EXPANSION FOR PRODUCT OF BESSEL'S FUNCTION AND I-FUNCTION OF MULTIVARIABLES

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ABSTRACT
In the research paper we present expansion for product of I-function of multivariables involving Bessel's function.
KEYWORDS : Bessel's function and multivariable of I-function.
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1. INTRODUCTION : The objective of this research paper is to introduce a expansion for product of Bessels's function and I-function of multivariables given by Y.N. Prasad [6] and Ronghe [8] represent of I-function of multivariables given as follows:

$$I[z_1,....z_r] = I^{o,n_2:o,n_3:.....:o,n_r(m',n');(m'',n'');.....,(m^{(r)},n^{(r)})}_{p_2,q_2:p_3,q_3:.....:p_r,q_r(p',q');(p'',q'');.....,(p^{(r)},q^{(r)})}$$

$$\left[z_1, z_2,...z_r \left| \begin{matrix} (a_{2_j};\alpha'_{2_j};\alpha''_{2_j})_{1,p_2} \; (a_{3_j};\alpha'_{3_j},\alpha''_{3_j},\alpha'''_{3_j})_{1,p_3} \; :....: \\ (b_{2_j};\beta'_{2_j};\beta''_{2_j})_{1,q_2} \; (b_{3_j};\beta'_{3_j},\beta''_{3_j},\beta'''_{3_j})_{1,q_3} \; :....: \end{matrix} \right. \right.$$

$$\left. \begin{matrix} (a_{r_j};\alpha'_{r_j},........,\alpha^{(r)}_{r_j})_{1,p_r} \; (a'_j;\alpha'_j)_{1,p'} \; :....: (a^{(r)}_j;\alpha^{(r)}_j)_{1,p^{(r)}} \\ (b_{r_j};\beta'_{r_j},........,\beta^{(r)}_{r_j})_{1,q_r} \; (b'_j;\beta'_j)_{1,q'} \; :....: (b^{(r)}_j;\beta^{(r)}_j)_{1,q^{(r)}} \end{matrix} \right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \phi_i(s_i),....,\phi_r(s_r)\,\psi(s_1....s_r)\, z_1^{s_1}.....z_r^{s_r} ds_1....ds_r \qquad (1.1)$$

where

$\omega = \sqrt{(-1)}$ and

$$\phi_i s_i = \frac{\prod (j=1,\to m^{(i)}),\Gamma(b^{(i)}_j - \beta^{(i)}_j s_i) \prod (j=1,\to n^{(i)}),\Gamma(1-a^{(i)}_j - \alpha^{(i)}_j s_i)}{\prod (j=m^{(i)}+1,\to q^{(i)}),\Gamma(1-b^{(i)}_j - \beta^{(i)}_j s_i) \prod (j=n^{(i)}+1,\to p^{(i)}),\Gamma(a^{(i)}_j - \alpha^{(i)}_j s_i)}$$

$\forall i\{1, 2,....., r\}$.

(1.2)

$$\psi(s_1, s_2,....,s_r) \frac{\prod (j=1,\to n_2),\Gamma\left(1-a_{2_j} + \Sigma(i=1-y)\left(\alpha^{(i)}_{2_j} s_i\right)\right)}{\prod (j=n_2+1,\to p_2),\Gamma\left(a_{2_j} - \Sigma(i=1\to 2)\left(\alpha^{(i)}_{2_j} s_i\right)\right)} \times$$

$$\frac{\prod(j=1,\to n_3),\Gamma\left(1-a_{3_j}+\Sigma(i=1\to 3)\left(\alpha_{3_j}^{(i)}s_i\right)\right)\ldots\ldots\ldots}{\prod(j=n_3+1,\to p_3),\Gamma\left(a_{3_j}-\Sigma(i=1\to 3)\left(\alpha_{3_j}^{(i)}s_i\right)\right)\ldots\ldots\ldots}$$

