

# Some Characterized Projective $\delta$ -cover

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## Abstract

In this paper we characterize some properties of projective  $\delta$ -cover and find some new results with  $\delta$ -supplemented module  $M$ . Let  $M$  be a fixed  $R$ -module. A  $\delta$ -cover in  $M$  is an  $\delta$ -small epimorphism from  $M$  onto  $P$ . This concept introduced by Zhou [14]. A  $\delta$ -cover is projective  $\delta$ -cover ( $M$ -projective  $\delta$ -cover) in case  $M$  is projective.

**Keywords:** Singular, non singular, simple, small,  $\delta$ -small, cover,  $\delta$ -cover, supplement,  $\delta$ -supplement.

## Introduction:

Throughout this paper  $R$  is an associative ring with unity and all modules are unitary  $R$ -modules. Let  $M$  be a fixed module, a sub module  $L$  of module  $M$  is denoted by  $L \leq M$ . Submodule  $L$  of  $M$  is called essential (large) in  $M$ , abbreviated  $K \leq_e M$ , if for every submodule  $N$  of  $M$ ,  $L \cap N$  implies  $N = 0$ . A sub module  $N$  of a module  $M$  is called small in  $M$ , denoted by  $N \ll M$ , if for every sub module  $L$  of  $M$ , the equality  $N + L = M$  implies  $L = M$ . For each  $X \subset M$ , the right  $\text{Ann}(X)$  in  $R$  is  $r_R(X) = \{r \in R : xr = 0 \text{ for all } x \text{ in } X\}$ . The sub module  $Z(M) = \{x \in M : r_R(x) \text{ is an essential in } R_R\}$  is singleton, is called singular submodule of  $M$ . The module  $M$  is called singular module if  $Z(M) = M$ . ( $M$  is non singular if  $Z(M) = 0$ ). A right  $R$ -module is called simple if  $M \neq 0$  and  $M$  has only proper submodules. A sub module  $N$  of  $M$  is called minimal in  $M$  if  $N \neq 0$  and for every submodules  $A$  of  $M$ ,  $A \subset N$  implies  $A = N$ .

An epimorphism  $f : M \rightarrow P$  is called small if  $\ker f \ll M$ . A small epimorphism  $f : M \rightarrow P$  is called projective cover if  $M$  is projective with  $\ker f \ll M$ . [Zhou] introduced the concept of  $\delta$ -small submodule as generalization of small submodules. Let  $K \leq M$ ,  $K$  is called  $\delta$ -small if whenever  $M = N + K$  and  $M/N$  is a singular, we have  $M = N$ . (denoted by  $\ll_{\delta}$ ). The sum of all  $\delta$ -small submodules is denoted by  $\delta(M)$ . A  $\delta$ -cover in  $M$  is an  $\delta$ -small epimorphism from  $M$  onto  $P$ . A  $\delta$ -cover is projective  $\delta$ -cover ( $M$ -projective  $\delta$ -cover) in case  $M$  is projective.

**Definition:** Let  $M$  be a fixed  $R$ -module. An  $R$ -module  $U$  is called (small)  $M$ -projective module, if for every (small) epimorphism  $f : M \rightarrow P$  and homomorphism  $g : U \rightarrow P$ , there exists a homomorphism  $v : U \rightarrow M$  such that  $f \circ v = g$ , i.e. following diagram is commute.

$$\begin{array}{ccc}
 & U & \\
 \swarrow & \downarrow g & \\
 M & \xrightarrow{f} P \rightarrow 0 & (\text{Kerf} \ll M)
 \end{array}$$

**Example:** Every proper sub module of the  $\mathbb{Z}$ -modules  $\mathbb{Z}_p^\infty$  is small in  $\mathbb{Z}_p^\infty$ .

**Remarks:** i) Every  $M$ -projective module is a small  $M$ -projective cover.

ii) Every self projective module  $M$  is self small projective module and converse is true for  $M$  is hollow.

**Lemma:** [Zhou] Let  $N$  be a sub module of  $M$ . The following are equivalent:

- i)  $N \ll_\delta M$
- ii) If  $M = X + N$ , then  $M = X \oplus Y$  for a projective semisimple sub module  $Y$  with  $Y \subseteq N$ .

