of the form

$$f(z)<z^{-p}+\frac{2\beta p(p-\eta)}{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}z^{n-p}. \tag{2.5}$$

3. Growth and Distortion

Theorem3.1: A function f defined by (1.1) is in the class $\Sigma C_{\vartheta,p,\mu}^{\alpha}(\eta,\beta)$, then for $0<|z|=r<1$, we have

$$|f(z)|\geq r^{-p}-\frac{2\beta p(p-\eta)}{\frac{\mu(\vartheta+\alpha)}{\vartheta}(1-p)[(1+\beta)-2\beta(p-\eta)]}r^{1-p} \tag{3.1}$$

$$|f(z)|\leq r^{-p}+\frac{2\beta p(p-\eta)}{\frac{\mu(\vartheta+\alpha)}{\vartheta}(1-p)[(1+\beta)-2\beta(p-\eta)]}r^{1-p} \tag{3.2}$$

$$\left|(f(z))'\right|\geq pr^{-p-1}-\frac{2\beta p(p-\eta)}{\frac{\mu(\vartheta+\alpha)}{\vartheta}[(1+\beta)-2\beta(p-\eta)]}r^{-p} \tag{3.3}$$

$$\left|(f(z))'\right|\leq pr^{-p-1}+\frac{2\beta p(p-\eta)}{\frac{\mu(\vartheta+\alpha)}{\vartheta}[(1+\beta)-2\beta(p-\eta)]}r^{-p} \tag{3.4}$$

With inequality for

$$|f(z)|\leq z^{-p}+\frac{2\beta p(p-\eta)}{\frac{\mu(\vartheta+\alpha)}{\vartheta}(1-p)[(1+\beta)-2\beta(p-\eta)]}z^{1-p} \tag{3.5}$$

Proof: Since $f\in\Sigma C_{\vartheta,p,\mu}^{\alpha}(\eta,\beta)$

$$|f(z)| = \left| z^{-p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \right|$$

Therefore

$$|\boldsymbol{z^{-p}}| - \left| \sum_{n=1}^{\infty} \boldsymbol{a_{n-p} z^{n-p}} \right| \leq |\boldsymbol{f(z)}| \leq |\boldsymbol{z^{-p}}| + \left| \sum_{n=1}^{\infty} \boldsymbol{a_{n-p} z^{n-p}} \right|$$

Thus for $0 < |z| = r < 1$, we have

$$\boldsymbol{r^{-p} - r^{n-p}} \left| \sum_{n=1}^{\infty} \boldsymbol{a_{n-p}} \right| \leq |\boldsymbol{f(z)}| \leq \boldsymbol{r^{-p} + r^{n-p}} \left| \sum_{n=1}^{\infty} \boldsymbol{a_{n-p}} \right| \tag{3.6}$$

Using (2.1) and (3.6) we easily arrive at the desired result (3.1) and (3.2). Furthermore, we observe that

$$\left| (f(z))' \right| = \left| -pz^{-p} + \sum_{n=1}^{\infty} a_{n-p}(n-p) z^{n-p} \right|$$

Therefore

$$p|z^{-p-1}| - \left| \sum_{n=1}^{\infty} a_{n-p}(n-p) z^{n-p-1} \right| \leq \left| (f(z))' \right| \leq p|z^{-p-1}| + \left| \sum_{n=1}^{\infty} a_{n-p}(n-p) z^{n-p-1} \right|$$

Thus for $0 < |z| = r < 1$, we have

$$pr^{-p-1} - (1-p) r^{-p} \sum_{n=1}^{\infty} a_{n-p} \leq \left| (f(z))' \right| \leq pr^{-p-1} + (1-p) r^{-p} \sum_{n=1}^{\infty} a_{n-p} \tag{3.7}$$

Using (2.1) and (3.7) we easily arrive at the desired result (3.3) and (3.4). Finally, we can see that the estimate for $H^{\alpha}_{\vartheta,p,\mu} f(z)$ and $\left(H^{\alpha}_{\vartheta,p,\mu} f(z) \right)'$ are sharp for the functions (3.5).