$$\frac{\ldots\ldots\ldots\prod(j=1,\to n_r),\Gamma\left(1-a_{r_j}+\Sigma(i=1\to r)\left(\alpha_{r_j}^{(i)}s_i\right)\right)}{\ldots\ldots\ldots\prod\left(j=n^{(r)}+1,\to p_r\right),\Gamma\left(a_{r_j}-\Sigma(i=1\to r)\left(\alpha_{r_j}^{(i)}s_i\right)\right)}\times$$

$$\frac{1}{\prod(j=1,\to q_2),\Gamma\left(1-b_{2_j}+\Sigma(i=1\to 2)\left(\beta_{r_j}^{(i)}s_i\right)\right)\ldots\ldots}\times$$

$$\frac{1}{\ldots\ldots\prod(j=1,\to q_r),\Gamma\left(1-b_{r_j}+\Sigma(i=1\to r)\left(\beta_{r_j}^{(i)}s_i\right)\right)}, \qquad (1.3)$$

Where $z_1,\ldots,z_r$ are not equal to zero &

$$A=\sum_{j=1}^{n_{(i)}}E_j-\sum_{j=n_{(i)}+1}^{p_{(i)}}E_j+\sum_{j=1}^{m_{(i)}}F_j-\sum_{j=m_1+1}^{q_{(i)}}F_j+\sum_{j=1}^{m_{(i)}}\beta_j-\sum_{j=m_{(i)}+1}^{q_{(i)}^{(r)}}\beta_{j_i}+\sum_{j=1}^{n_{(i)}}\alpha_j-\sum_{j=n_{(i)}+1}^{p_{(i)}^{(r)}}\alpha_{j_i}>0 \qquad (1.4)$$

$$B=\sum_{j=1}^{n_1}E_j^{'}-\sum_{j=n_1+1}^{p}E_j^{'}+\sum_{j=1}^{m_{(i)}}F_j^{'}-\sum_{j=m_{(i)}+1}F_j^{'}+\sum_{j=1}^{m_{(i)}}\delta_j-\sum_{j=m_{(i)}+1}^{q_{(i)}^{(r)}}\delta_{j_i}+\sum_{j=1}^{n_{(i)}}\gamma_j-\sum_{j=n_3+1}^{p_{(i)}^{(r)}}\gamma_{j_i}>0 \qquad (1.5)$$

$$|\arg z_1|<A_i\frac{\pi}{2},\ldots\ldots\ldots,|\arg z_r|<B_i\frac{\pi}{2},\ \forall i\in\{1,2,\ldots\ldots,r\}$$

2. FORMULAE REQUIRED :

The following known results well be used throughout the paper, huke [4] gives the formulae

$$\int_0^\infty x^{\rho-1}J_u(x)\,dx=\Gamma\left(\frac{u+\rho}{2}\right).\left[\Gamma 1-\left(\frac{\rho-u}{2}\right)\right]^{-1} \qquad (2.1)$$

Where

$$\mathrm{Re}(p+u)>0,\ \mathrm{Re}(p)<0$$

$$\int_0^\infty y^{\sigma-1}\cos y\,J_v(y)\,dy=\frac{2^{\sigma-1}\sqrt{\pi}\,\Gamma\left(\frac{1}{2}-\sigma\right)\left[\left(\frac{v+\sigma}{2}\right)\right]}{\Gamma\left\{1-\left(\frac{v-\sigma}{2}\right)\right\}\Gamma\left(\frac{1-v-\sigma}{2}\right)\Gamma\left(\frac{1+v-\sigma}{2}\right)} \qquad (2.2)$$

Provided

$$\mathrm{Re}(\sigma+v)>0,$$

And

Orthogonal property for Bessels's function [3],

$$\int_0^\infty x^{-1}J_{u+2m+1}\,J_{\mu+2n+1}(x)\,dx=\begin{cases}o, & \text{if } m\neq n,\\ \dfrac{(\mu+2n+1)^{-1}}{2}, & \text{if } m=n,\end{cases} \qquad (2.3)$$