**Proof:** [14]

**Lemma:** If each  $f_i: N_i \rightarrow M_i$  are  $M$ -projective  $\delta$ -covers for  $i = 1, 2, 3, \dots, n$ , then

$\bigoplus_{i=1}^n f_i: \bigoplus_{i=1}^n N_i \rightarrow M_i$  is  $M$ -projective  $\delta$ -cover.

**Proof:** [12]

**Lemma:** If  $N$  is a direct summand of module  $M$  and  $A \ll_\delta M$ , then  $A \cap N \ll_\delta N$ .

**Lemma:** Let  $K$  be a sub module of a  $M$ -projective module  $U$ . If  $U/K$  has a  $M$ -projective  $\delta$ -cover, then it has a  $M$ -projective  $\delta$ -cover of the form  $f: \frac{U}{L} \rightarrow \frac{U}{K}$  with  $\ker f = \frac{K}{L}$  where  $L \subseteq K$ .

**Proof:** Let  $K$  be a sub module of a  $M$ -projective module  $U$ . Let  $f: M \rightarrow \frac{U}{K}$  be a  $M$ -projective  $\delta$ -

cover of  $\frac{U}{K}$ , and  $\pi: U \rightarrow \frac{U}{K}$  is a canonical epimorphism,  $U$  is  $M$ -projective module, there exists an homomorphism  $v: U \rightarrow M$  s.t.  $f \circ v = \pi$ .

$$\begin{array}{ccc}
 & L & \\
 & \downarrow & \\
 & K & \\
 & \downarrow & \\
 & U & \\
 \swarrow & \downarrow \pi & \\
 M & \xrightarrow{f} \frac{U}{K} \rightarrow 0 & \\
 \swarrow v & & 
 \end{array}$$

Then  $M = \ker g \oplus \text{Im } v$ . By lemma [Zhou]  $M = N \oplus \text{Im } v$  for semi simple sub module  $N$ , with  $N \subseteq \text{Ker } f$  since  $\ker(f|_{\text{Im } v}) \prec\prec_{\delta} \text{Im } v$ . So  $f|_{\text{Im } v}$  is also  $M$ -projective  $\delta$ -cover of  $\frac{U}{K}$ . But

$\frac{U}{\ker v} \cong \text{Im } v$  by isomorphism theorem. Since  $f \circ v = \pi$  and  $\ker v \subseteq K$ . If we consider the isomorphism  $v': \frac{U}{\ker v} \rightarrow \text{Im } v$  defined by  $v'(\ker v + u) = u \quad \forall u \in U, \text{Im } v \leq^{\oplus} U$ . Then we obtain

$$\ker(f|_{\text{Im } v}, v') \prec\prec_{\delta} \frac{U}{\ker v} //$$

**Lemma:** A pair  $(M, f)$  is a  $M$ -projective  $\delta$ -cover of finitely generated module  $U$ , The there exists a finitely generated direct summand  $M'$  of  $M$  such that  $f|_{M'}$  is a  $M$ -projective  $\delta$ -cover of  $U$ .

**Theorem:** An  $R$  module  $M$  has a  $M$ -projective  $\delta$ -cover, then for every epimorphism  $f: M \rightarrow P$ , the following are equivalent:

- i)  $f: M \rightarrow P$  is a  $M$ -projective cover.
- ii)  $M$  is projective, for every epimorphism  $f': M' \rightarrow P$ , with  $M' \leq^{\oplus} M$ , there exists a necessarily split epimorphism  $h: M' \rightarrow M$  such that  $f \circ h = f'$ .
- iii) For every small epimorphism  $g: M \rightarrow N$ , there exists an epimorphism  $h: P \rightarrow M$  such that  $f \circ h = g$

**Corollary:** Let  $f: M \rightarrow P$  and  $f': M' \rightarrow P, M' \leq^{\oplus} M$ , be a  $M$ -projective cover. Then there is an isomorphism  $h: M \rightarrow M'$  such that  $f' \circ h = f$ . In fact if  $h: M \rightarrow M'$  is a homomorphism with  $f' \circ h = f$ , then  $h$  is an isomorphism.