Corollary3.2: Under the hypothesis of Theorem 3.1, $\boldsymbol{H^{\alpha}_{\vartheta,p,\mu} f(z)}$ is included in the disk with center at origin and radius $\boldsymbol{R_1}$ given by

$$R_1 = 1 + \frac{2\beta p(p-\eta)}{\frac{\mu(\vartheta+\alpha)}{\vartheta}(1-p)[(1+\beta) - 2\beta(p-\eta)]}$$

and $\left(\boldsymbol{H^{\alpha}_{\vartheta,p,\mu} f(z)} \right)'$ is included in the disk with center at origin and radius $\boldsymbol{R_2}$ given by

$$R_2 = p + \frac{2\beta p(p-\eta)}{\frac{\mu(\vartheta+\alpha)}{\vartheta}[(1+\beta) - 2\beta(p-\eta)]}$$

4. Modified Hadamard Products:

Let the functions $f_i(z), (i = 1,2)$ be defined by

$$f_i(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,i} z^{n-p}, (p \in N = 1,2,3 \dots) \tag{4.1}$$

The modified hadamard product of $f_i(z)$ is defined by

$$(f_1 * f_2)(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,1} a_{n-p,2} z^{n-p}, (p \in N = 1,2,3 \ldots)$$

Theorem 4.1: Let the functions $\boldsymbol{f_i(z)}, (\boldsymbol{i = 1, 2})$ be defined by (4.1) be in the class $\boldsymbol{\Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta, \beta)}$. Then $\boldsymbol{(f_1 * f_2)(z) \in \Sigma C^{\alpha}_{\vartheta,p,\mu}(\delta, \beta)}$, where

$$\delta \leq p - \frac{2\beta(1+\beta)p(p-\eta)^2}{\frac{\mu(\vartheta+\alpha)}{\vartheta}(1-p)[(1+\beta) - 2\beta(p-\eta)]^2 + 4\beta^2 p(p-\eta)^2}.$$

The result is sharp for the functions

$$f_i(z) = z^{-p} + \frac{2\beta p(p-\eta)}{\frac{\mu(\vartheta+\alpha)}{\vartheta}(1-p)[(1+\beta) - 2\beta(p-\eta)]} z^{1-p}, i = 1,2; p \in N \tag{4.2}$$

Proof: To prove the theorem, we need to find largest δ such that

$$\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n} (n-p)[n(1+\beta) - 2\beta(p-\delta)] a_{n-p,1} a_{n-p,2}}{2\beta p(p-\delta)} \leq 1$$

Since $f_i(z) \in \Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta, \beta)$, then

$$\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n} (n-p)[n(1+\beta) - 2\beta(p-\eta)] a_{n-p,i}}{2\beta p(p-\eta)} \leq 1, i = 1,2.$$

By Cauchy-Schwarz inequality, we get

$$\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n} (n-p)[n(1+\beta) - 2\beta(p-\eta)] \sqrt{a_{n-p,1} a_{n-p,2}}}{2\beta p(p-\eta)} \leq 1 \tag{4.3}$$

We want to show that

$$\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n} (n-p)[n(1+\beta) - 2\beta(p-\delta)] a_{n-p,1} a_{n-p,2}}{2\beta p(p-\delta)} \leq$$

$$\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n} (n-p)[n(1+\beta) - 2\beta(p-\eta)] \sqrt{a_{n-p,1} a_{n-p,2}}}{2\beta p(p-\eta)},$$

This is equivalent to

$$\sqrt{a_{n-p,1} a_{n-p,2}} \leq \frac{[n(1+\beta) - 2\beta(p-\eta)](p-\delta)}{[n(1+\beta) - 2\beta(p-\delta)](p-\eta)},$$

Using (4.3)

$$\sqrt{a_{n-p,1} a_{n-p,2}} \leq \frac{2\beta p(p-\eta)}{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)} \sum_{n=1}^{\infty} \frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n} (n-p)[n(1+\beta) - 2\beta(p-\eta)]}$$

$$\leq \frac{[n(1+\beta) - 2\beta(p-\eta)](p-\delta)}{[n(1+\beta) - 2\beta(p-\delta)](p-\eta)}$$

Or, equivalent that

$$\delta \leq p - \frac{2n\beta(1+\beta)p(p-\eta)^2}{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[(1+\beta)-2\beta(p-\eta)]^2+4\beta^2p(p-\eta)^2}$$

Letting n=1

$$\delta \leq p - \frac{2\beta(1+\beta)p(p-\eta)^2}{\frac{\mu(\alpha+\vartheta)}{\vartheta}(1-p)[(1+\beta)-2\beta(p-\eta)]^2+4\beta^2p(p-\eta)^2}$$

which completes the proof.