Provided; $\mathrm{Re}(\mu+m+n)>-1,$

3. MAIN INTEGRAL :

We shall evaluate the integrals that will used in the expansions

First integrals :

$$\int_0^\infty x^{\rho-1} J_u(x) \times I^{o,n_2:o,n_3:\ldots:o,n_r(m',n');(m'',n'');\ldots,(m^{(r)},n^{(r)}):P}_{p_2,q_2:p_3,q_3:\ldots:p_r,q_r(p',q');(p'',q'');\ldots,(p^{(r)},q^{(r)}):Q}\left[\begin{matrix} z_1 x^{rh_1} & S : T \\ \vdots & \\ z_r x^{rh_r} & S^{(i)} : T^{(i)} \end{matrix}\right] dx$$

$$= 2^{\rho-1} I^{o,n_2:o,n_3:\ldots:o,n_r(m',n');(m'',n'');\ldots,(m^{(r)},n^{(r+1)}):P}_{p_2,q_2:p_3,q_3:\ldots:p_r,q_r(p',q');(p'',q'');\ldots,(p^{(r+2)},q^{(r)}):Q}\left[\begin{matrix} z_1 2^{rh_1} \\ \vdots \\ z_r 2^{rh_r} \end{matrix}\middle| \left(1-\frac{\rho+u}{2}:h_1,\ldots,h_r\right), \begin{matrix} S \\ \vdots \\ S^{(i)} \end{matrix}, \left(1-\frac{\rho-u}{2}:h_1,\ldots,h_r\right) \begin{matrix} T \\ \vdots \\ T^{(i)} \end{matrix}\right] \quad (3.1)$$

Provided $h_1,\ldots,h_r > 0$,

$$\mathrm{Re}\left[p++u+2h_1 \min_{1\le j\le n_{(i)}} R_{(i)}\left(\frac{b_j}{\beta_j}\right)+\ldots+2h_r \max_{1\le j\le n_{(i)}} R_{(i)}\left(\frac{d_j}{\delta_j}\right)\right] > 0$$

(3.1.1)

$$\mathrm{Re}\left[p+2h_1 \max_{1\le j\le n_{(i)}} R_{(i)}\left(\frac{a_j-1}{\alpha_j}\right)+\ldots+2h_r \max_{1\le j\le n_{(i)}} R_{(i)}\left(\frac{c_j-1}{\gamma_j}\right)\right] < 0$$

(3.1.2)

Sets of parameter are as follows,

Where

$$\left.\begin{aligned} S &= \left(e_p : E_p, E_p'\right) \\ S^{(i)} &= \left(f_q : F_q, F_q'\right) \end{aligned}\right\}$$

(3.1.3)

$$\left.\begin{aligned} P &= m_{(i)}, n_{(i)}; m_{(r)}, n_{(r)} \\ Q &= p_{(i)}^{(i)}, q_{(i)}^{(i)}; q_{(i)}^{(r)}, q_{(i)}^{(r)}; r \end{aligned}\right\}$$

(3.1.4)

$$T = \left[\left(a_j, \alpha_j\right)_1, n_{(i)}\right], \left[\left(b_{j_i}, \beta_{j_i}\right)_{m_{(i)}+1}, q_{(i)}^{(r)}\right]; \left[\left(d_{j_i}, \delta_{j_i}\right)_{m_{(i)}+1}, q_{(i)}^{(r)}\right]$$

(3.1.5)

$$T^{(i)} = \left[\left(b_j, \beta_j\right)_1, m_{(i)}\right], \left[\left(b_{j_i}, \beta_{j_i}\right)_{m_{(i)}+1}, q_{(i)}^{(r)}\right];; \left[\left(d_j, \delta_j\right)_{1+m_{(i)}}\right]\left[\left(d_{j_i}, \delta_{j_i}\right)_{m_{(i)}+1}, q_{(i)}^{(r)}\right]$$