**Proposition:** Let  $f: M \rightarrow P$  be a  $M$ -projective  $\delta$ -cover. If  $U$  is  $M$ -projective and  $g: U \rightarrow P$  is an homomorphism, then there exists decomposition  $M = A \oplus B$  and  $U = X \oplus Y$  such that

- i)  $A \cong X$
- ii)  $f|_A: A \rightarrow P$  is a  $M$ -projective  $\delta$ -cover.
- iii)  $h|_X: X \rightarrow P$  is a  $M$ -projective  $\delta$ -cover.
- iv)  $B$  is a Projective semi simple with  $B \subseteq \ker f$  and  $Y \subseteq \ker h$

**Proof:** Since  $U$  is  $M$ -projective,

$$\begin{array}{ccc} & U & \\ & \swarrow h & \downarrow g \\ M & \xrightarrow{f} & P \rightarrow 0 \end{array}$$

Then there exists  $h: U \rightarrow M$  such that  $f \circ h = g$ . Thus we have  $M = \text{Im } h + \ker f$  and  $\ker f \prec\prec_{\delta} M$ , we have  $M = \text{Im } h + B$  for a semi simple module  $B$  with  $B \subseteq \ker f$ , by lemma 9.  $f|_A: A \rightarrow P$  is a  $M$ -projective  $\delta$ -cover.

Since direct summand of projective module is projective, so  $A$  is projective and homomorphism  $h: U \rightarrow A$  splits, then there exists  $t: A \rightarrow U$  such that  $h \circ t = I_A$ . Thus  $U = X \oplus Y = \text{Im}t + \ker h$  this implies  $A \cong t(A) = X$ . Since  $\ker(hI_A) \prec\prec_\delta A$  ( $M = A \oplus B$ ), we have  $\ker(hI_X) = \ker(hI_A) \prec\prec_\delta t(A) = X$ .  $g(X) = (f \circ h)(X + Y) = (f \circ h)(U) = P$ . Thus  $hI_X: X \rightarrow P$  is a  $M$ -projective  $\delta$ -cover. //

**Lemma:** Let  $U$  be a  $M$ -projective module and  $N \leq^\oplus M$ , then the following are equivalent;

- i)  $\frac{M}{N}$  has a  $M$ -projective  $\delta$ -cover.
- ii)  $M = M_1 \oplus M_2$  for some  $M_1$  and  $M_2$ , with  $M_1 \subset N$  and  $M_2 \cap N \prec\prec_\delta M$ .

**Proof:** i) $\Rightarrow$  ii) Assume that  $\frac{M}{N}$  has a  $M$ -projective  $\delta$ -cover. Let  $g: U \rightarrow \frac{M}{N}$  be a  $M$ -projective  $\delta$ -cover and  $\pi: M \rightarrow \frac{M}{N}$  is canonical epimorphism, then there exists an homomorphism  $h: U \rightarrow M$  such that the diagram

$$\begin{array}{ccc} & U & \\ & \swarrow h & \downarrow g \\ M & \xrightarrow{\pi} & \frac{M}{N} \rightarrow 0 \end{array}$$

is commute. Therefore  $M = \text{Im}h + \ker \pi = \text{Im}h + N$ . By lemma [Zhou 3.1] there exists a decomposition  $M = M_1 \oplus M_2$  such that  $\pi|_{M_2}: M_2 \rightarrow \frac{M}{N}$  is a  $M$ -projective  $\delta$ -cover and  $M_1 \subseteq \ker \pi = N$ . Thus  $M_2 \cap N = \ker(\pi|_{M_2}) \prec\prec_\delta M$ . Since  $M_2 \leq^\oplus M$  then  $M_2 \cap N \prec\prec_\delta M$ .

ii) $\Rightarrow$  i) it is clear.

**Lemma:** If  $f: U \rightarrow M$  and  $g: M \rightarrow N$  are  $\delta$ -covers, then  $g \circ f$  is a  $\delta$ -cover.

Proof: [12]

**Lemma:** Let  $M, N, P$  be  $R$ -modules, for some homomorphisms  $f: M \rightarrow P$ ,  $g: M \rightarrow N$  and  $h: N \rightarrow P$  such that  $h \circ g = f$  then,

- i)  $f$  is a small epimorphism if and only if  $N = \ker h + \text{Im}g$ .
- ii) A pair  $(M, f)$  is a projective  $\delta$ -cover if and only if  $g(M)$  is a  $\delta$ -supplement of  $\ker h$  in  $N$  and  $\ker g \prec\prec_\delta M$

**Proof:** i) it is clear by lemma R

(ii)  $\Rightarrow$  Suppose a pair  $(M, f)$  is a  $\delta$ -cover, by (i) we have  $N = \ker h + \text{Im}g$  i.e.  $f$  is small epimorphism, we get  $g(\ker f) = \ker h + \text{Im}g$  and  $\ker f \prec\prec_\delta M$ . By lemma [1,1 K. Al-Thakman]  $g(\ker f) \prec\prec_\delta \text{Im}g$ , hence  $\text{Im}g$  is  $\delta$ -supplement of  $\ker h$  in  $N$ .