Corollary4.2: For $f_i(z), (i = 1, 2)$ as in the Theorem 4, we have

$$h(z) = z^{-p} + \sum_{n=1}^{\infty} \sqrt{a_{n-p,1}a_{n-p,2}}\, z^{n-p}$$

belongs to the class $\Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$. The result is sharp with the function given by (4.2).

Proof. The result follows from the inequality (4.3). Similarly we can prove the following results.

Theorem 4.3. Let the function $f_i(z)$, defined by (4.1) be in the class $\boldsymbol{\Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)}$. Then $(f_1 * f_2)(z) \in \boldsymbol{\Sigma C^{\alpha}_{\vartheta,p,\mu}(\tau,\beta)}$, where

$$\tau \leq p - \frac{2\beta(1+\beta)p(p-\eta_1)(p-\eta_2)}{\frac{\mu(\alpha+\vartheta)}{\vartheta}(1-p)[(1+\beta)-2\beta(p-\eta_1)][(1+\beta)-2\beta(p-\eta_2)]+4\beta^2(p-\eta_1)(p-\eta_2)}$$

The result is sharp with the functions

$$f_i(z) = z^{-p} + \frac{2\beta p(p-\eta_i)}{\frac{\mu(\vartheta+\alpha)}{\vartheta}(1-p)[(1+\beta)-2\beta(p-\eta_i)]} z^{1-p}, i = 1,2; p \in N.$$

5. Inclusion Properties:

Theorem 5.1: Let the function $f_i(z), (i = 1, 2)$ defined by (4.1) be in the class $\boldsymbol{\Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)}$. Then the function

$$h(z) = z^{-p} + \sum_{n=1}^{\infty}\left({a^2}_{n-p,1}+{a^2}_{n-p,2}\right)$$

belongs to the class $\Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$, where

$$\delta \leq p - \frac{4\beta(1+\beta)p(p-\eta)^2}{\frac{\mu(\alpha+\vartheta)}{\vartheta}(1-p)[(1+\beta)-2\beta(p-\eta)]^2+8\beta^2p(p-\eta)^2}$$

The result is sharp with the function given by (4.4).

Proof: By virtue of the Theorem 2.1, we obtain

$$\sum_{n=1}^{\infty}\left\{\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)}\right\}^2 {a^2}_{n-p,1}$$

$$\leq \left\{\sum_{n=1}^{\infty}\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)} a_{n-p,1}\right\}^2 \leq 1 \tag{5.1}$$

and

$$\sum_{n=1}^{\infty}\left\{\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)}\right\}^2 a^2{}_{n-p,2}$$

$$\leq\left\{\sum_{n=1}^{\infty}\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)}a_{n-p,2}\right\}^2\leq 1. \tag{5.2}$$

It follows from (5.1) and (5.2) that

$$\frac{1}{2}\sum_{n=1}^{\infty}\left\{\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)}\right\}^2\left(a^2{}_{n-p,1}+a^2{}_{n-p,2}\right)$$

$$\leq 1.$$

Therefore we need to find largest δ such that

$$\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\delta)]}{2\beta p(p-\delta)}$$

$$\leq\frac{1}{2}\left\{\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)}\right\}^2$$

$$\delta\leq p-\frac{4n\beta(1+\beta)p(p-\eta)^2}{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[(1+\beta)-2\beta(p-\eta)]^2+8\beta^2p(p-\eta)^2}.$$

Letting n=1, we get

$$\delta\leq p-\frac{4\beta(1+\beta)p(p-\eta)^2}{\frac{\mu(\alpha+\vartheta)}{\vartheta}(1-p)[(1+\beta)-2\beta(p-\eta)]^2+8\beta^2p(p-\eta)^2}.$$

This completes the proof.

6. Radii of Close to convex, Starlikeness and Convexity:

Theorem6.1. Let the function $f(z)$ defined by (1.1) be in the class $\Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$. Then $f(z)$ is p-valently close to convex of order δ $(0\leq\psi<p)$ in $|z|<r_1$ where

$$r_1=\inf_{n\in N}\left\{\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(p-\psi)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)}\right\}^{1/n} \tag{6.1}$$

The result is sharp with the extremal function $f(z)$ given by (2.5).

