

On Common Fixed Point Theorem in Intuitionistic Fuzzy Metric Spaces

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Abstract:
Fixed point theory has fascinated hundreds of researchers since 1922 with the celebrated Banach's fixed point theorem. There exists a vast literature on the topic and this is a very active field of research at present**.** The main purpose of this paper is to prove some generalized common fixed point theorem in intuitionistic fuzzy metric spaces.

Key Words: - Fixed point, common fixed point intuitionistic fuzzy metric space, weakly compatible mapping.

INTRODUCTION

The concept of fuzzy set was first introduced by Zadeh [18] in 1965. Many authors have introduced the concept of fuzzy metric space in different ways ([7], [8]). George and Veeramani [5] modified the notion of fuzzy metric space introduced by Kramosil and Michalek [10]. In 1986, Atanassov [3] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [18].Park [13] ,in 2004 introduced the concept of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms. In 2006,using the notion of intuitionistic fuzzy sets, Alaca defined the concept of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space which is introduced by Kramosil and Michalek [10]. Turkogulu et.al [17] generalized Jungck's [7] common fixed point theorem in intuitionistic fuzzy metric space.They first introduced the concept of weakly commuting and R weakly commuting mappings in intuitionistic fuzzy metric space. The concept of weakly compatible mapping is most general as each pair of compatible mapping is weakly compatible but the converse is not true.

PRELIMINARIES

Definition 2.1. A continuous t-norm is a binary operation $*$ on $[0,1]$ satisfying the following conditions:

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a$ for all $a \in [0,1]$;

(iv) $a * b \leq c * d$ Whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2.2. A continuous t-conorm is a binary operation $\lozenge$ on $[0,1]$ satisfying the following conditions:

(i) $\lozenge$ is commutative and associative;

(ii) $\lozenge$ is continuous;

(iii) $a \lozenge 0 = a$ for all $a \in [0,1]$;

(iv) $a \lozenge b \leq c \lozenge d$ Whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2.3. An intuitionistic fuzzy metric space is a 5-tuple $(X, M, N, *, \lozenge)$ where X is a nonempty set, $*$ is a continuous t-norm, $\lozenge$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) > 0$ for all $x, y \in X$;

(ii) $M(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$ iff $x = y$;

(iii) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;

(v) $M(x, y, .): [0, \infty) \rightarrow [0, 1]$ is continuous for all $x, y \in X$;

(vi) $N(x, y, t) \geq 0$ for all $x, y \in X$;

(vii) $N(x, y, t) = 0$ for all $x, y \in X$ and $t > 0$ iff $x = y$;

(vii) $N(x, y, t) = N(y, x, t)$ for all $x, y \in X$ and $t > 0$;

(ix) $N(x, y, t) \lozenge N(y, z, s) \geq N(x, z, t + s)$ for all $x, y, z \in X$ and $s, t > 0$;

(x) $N(x, y, .): [0, \infty) \rightarrow [0, 1]$ is continuous for all $x, y \in X$;

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Definition 2.4. An intuitionistic fuzzy metric $(X, M, N, *, \lozenge)$ on X is said to be stationary if M and N does not depend on t, i.e. the function $M_{x,y}(t) = M(x, y, t)$ and $N_{x,y}(t) = N(x, y, t)$ is constant.

Definition 2.5. In *an* intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$.

(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$, $\lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$,for all $t > 0$ and $p > 0$

(b) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \epsilon X$.if $\lim_{n\to\infty} M(x_n, x, t) = 1$, $\lim_{n\to\infty} N(x_n, x, t) = 0$,for all $t > 0$,

Definition 2.6. An Intuitionistic fuzzy metric space is called complete iff every Cauchy sequence in X is convergent .

Definition 2.7. The F mappings and g are called compatible where $F: X \times X \rightarrow X$ and $g: X \rightarrow X$ if

$$\lim_{n\to\infty} d(g(F(x_n\, y_n)), F\big(g(x_n), g(y_n)\big) = 0 \text{ and}$$

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(g(y_n), g(x_n)) = 0$$

Whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X,such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for all $x, y \in X$are satisfied.

