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Projective and Injective Weak Distributive Lattice

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Abstract:

This Paper is Concerned with solving problems existence with quasi projective and quasi injective objects and retracts respectively over problems with projective and injective objects and retracts in the category whose objects are the complete quasi lattice and morphism are the complete quasi lattice homomorphism from the point of view .in this paper we mentioned here some necessary and sufficient conditions for the given lattice be quasi projective and quasi injective and retracts respect.

1. Basic Definitions and Theorems :

There are used following symbols:

Categories are denoted by 2I, B, C....... Objects of categories by A, B, C......morphism by letters f, g, h, If A, B are the objects of category 2Ithen H (A, B) denotes the set of all morphism from A to B. Identity morphism from A to A is given by id_A. For f $\in H(A,B)$ and g $\in H(B,C)$ then there exist h such that h $\in H(A,C)$ then composition of given monomorphism is denoted by goh or

$$gh = f.$$

If X, Y are sets then f: X→Y denotes mapping of the set X into Set Y, Put

$f(X) = \{f(t) \mid t\in X\}$ $y\in Y$, $f^{-1}(y) = \{ x \mid x\in X \; f(x) = y\}$

For U⊆X, f | U denotes the restriction of the mapping f on the set U. If f: X→Y and $f(x_1)\neq f(x_2)$ then it is injective for two elements x, y∈X, $x_1\neq x_2$ and surjective if f(x) = y if f is both then it is bijective.

In Partially ordered set that is a set which is reflexive, anti-symmetric and transitive if A is an ordered set ∅ ≠X⊆ A the least upper bound of subset X in the set A if it exist then denoted by $sup_A X$ and greatest lower bound of X is $inf_A X$ can also use as x∨ y and x∧y instead of sup {x, y} and inf {x, y}, X-Y denotes difference of X and Y.

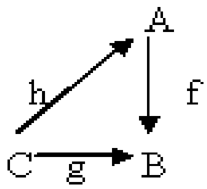
For ordered set A, x and y ∈ A and x, y are incomparable that is neither x≤y <x, y> denotes closed interval {t| t∈ A, x≤ t≤ y} (x, y) denotes open interval {t| t∈ A, x<t<y} we can say y covers x. If y>x, <x, y> ={x, y} smallest and greatest element are 0_A and 1_A.

If A, B are ordered sets then A+B denote cardinal sum and A⊕B denote direct sum or ordinal sum. AxB is their cardinal product [1]. If a∈A, b∈ B then [a, b] ∈ AXB and A⊗B is direct or ordinal product.

Let 2I be a category an object A∈ 2I is called projective if for arbitrary B,C ∈ 2I for arbitrary epimorphism g ∈ H_{2I} (B,C) and arbitrary morphism f ∈ H_{2I}(A,C) we have h ∈ H_{2I}(A,B) so that gh=f.

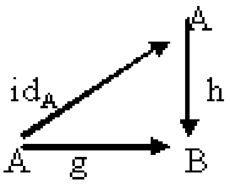
Let A∈2I is called projective retract if for every B ∈2I and for arbitrary epimorphism g∈ H_{2I} (A, B) we have h∈ H_{2I} (A, B) such that gh=idA.

Let A∈2I is called injective object if for arbitrary B, C ∈2I arbitrary monomrphism g∈H_{2I}(C, B) and arbitrary monomrphism f∈H_{2I} (B, A) such that hg=f.



Let A∈2I is called the injective retract if for .arbitrary B∈2I and arbitrary monomorphism g: A → B there exist h∈H_{2I} (B, A),

hg = id_A.



From above it is obvious that every projective object is a projective retract and every injective object is an injective retract.

On the above discussion in this paper we study generalization of distributive complete lattice and hence weakly distributive complete lattice over distributive and hence weakly distributive modules. By { [1],[5],[7],[8],distributive modules over R-module}, let L be an R module (Lattice) and C⊆L the sub-lattice C is said to be distributive sub-lattice of L if C=C∩A+C∩B for all sub-sub-lattice A,B⊆ L and L is distributive if each sub-lattice is distributive. C is a weak distributive sub-lattice of L if C= C∩A+C∩B for all sub-lattice A, B⊆ L such that A+B=C. A lattice L is said to be weakly distributive if every sub-lattice of L is a weak distributive sub-lattice of L. A ring R is weakly distributive if R is weakly distributive left R-module. Weakly distributive lattice is generalization of weakly distributive module of distributive module by which we obtain a weakly distributive lattice is distributive if and only if every sub-lattice is weakly distributive.

In section 2 shown that homomorphic image of weakly distributive lattice is weakly distributive. We prove that any π -projective and π - injective and direct injective duo lattice is weakly distributive. Any commutative lattice is weak distributive.

In section 3 we prove that the sum and intersection of two direct summands of a weakly distributive lattice is again a direct summand and the summand intersection property.

1. Let A and B are complete lattice .the mapping f : A→B is called the complete homomorphism if f (sup_AX)=sup_B{f(X)}, f(inf_AX) = inf_B{f(X)} for arbitrary subset ∅ ≠X⊆ A
2. Let A be a complete lattice. A subset ∅ ≠X⊆ A is called the closed sub lattice of the lattice A if sup_AY∈X , inf_AY∈X holds for every subset ∅ ≠Y⊆ X it obvious that a closed sub lattice of the complete lattice is complete lattice
3. Let A, B are complete lattice f : A→B a complete homomorphism then f(A) is a closed sub lattice of the lattice B.