Proof: For $0\leq\psi<p$, it suffices to show that

$$\left|\frac{f'(z)}{z^{-p-1}}+p\right|\leq(p-\psi),|z|<r_1.$$

Indeed, we have

$$\left|\frac{f'(z)}{z^{-p-1}}+p\right| \le \sum_{n=1}^{\infty}(n-p)a_{n-p}|z|^{n}.$$

The above expression is less than $(p-\psi)$ if

$$\sum_{n=1}^{\infty}\frac{(n-p)}{(p-\psi)}a_{n-p}|z|^{n} \le 1 \tag{6.2}$$

Using the fact, that $f(z) \in \Sigma C_{\vartheta,p,\mu}^{\alpha}(\eta,\beta)$ and Theorem 2.1, (6.2) is true if

$$\frac{(n-p)}{(p-\psi)}z^{n} \le \frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)}, n \ge 1. \tag{6.3}$$

Solving (6.3), we get the desired result (6.1)

Theorem6.2. Let the function $f(z)$ defined by (1.1) be in the class $\Sigma C_{\vartheta,p,\mu}^{\alpha}(\eta,\beta)$. Then $f(z)$ is p-valently starlike of order δ $(0 \le \psi < p)$ in $|z| < r_2$ where

$$r_2 = \inf_{n \in N}\left\{\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(p-\psi)(n-p)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)(n-p-\psi)}\right\}^{1/n}$$

The result is sharp with the extremal function $f(z)$ given by (2.5).

Proof: The proof is analogous to that of Theorem.1, and we omit the details.

Theorem6.3 Let the function $f(z)$ defined by (1.1) be in the class $\Sigma C_{\vartheta,p,\mu}^{\alpha}(\eta,\beta)$. Then $f(z)$ is p-valently convex of order δ $(0 \le \psi < p)$ in $|z| < r_2$ where

$$r_3 = \inf_{n \in N}\left\{\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(p-\psi)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)(n-p-\psi)}\right\}^{1/n}$$

The result is sharp with the extremal function $f(z)$ given by (2.5).

Proof: The proof is analogous to that of Theorem.1, and we omit the details.

7. Extreme Points:

Theorem 7.1 Let $f_{-p}(z) = z^{-p}$

And

$$f_{n-p}(z) = z^{-p} + \sum_{n=1}^{\infty}\frac{2\beta p(p-\eta)}{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}z^{n-p}, n = 1.2.3..$$

Then $f(z) \in \Sigma C_{\vartheta,p,\mu}^{\alpha}(\eta,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty}\lambda_{n-p}f_{n-p}(z) \tag{7.1}$$

Where

$$\lambda_{n-p} \geq 0, \qquad \sum_{n=0}^{\infty} \lambda_{n-p} = 1$$

(7.2)

Proof: Let

$$f(z) = \sum_{n=1}^{\infty} \lambda_{n-p} f_{n-p}(z)$$

$$= z^{-p} + \sum_{n=1}^{\infty} \frac{2\beta p(p-\eta)}{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)} \frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]} \lambda_{n-p} z^{n-p},$$

Then, in view of (7.2), it follows that

$$f(z) = \sum_{n=1}^{\infty} \frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)} \frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)} \frac{2\beta p(p-\eta)}{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)} \frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]},$$

$$= \textstyle\sum_{n=1}^{\infty} \lambda_{n-p} = 1 - \lambda_{-p} \leq 1.$$

So, by Theorem 2.1, the function $f(z)$ belongs to the class $\Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$. Conversely, let the function $f(z)$ defined by (1.1) belongs to the class $\Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$. Then

$$a_{n-p} \leq \frac{2\beta p(p-\eta)}{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)} \frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}, n,p \in N.$$

Setting

$$\lambda_{n-p} = \frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)} \frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)} a_{n-p}, n,p \in N.$$

And $\lambda_{-p} = 1 - \lambda_{n-p}$,

It follows that

$$f(z) = \sum_{n=1}^{\infty} \lambda_{n-p} f_{n-p}(z)$$

We see that $f(z)$ can be expressed in the form (7.1). This completes the proof.

8. Convex Linear Combination

Theorem 8.1: The class $\Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$ is close under convex linear combinations.