Definition 2.8. The mappings F and g where$F: X \times X \rightarrow X$ and $g: X \rightarrow X$ has g mixed monotone property of intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ if F is monotone g nondecreasing in first argument and g is monotone g-nonincreasing in second argument.

Definition 2.9. The mappings F and g on intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ are said to be compatible where $F: X \times X \rightarrow X \; and \; g: X \rightarrow X$ if

$$\lim_{n\to\infty} M(g(F(x_n, y_n)), F(g(x_n), g(y_n)), t) = 1$$

$$\lim_{n\to\infty} M(g(F(y_n, x_n)), F(g(y_n), g(x_n)), t) = 1$$

$$\lim_{n\to\infty} N(g(F(x_n, y_n)), F(g(x_n), g(y_n)), t) = 0$$

$$\lim_{n\to\infty} N(g(F(y_n, x_n)), F(g(y_n), g(x_n)), t) = 0$$

Whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for all $x, y \in X$ are satisfied.

Definition 2.10. The mappings $F: X \times X \to X$ and $g: X \to X$ on intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ are said to be weakly compatible if they commute at three coincidence points i.e.

$F(x, y) = g(x)$ for some $x \in X$ then $F(g(x), g(y)) = g(F(x, y))$ and

$F(y, x) = g(y)$ for some $y \in X$ then $F(g(y), g(x)) = g(F(y, x))$.

Definition 2.11. The mappings F and g of Intuitionistic fuzzy metric space where $F: X \times X \to X$ and $g: X \to X$ satisfy E.A. property if there exist sequences $\{x_n\}$ and $\{y_n\}$ are sequences in X,such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = g(u)$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = g(v)$ for all $u, v \in X$ and $t > 0$.

MAIN RESULT

Theorem 3.1. Let S and T be two continuous self mappings of an intuitionistic fuzzy metric space $(X, M, N, *, \lozenge)$ .Let A be a self mapping of X satisfying $\{A, S\}$ and $\{A, T\}$ are R-weakly commuting and

$$A(X) \subseteq S(X) \cap T(X) \quad (1)$$

and $M(Ax, Ay, t) \geq r[min\{M(Sx, Ty, t), M(Sx, Ax, t), M(Sx, Ay, t), M(Ty, Ay, t)\}]$ (2.1)

and $N(Ax, Ay, t) \leq r'[max\{N(Sx, Ty, t), N(Sx, Ax, t), N(Sx, Ay, t), N(Ty, Ay, t)\}]$ (2.2)

for all $x, y \in X$,

where $r: [0,1] \to [0,1]$ and $r': [0,1] \to [0,1]$ is a continuous function such that $r(t) > t$ and $r'(t) < t$ for each $t < 1, r(t) = 1$ and $r'(t) = 0$ $for\ t = 1$. (3)

The sequences $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \to x, y_n \to y, t > 0$ implies $M(x_n, y_n, t) \to M(x, y, t)$ and $N(x_n, y_n, t) \to N(x, y, t)$ .Then A, S, T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be any arbitrary point. Since $A(X) \subseteq S(X)$ there is a point $x_1 \in X$ such that $Ax_0 = Sx_1$. Also since $A(X) \subseteq T(X)$there is another point $x_2 \in X$ such that $Ax_1 = Tx_2$.In general we get points $x_{2n-1}$and x_{2n+2}in X such that $Sx_{2n+1} = Ax_{2n}$; $Tx_{2n+2} = Ax_{2n+1} for\ n = 0,1,2 \dots \dots$..