$\Leftarrow$  Assume that the  $g(M)$  is a  $\delta$ -supplement of  $\ker h$  in  $N$ , then  $N = \text{Im}g + \ker h$  and  $\text{Im}g \cap \ker h \prec\prec_\delta \text{Im}g$ . Since  $f$  is epimorphism, consider  $\ker f + S = M$  and  $\frac{M}{S}$  is singular. So  $g(\ker f) + g(S) = g(M)$  but  $g(\ker f) = \ker h \cap \text{Im}g$ , Hence  $g(M) = g(\ker f) \cap \text{Im}g + g(S)$ ,

since  $\frac{g(M)}{g(S)}$  is singular, being a homomorphic image of singular module and  $\text{Im } g \cap \ker h \prec_{\delta} \text{Im } g$ . We have  $g(M) = g(S)$  and so  $M = S + \ker g$ , by assumption  $\ker g \prec_{\delta} M$  and  $\frac{M}{S}$  is singular, so  $M = S$ . Hence  $\ker f \prec_{\delta} M$ . //

**Theorem:** If  $M = M_1 + M_2$  then the following are equivalent:

- i)  $M_2$  is a small-  $M_1$ -projective.
- ii) For any sub module N of M such that  $M_1$  is a  $\delta$ -supplement of N in M. There exists a sub  $N_1$  of N such that  $M = M_1 \oplus N_1$ .

**Proof:** [14].

**Proposition:** If U is a sub module of R-module M, then following are equivalent:

- i)  $\frac{M}{M_1}$  has a M-projective  $\delta$ -cover,
- ii) If  $M_2 \leq M$  and  $M = M_1 + M_2$ ,  $M_2$  has a  $\delta$ -supplemented  $M'_1 \subseteq M_2$  such that  $M'_1$  has a M-projective  $\delta$ -cover.
- iii)  $M_2$  has a  $\delta$ -supplemented  $M'_1$ , which has a M-projective  $\delta$ -cover.

**Proof:** (i)  $\Rightarrow$ (ii) Assume that  $\frac{M}{M_1}$  has a M-projective  $\delta$ -cover. Therefore  $f : U \rightarrow \frac{M}{M_1}$  be a M-

projective  $\delta$ -cover. Since  $M = M_1 + M_2$ ,  $g : M_2 \rightarrow \frac{M}{M_1}$  is an epimorphism. Given that U is M-

projective module, then there exists an homomorphism  $h : U \rightarrow M_2$  such that  $f = g \circ h$ . By lemma Q]

$M = M_1 + \text{Im } h = M_1 + h(U)$ , where  $h(U) \prec_{\delta} M_2$ . Since  $\ker f \prec_{\delta} U$ , we have  $M_1 \cap h(U) = h(\ker f) \prec_{\delta} h(U)$  and  $h(U)$  is  $\delta$ -supplement of  $M_1$  in M. Since  $\ker h \leq \ker f \prec_{\delta} U$ ,  $h : U \rightarrow h(U)$  is M-projective  $\delta$ -cover.

(ii)  $\Rightarrow$ (iii) it is clear.

(iii)  $\Rightarrow$ (i) Let  $f : U \rightarrow M'_1$  be a M-projective  $\delta$ -cover. Since  $M'_1$  is a  $\delta$ -supplement of M, the natural

epimorphism  $g : M'_1 \rightarrow \frac{M'_1}{M_1 \cap M'_1} \cong \frac{M_1 + M'_1}{M_1} = \frac{M}{M_1}$  is M-projective  $\delta$ -cover. Hence

$f : U \rightarrow \frac{M}{M_1}$  is a M-projective  $\delta$ -cover, by lemma [A], where  $h : \frac{M'_1}{M_1 \cap M'_1} \rightarrow \frac{M_1 + M'_1}{M_1}$  is an

isomorphism. //

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