Proof: Suppose that the function $f_1(z)$ and $f_2(z)$ defined by

$$f_j(z) = z^{-p} + \sum_{n=1}^{\infty} |a_{n-p,j}| z^{n-p}, (j = 1,2)$$

Are in the class $\Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$. Setting

$$f(z) = \gamma f_1(z) + (1-\gamma) f_2(z), \qquad 0 \leq \gamma \leq 1.$$

(8.1)

We find from (8.1) that

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} \{\gamma |a_{n-p,1}| + (1-\gamma)|a_{n-p,2}|\} z^{n-p}, (0 \leq \gamma \leq 1).$$

In view of Theorem 2.1, we have,

$$\sum_{n=1}^{\infty}\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]\{\gamma|a_{n-p,1}| + (1-\gamma)|a_{n-p,2}|\},$$

$$=\gamma\sum_{n=1}^{\infty}\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]|a_{n-p,1}|$$

$$+(1-\gamma)\sum_{n=1}^{\infty}\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]|a_{n-p,2}|$$

$$\leq \gamma[2\beta p(p-\eta)] + (1-\gamma)[2\beta p(p-\eta)] = 2\beta p(p-\eta)$$

This shows that $f\ \Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$.

9. Closure Theorem:

Let the functions $f_k(z), k = 1,2,3, \dots s$, defined by

$$f_k(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p,k} z^{n-p}, \left(a_{n-p,k} \geq 0\right). \tag{9.1}$$

We shall prove the following closure theorem.

Theorem9.1. Let the function $f_k(z), k = 1,2,3, \dots, s$, defined by (9.1) be in the class $\boldsymbol{\Sigma} C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$. Then the function $\mathrm{F} \in \boldsymbol{\Sigma} C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$ where

$$F(z) = \sum_{n=1}^{\infty} b_k f_k(z),\ b_k \geq 0\ and\ \sum_{k=1}^{s} b_k = 1\ . \tag{9.2}$$

Proof: From (9.2), we can write

$$F(z) = \sum_{n=1}^{\infty} b_j f_j(z) = \sum_{j=1}^{s} b_j \left(z^{-p} + \sum_{n=1}^{\infty} a_{n-p,j} z^{n-p}\right) = z^{-p} + \sum_{n=1}^{\infty}\sum_{j=1}^{s} b_j\, a_{n-p,j} z^{n-p}$$

Now, $F(z) \in \Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$, Since

$$F(z) = \sum_{n=1}^{\infty}\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)}\sum_{j=1}^{s} b_j\, a_{n-p,j}$$

$$=\sum_{j=1}^{s} b_j \sum_{n=1}^{\infty}\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)} a_{n-p,j}$$

$$\leq \sum_{j=1}^{s} b_j = 1.$$

This completes the proof.

Theorem9.2. Let the function $f_k(z), k = 1,2,3, \dots, s$, defined by (9.1) be in the class $\Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$. Then the function

$$h(z) = \frac{1}{m}\sum_{k=1}^{m} f_k(z)$$

Belong to the class $\Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$.

Proof: We have

$$h(z) = \frac{1}{m}\sum_{k=1}^{m} f_k(z) = z^{-p} + \sum_{n=1}^{\infty}\frac{1}{m}\sum_{k=1}^{m} a_{n-p,k} z^{n-p} = z^{-p} + \sum_{n=1}^{\infty} e_k z^{n-p},$$

Where $e_k = \frac{1}{m}\sum_{k=1}^{m} a_{n-p,k}$. Since $f_k(z) \in \Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$, from Theorem (2.1), we have

$$\sum_{n=1}^{\infty}\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)} a_{n-p,j} \leq 1.$$

Now, $\mathrm{h}(z) \in \Sigma C^{\alpha}_{\vartheta,p,\mu}(\eta,\beta)$, since

$$\sum_{n=1}^{\infty}\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)} e_k$$

$$= \sum_{n=1}^{\infty}\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)} \frac{1}{m}\sum_{k=1}^{m} a_{n-p,k}$$

$$= \frac{1}{m}\sum_{k=1}^{m}\sum_{n=1}^{\infty}\frac{\frac{\Gamma(\vartheta)}{\Gamma(\alpha+\vartheta)}\frac{\Gamma(n+\vartheta+\alpha)(\mu)_n}{\Gamma(n+\vartheta)(1)_n}(n-p)[n(1+\beta)-2\beta(p-\eta)]}{2\beta p(p-\eta)} a_{n-p,k}$$

$$= \frac{1}{m}\sum_{k=1}^{m} 1 = 1,$$

This completes the proof.

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