Let $M_{2n} = M(Ax_{2n+1}, Ax_{2n}, t)$

$\geq r[\min\{M(Sx_{2n+1}, Tx_{2n}, t), M(Sx_{2n+1}, Ax_{2n+1}, t), M(Sx_{2n+1}, Ax_{2n}, t), M(Tx_{2n}, Ax_{2n}, t)\}]$

$= r[\min\{M(Sx_{2n+1}, Ax_{2n-1}, t), M(Ax_{2n}, Ax_{2n+1}, t), M(Ax_{2n}, Ax_{2n}, t), M(Ax_{2n-1}, Ax_{2n}, t)\}]$

$= r[\min\{M_{2n-1}M_{2n}, 1, M_{2n-1}\}]$. (4.1)

$N_{2n} = N(Ax_{2n+1}, Ax_{2n}, t)$

$\leq r'[\max\{N(Sx_{2n+1}, Tx_{2n}, t), N(Sx_{2n+1}, Ax_{2n+1}, t), N(Sx_{2n+1}, Ax_{2n}, t), N(Tx_{2n}, Ax_{2n}, t)\}]$

$= r'[\max\{N(Sx_{2n+1}, Ax_{2n-1}, t), N(Ax_{2n}, Ax_{2n+1}, t), N(Ax_{2n}, Ax_{2n}, t), N(Ax_{2n-1}, Ax_{2n}, t)\}]$

$= r'[\max\{N_{2n-1}N_{2n}, 1, N_{2n-1}\}]$. (4.2)

If $M_{2n-1} > M_{2n}$ then $M_{2n} \geq r(M_{2n}) > M_{2n}$ and If $N_{2n-1} < N_{2n}$ then $N_{2n} \leq r'(N_{2n}) < N_{2n}$ a contradiction, therefore, $M_{2n-1} \leq M_{2n}$and $N_{2n-1} \geq N_{2n}$. From (4.1) and From (4.2) we get

$$M_{2n} \geq r(M_{2n-1}) > M_{2n-1} \quad (5.1)$$

$$N_{2n} \leq r'(N_{2n-1}) < N_{2n-1} \quad (5.2)$$

Thus $\{M_{2n}, n \geq 0\}$ is an increasing sequence of positive real numbers in [0,1] and therefore tends to a limit $L \leq 1$ and $\{N_{2n}, n \geq 0\}$ is an decreasing sequence of positive real numbers in [0,1] and therefore tends to a limit $L \geq 1$. We claim that $L = 1$.If $L < 1$,on taking $n \to \infty$ in (5.1) we get $L \leq r(L) > L$, a contradiction . Hence $L = 1$ and if $L > 1$, on taking $n \to \infty$ in (5.2) we get $L \geq r'(L) < L$, a contradiction. Hence $L = 1$.

Now , for any integer p,

$$M(Ax_n, Ax_{n+p}, t) \geq M(Ax_n, Ax_{n+1}, t) * \dots \dots \dots .. * M(Ax_{n+p-1}, Ax_{n+p}, t/p)$$

$$\geq M(Ax_n, Ax_{n+1}, t/p) * \dots \dots \dots .. * M(Ax_n, Ax_{n+1}, t/p)$$

$$N(Ax_n, Ax_{n+p}, t) \leq N(Ax_n, Ax_{n+1}, t) \lozenge \dots \dots \dots .. \lozenge N(Ax_{n+p-1}, Ax_{n+p}, t/p)$$

$$\leq N(Ax_n, Ax_{n+1}, t/p) \lozenge \dots \dots \dots .. \lozenge N(Ax_n, Ax_{n+1}, t/p)$$

Since $\lim_{n\to\infty} M(Ax_n, Ax_{n+1}, t) = 1$ and $\lim_{n\to\infty} N(Ax_n, Ax_{n+1}, t) = 0$, for $t > 0$,it follows that $\lim_{n\to\infty} M(Ax_n, Ax_{n+p}, t) \geq 1 * 1 * \dots \dots \dots .* 1 = 1$ and $\lim_{n\to\infty} N(Ax_n, Ax_{n+p}, t) \leq 0 \lozenge 0 \lozenge \dots \dots .\lozenge 0 = 0$