(3.1.6)

Second integral :

$$\int_0^\infty x^{\sigma-1} \cos x \, J_u(x) I^{o,n_2:o,n_3:\ldots:o,n_r(m',n');(m'',n'');\ldots,(m^{(r)},n^{(r)}):P}_{p_2,q_2:p_3,q_3:\ldots:p_r,q_r(p',q');(p'',q'');\ldots,(p^{(r)},q^{(r)}):Q}$$

$$\left[\begin{matrix} z_1 x^{rk_1} & S : T \\ \vdots & \\ z_r x^{rk_r} & S^{(i)} : T^{(i)} \end{matrix}\right] dx =$$

$$= 2^{\sigma-1}\sqrt{\pi}\, I^{o,n_2:o,n_3:\ldots:o,n_r(m',n');(m'',n'');\ldots;\left(m^{(r)},n^{(r+1)}\right):P}_{p_2,q_2:p_3,q_3:\ldots:p_r+3,q_r+2(p',q');(p'',q'');\ldots;\left(p^{(r+2)},q^{(r)}\right):Q}$$

$$\left[\begin{matrix} z_1 2^{rk_1} \\ \vdots \\ z_r 2^{rk_r} \end{matrix}\left|\begin{matrix} U : T \\ \\ U^{(i)} T^{(i)} \end{matrix}\right.\right] \tag{3.2}$$

Provided, $k_1, \ldots, k_r > 0$

$$\operatorname{Re}\left[\sigma + u + 2k_1 \min_{1\le j\le m_{(i)}} R_{(i)}\left(b_j/\beta_j\right) + \ldots + 2k_r \min_{1\le j\le m_{(i)}} R_{(i)}\left(d_j/\beta_j\right)\right] > 0$$

(3.2.1)

Where

$$U = \left(1 - \frac{U+\sigma}{2} : k_1, \ldots, k_r\right), \left(e_p : E_p, E_p'\right)\left(\frac{1-u-\sigma}{2} : k_1, \ldots, k_r\right),$$
$$\left(\frac{1+u-\sigma}{2} : k_1, \ldots, k_r\right)$$

(3.2.2)

$$U^{(i)} = \left(\frac{1}{2} - \sigma : 2k_1, \ldots, 2k_r\right), \left(f_q : F_q, F_q'\right)\left(\frac{u-\sigma}{2} : k_1, \ldots, k_r\right)$$

And $P, Q, S, S^{(i)}, T$ and $T^{(i)}$ given in the equation (3.1.3), (3.14), (3.15), (3.16) respectively.

Proof : To establish (3.1), express I-function of multivariable as in (1.1), change the order of integration evaluate the inner integral with the help of integration evaluate the inner integral with the help of (2.1) and again applying (1.1), we get the required integral (3.1). Apply the similar technique to prove the result (3.2)

4. Multi-Dimensional series expansion :

The Multi-Dimensional series expansion to be established is

$$x^{\rho} y^{\sigma} \cos y\, I^{o,n_2:o,n_3:\ldots:o,n_r(m',n');(m'',n'');\ldots;\left(m^{(r)},n^{(r)}\right):P}_{p_2,q_2:p_3,q_3:\ldots:p_r,q_r(p',q');(p'',q'');\ldots;\left(p^{(r)},q^{(r)}\right):Q}$$

$$\left[\begin{matrix} z_1 x^{rh_1} y^{rk_1} \\ \vdots \\ z_r x^{rh_r} y^{rk_r} \end{matrix}\left|\begin{matrix} S : T \\ \\ S : T^{(i)} \end{matrix}\right.\right] 2^{\rho+\sigma}\sqrt{\pi} \sum_{S,t=0}^{\infty} (v + 2s + 1)(u + 2st + 1) J_{v+2s+1(x)} J_{u+2t+1(y)} \times$$