2. Weakly Distributive Lattice:

It is well-known that, if f*: L → T* is an isomorphism, then there is a one-to-one correspondence between the sub-module (sub-lattice) of L and the sub-module of T. Therefore, any module (lattice) isomorphic to a weakly distributive module (lattice), is itself weakly distributive.

Lemma 2.1. Let L be a weakly distributive lattice and f: L → T be a homomorphism. Then Im f is a weakly distributive lattice.
Proof. Let f: L → T be a homomorphism. Then Im f ≅ L/K where K = ker f. Let
U/K+V/K = L/K for some A, B ⊆ L and A/K ⊆ L/K. Then we have U+V = L.
Since L is weakly distributive, A = A ∩ U + A ∩ V. Therefore,
A/K = (A ∩ U + A ∩ V)/K = [(A/K) ∩ (U/K)] + [(A/K) ∩ (V/K)].
Hence Im *f* ≅ *L/K* is weakly distributive.

Examples of weak distributive sub-modules (lattice) can be found in the following lemma. Recall that an *R*-module *L* is called *π-projective* if whenever L = A + B , there exists *α* ∈ End*(*L*)* such that *α(*L*)* ⊆ *X* and *(*1 − *α)(*L*)* ⊆ *Y* .

R-module L is called π-injective if, for all sub-modules (lattice) U and V of L with U ∩ V = 0, there exists f ∈ S with U ⊂ Ker (f) and
V ⊂ Ker (1 − f). A module L is called a self-generator if it generates all its sub-lattice.

Lemma 2.2: Let L be a π-projective module. Every fully invariant sub-module of L is a weak distributive sub-module of L.
Proof. Let U be a fully invariant sub-module of L and suppose L = A+ B. Then there exists an endomorphism f ∈ End (M) such that f (L) ⊆ A and (1 − f) (L) ⊆ B. Since
U is a fully invariant sub-module of L, we have
f (U) ⊆ U ∩ A, (1 − f)(U) ⊆ U ∩ B.
Then

$$U \subseteq f(U) + (1 - f)(U) \subseteq U \cap A + U \cap B$$

And so U = U ∩ A + U ∩ B. That is U is a weak distributive sub-module (lattice) of L.

Further the mapping f: A→B is called the complete homomorphism if

$sup_A X \in U$ then $f(U) = f(sup_A X) = sup_B\{(f(x)\}$ and $inf_A X \in U$ then $f(inf_A X) = inf_B\{f(X)\}$

Theorem 2.3: An R-module L (lattice) is projective (resp. injective) distributive if and only if L is projective and injective weakly distributive.

Let L = ⊕i∈I Ti be a semi simple module (lattice) where T_i is a simple lattice for each i ∈ I.

then L is a weakly distributive lattice if and only if $Hom(T_i, T_j) = 0$ for every i, j ∈ I such that i ≠ j .

Proof: Let *i, j* ∈ *I* and *i* ≠*j*. Then the sub module (lattice) Ti ⊕ T_j of *M* is a direct summand of *L* and hence a weakly distributive module. And also T_i and
T_j are not isomorphic. Clearly this implies that Hom*(Ti, Tj)* = 0. For the converse let *A* be a sub module of *M*. First we shall prove that *A* is a fully invariant sub module of *M*. Since is semi simple *A* = ⊕*j*∈*J* Tj for some *J* ⊂ *I.* Let *f* ∈ End *(*L*)*. Then *f (*Tj*)* ⊂ Tj for each
j ∈ *J* , by hypothesis. Therefore f *(A)* ⊂ *A*. That is *A* is a fully invariant sub module of L. Since L is semi simple it is self-projective, hence L is *π* projective. Then Lemma 2.2 implies that A is a weak distributive sub lattice of L.
Proposition 2.4: An R-module L is distributive if and only if every sub lattice of *L is a* weakly distributive lattice.
Proof. The necessity part is clear. For sufficiency, let *A, B* and *C* be sub lattice of L.
Then,
$A \cap (B + C) = [A \cap (B + C)] \cap (B + C)$
$= [A \cap (B + C)] \cap B + [A \cap (B + C)] \cap C$
$= A \cap B + A \cap C,$
Since *B* + *C* is weakly distributive lattice.

3. Summand Sum Property :

Lemma 3.1. Weakly distributive lattice satisfy the summand sum property and the summand intersection property.

Proof. Let A and B be direct summands of L. Suppose L = A ⊕ A'
= B ⊕ B'
Then
A = A ∩ B ⊕ A ∩ B'
We get L = A⊕ A'
= A ∩ B ⊕ A ∩ B'⊕ A'
Hence A ∩ B is a
Direct summand of L, and so L has the summand intersection property.
To prove that L has the summand sum property, we need to show that A+B is a direct
Summand of L. We have
A + B = A ∩ B + A ∩ B'+ B ∩ A + B ∩ A'

= (A ∩ B+ A'∩ B) + A ∩ B' = B ⊕ A∩ B'
Now we get L = B ⊕B'
= B ⊕B' ∩A ⊕B'∩A'
= (A +B) ⊕ B'∩A' This completes the proof.

Conclusion: From the above it has been proved that and lattice satisfy the summand sum property and the summand intersection property.

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