Thus $\{Ax_n\}$ is a Cauchy sequence and by the completeness of X, $\{Ax_n\}$ converges to a point $z \epsilon X$.Clearly the subsequences $\{Sx_{2n+1}\}$ and $\{Tx_{2n}\}$ of $\{Ax_n\}$,also converge to the same limit .Thus $Sx_{2n+1} \to z$ and $Tx_{2n} \to z$. Since A is R-weakly commuting with S, we get

$$M(ASx_{2n+1}, SAx_{2n+1}, t) \geq M(Ax_{2n+1}, Sx_{2n+1}, t/R) \text{ and}$$

$$N(ASx_{2n+1}, SAx_{2n+1}, t) \leq N(Ax_{2n+1}, Sx_{2n+1}, t/R).$$

Which by continuity of S gives$\lim_{n\to\infty} ASx_{2n+1} = \lim_{n\to\infty} SAx_{2n+1} = Sz$.

Now we prove that $Sz = z$. Suppose $Sz \neq z$ then there exists $t > 0$ such that

$M(Sz, z, t) < 1$ and $N(Sz, z, t) > 0$.Using (2.1) and (2.2) we have

$$M(ASx_{2n+1}, Ax_{2n}, t)$$

$$\geq r[min\, M(S^2x_{2n+1}, Tx_{2n}, t), M(S^2x_{2n+1}, ASx_{2n+1}, t) M(S^2x_{2n+1}, Ax_{2n}, t), M(Ax_{2n}, Tx_{2n}, t)\}].$$

$$N(ASx_{2n+1}, Ax_{2n}, t)$$

$$\leq r'[max\, N(S^2x_{2n+1}, Tx_{2n}, t), N(S^2x_{2n+1}, ASx_{2n+1}, t), N(S^2x_{2n+1}, Ax_{2n}, t), N(Ax_{2n}, Tx_{2n}, t)\}].$$

In the limiting case we get,

$$M(Sz, z, t) \geq r[min\{ M(Sz, z, t), M(Sz, Sz, t), M(Sz, z, t), M(z, z, t) \}]$$

$$= r[M(Sz, z, t)] > M(Sz, z, t)$$

and $N(Sz, z, t) \leq r'[max\{ N(Sz, z, t), N(Sz, Sz, t), N(Sz, z, t), N(z, z, t) \}]$

$$= r[N(Sz, z, t)] < N(Sz, z, t) .$$

Which is a contradiction .Thus z is a fixed point of S .Similarly we can show that z is a fixed point of A. Now we claim that z is also a fixed point of T.Suppose it is not so. Then for any $t > 0$, $M(z, Sz, t) < 1$and $N(z, Sz, t) > 0$ and

$$M(Az, A\,Tx_{2n,} t) \geq r[\min\{ M(Sz, T^2x_{2n,} t), M(Sz, Az, t), M(Sz, ATx_{2n}, t)$$

$$M(T^2x_{2n}, ATx_{2n}, t)\}].$$

$$N(Az, A\,Tx_{2n,} t) \leq r'[\max\{ N(Sz, T^2x_{2n,} t), N(Sz, Az, t), N(Sz, ATx_{2n}, t)$$

$$N(T^2x_{2n}, ATx_{2n}, t)\}].$$

On taking limit $n \to \infty$ it gives

$M(z,Tz,t) \geq r[\min\{ M(z,Tz,t), M(z,z,t), M(z,Tz,t), M(Tz,Tz,t)\}]$,

i.e, M(z,Tz,t) $\geq r[M(z,Tz,t)]$

$N(z,Tz,t) \leq r[\max\{ N(z,Tz,t), N(z,z,t), N(z,Tz,t), N(Tz,Tz,t)\}]$,

i.e, $N(z,Tz,t) \leq r[N(z,Tz,t)]$

which contradicts (3) . So $M(z,Tz,t) = 1$and $N(z,Tz,t) = 0$ implying z is also fixed point of T.

Using (2.1) and (2.2) the uniqueness of the fixed point can be shown easily. Thus z is the unique common fixed point of A, S and T and this completes the proof.

Taking $T = S$ in the above theorem we get the following corollary unifying vasuki's theorem which in turn also generalizes the result of pant [12].