$$I^{o,n_2:o,n_3:\ldots:o,n_r(m',n');(m'',n'');\ldots;\left(m^{(r+1)},n^{(r+2)}\right):P}_{p_2,q_2:p_3,q_3:\ldots:p_r,q_r(p',q');(p'',q'');\ldots;\left(p^{(r+1)},q^{(r+2)}\right):Q} \times$$

$$I^{o,n_2:o,n_3:\ldots:o,n_r(m',n');(m'',n'');\ldots;\left(m^{(r+1)},n^{(r+2)}\right):P}_{p_2,q_2:p_3,q_3:\ldots:p_r,q_r(p',q');(p'',q'');\ldots;\left(p^{(r+1)},q^{(r+2)}\right):Q}\left[\begin{matrix} z_1 2^{r(h_1+k_1)} \\ \vdots \\ z_r 2^{r(h_r+k_r)} \end{matrix}\left|\begin{matrix} x : T \\ \\ x^{(i)} : T^{(i)} \end{matrix}\right.\right] \tag{4.1}$$

Valid under the condition of (3.1) and (3.2)

Where

$$x = \left(\frac{1-\rho-\nu-2s}{2}: h_1,\ldots\ldots,h_r\right),\left(\frac{1-\sigma-\mu-2t}{2}: k_1,\ldots\ldots,k_r\right),\left(e_p : E_p, E_p^{'}\right)$$

$$\left(\frac{3+\nu-\rho+2s}{2}: h_1,\ldots\ldots,h_r\right),\left(\frac{2+\mu+2t-\sigma}{2}: k_1,\ldots\ldots,k_r\right),$$

(4.1.1)

$$\left(\frac{-\mu+2+\sigma}{2}: k_1,\ldots\ldots,k_r\right)$$

$$x^{(i)} = \left(\frac{1}{2}-\sigma; 2k_1,\ldots\ldots,2k_r\right),\left(f_q, F_q, F_q^{'}\right),\left(\frac{1+\mu+2t}{2}: k_1,\ldots\ldots,k_r\right)$$

(4.1.2)

and $P, Q, S, S^{(i)}, T$ and $T^{(i)}$ are given in the equation (3.1.3), (3.1.4), (3.1.5), (3.1.6) respectively.

Proof : To establish the multi dimensional series expansion (4.1), let

$$x^{\rho} y^{\sigma} \cos y\, I^{o,n_2:o,n_3:\ldots\ldots:o,n_r(m',n');(m'',n'');\ldots\ldots,\left(m^{(r+1)},n^{(r+2)}\right):P}_{p_2,q_2:p_3,q_3:\ldots\ldots:p_r,q_r(p',q');(p'',q'');\ldots\ldots,\left(p^{(r+1)},q^{(r+2)}\right):Q}\left[\begin{array}{c|c} z_1 x^{rh_1} y^{rk_1} & S : T \\ \vdots & \\ z_r x^{rh_r} y^{rk_r} & S^{(i)} : T^{(i)} \end{array}\right]$$

$$= \sum_{S,t=0}^{\infty} C_{s,t} J_{\nu+2s+1}(x) J_{\mu+2t+1}(y) \qquad (4.2)$$

Multiply both sides of equation (4.2) by $y^{-1}J_{\mu+2\nu+1}(y)$ and integrating w.r.t. y from 0 to ∞, using (3.2) & orthogonal property of Bessel's function (2.3) multiplying both side of the resulting expression by $x^{-1}J_{\nu+2\mu+1}(x)$ and integrating w.r.t. x from 0 to ∞, then using (3.1) and (2.3), we obtain the value of constant $C_{s,t}$.

Finally substituting the value of $C_{s,t}$ in (4.2), the multi dimensional series expansion formula is obtained on applying the same procedure as above, we can establish three other forms of two dimensional expansions of this class with the help of alternative forms of (3.1) and (3.2). Since an specializing the parameters of multivariable I-function yield almost all special functions appearing in Applied mathematics and physical sciences.

Therefore the result presented in the paper is of a general character and hence may encompass several cases of interest.

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