Corollary 3.2. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space and S be a continuous self mapping of X. Let A be another self mapping of X satisfying the pair $\{A, S\}$ is R-weakly commuting with $A(X) \subseteq S(X)$ and

$$M(Ax,Ay,t) \geq r\ [min\{ M(Sx,Sy,t), M(Sx,Ax,t), M(Sx,Ay,t), M(Sy,Ay,t) \}]$$

$$N(Ax,Ay,t) \leq r^{'}[max\{N(Sx,Sy,t), N(Sx,Ax,t), N(Sx,Ay,t), N(Sy,Ay,t) \}]$$

For all $x, y \in X$, where $r: [0,1] \to [0,1] and\ r': [0,1] \to [0,1]$ is a continuous function such that $r(t) > t$ and $r'(t) < t$.

For each $0 \leq t < 1$ and $r(t) = 1\ and\ r^{'(t)} = 0$ for $t = 1$. The sequences $\{x_n\}$ and $\{ y_n\}$ in X are such that $x_n \to x, y_n \to y, t > 0$ implies $M(x_n, y_n, t) \to M(x,y,t) and\ N(x_n, y_n, t) \to N(x,y,t)$ .Then A, S have a unique common fixed point in X.

REFERENCES

[1] Alaca, C., Turkoglu,D.and Yildiz, C., Common fixed point theorem in intuitionistic fuzzy metric spaces ,J.Appl.math,&computing, 22(2006),411-424.

[2] Alaca, C., Turkoglu,D.and Yildiz, C., Fixed point in intuitionistic fuzzy metric spaces, Chaos, Solitons & Fractals, 29(2006),1073-1078.

[3] Atanssov, K.., Intuitionstic fuzzy sets, Fuzzy Sets and Systems 2(1) (1986), 87-96.

[4] Deng, Zi-Ke,Fuzzy pseudo-metric spaces,J.Math.Analy.appl.86(1982),74-95.

[5] George, .A andVeeramani, P., On some results in fuzzy metric spaces, Fuzzy Sets and Systems, 64(1994), 395-399.

[6] Grabiec, M., Fixed points in fuzzy metric space, Fuzzy Sets and Systems, 27(1988), 385-389.

[7] Jungck, G.:Commuting mappings and fixed points, Amer. Math. Monthly, 183 (1976), 261-263.

[8] Kaleva, O. And Sheikkala, S.,on fuzzy metric space. Fuzzy Sets and Systems, 12(1984), 215-229.

[9] Khichi, S., Singh, A.,Intuitionistic Fuzzy Metric Spaces and Common Fixed Point Theorems Using Occasionally Weakly Compatible (OWC) Self Mappings Volume 4,Number 2 (2014), pp. 135-144.

[10] Kramosil, I. and Michalek, J., Fuzzy metric and statical metric spaces, Cybernetics, 11 (1975), 336-344.

[11] Lakshmikantham, V. and Ciric, L., Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis 70 (12) (2009), 4341-4349.

[12] Pant, R. P., Common fixed points of noncommuting mappings, J. Math. Anal. Appl.188:2(1994), 436-440.

[13] Park, J. H. 2004. Intuitionistic fuzzy metric spaces, Chaos Solitons& Fractals, 22(2004), 1039-1046.

[14] Park, J. H., Park, J. S. and Kwun, Y. C. 2005., A common fixed point theorem in intuitionstic fuzzy metric spaces, In: Advance in Natural Computing Data Mining (Proceeding 2nd ICNC and 3rd FSKK) (2005), 293-300.

[15] Schweizer, B. and Sklar A. 1983. Probabilistic metric spaces, Amesterdam North Holland 1983.

[16] Shrivastava, R.,Tiwari,P.,et.al. Coupled fixed point theorems in intuitionistic fuzzy metric space,South Asian Journal of Mathematics4:3(2013),301-324.

[17]Turkoglu, D., Alaca, C., Cho, Y. J. and C. Yildiz, 2006.Common fixed point theorem in intuitionistic fuzzy metric spaces, Journal of applied mathematics and computing, 22(1-2) (2006), 411-424

[18] Zadeh,L.A.: Fuzzy sets ,Inform control 8(1965), 338-353.