SOME RESULTS USING IMPLICIT RELATION AND E.A PROPERTY IN COMPLEX VALUED METRIC SPACE

[1]Rituja Nighojkar, [2]Mayank Sharma & [3] S.S Pagey

Department of Engg. Mathematics L.N.C.T College Bhopal (M.P)
Institute for Excellence in Higher Education Bhopal (M.P)

Abstract: - The aim of this paper is to prove common fixed point theorem for four maps via pair wise commuting maps in complex valued metric space satisfying implicit relation and E.A property.

Keywords: - Complex valued metric spaces, E.A property, implicit relation, weakly compatible mappings.
2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction and Preliminaries :

Fixed point theory is one of the famous and traditional theories in mathematics. These theorems have applications not only in different branches of mathematics, but also in economics, chemistry, biology, computer science and others. The main tools in fixed point theory is the Banach contraction theorem [1] which states that " (X,d) is a complete metric space ans $T: X \to X$ is a contraction mapping. Then T has a unique fixed point.So the banach fixed point theorem in a complete metric space has been generalized in many spaces.

Recently,Azam et al.[2] introduced the notation of complex-valued metric spaces and the theorem proved by Azam et.al[2] and Bhatt et.al[3] uses the rational inequality in a complex valued metric space as contractive condition. M. Aamri, D.El. Moutawakil, in [4] introduced the property E.A and proved Some new common fixed point theorems. Many results using the property E.A are proved in various spaces.
Now purpose of this paper is to study common fixed point results for a contractive condition and in this paper we utilize implicit relation and E.A property to prove our result

An ordinary metric d is a real valued function from a set $X \times X$ into $\mathbb{R}$, where X is a non-empty set. That is, $d: X \times X \to \mathbb{R}$. A complex number $Z \in \mathbb{C}$ is an ordered pair of real numbers, Whose first co-ordinatr is called Re(z) and second co-ordinate is called Im(z). Thus a complex-valued metric d is a function from a set $X \times X$ into $\mathbb{C}$, Where X is a nonempty set and $\mathbb{C}$ is the set of complex number. That is,$d: X \times X \to \mathbb{C}$.Let $z_1, z_2 \in \mathbb{C}$, define a partial order $\preccurlyeq$ on $\mathbb{C}$ as follows:

$z_1 \preccurlyeq z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$.
$It\ follows\ that$ $z_1 \preccurlyeq z_2$ if one of the following conditions is satisfied:

(*i*) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$.

(*ii*) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$.

(*iii*) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$.

(*iv*) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$.

In *(i)*,*(ii)*,*(iii)*, we have $|z_1| < |z_2|$. In *(iv)*, we have $|z_1| = |z_2|$. So $|z_1| \leq |z_2|$. In particular, $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i),(ii),(iii) is satisfy. In this case $|z_1| < |z_2|$. we will write $z_1 \prec z_2$ if only (iii) satisfy. Further,

$$0 \preccurlyeq z_2 \precnsim z_2 \Rightarrow |z_1| < |z_2|,$$

$$z_1 \preccurlyeq z_2\ and\ \text{write}\ z_2 \prec z_3 \Rightarrow \text{write}\ z_1 \prec z_3.$$

Azam et.al [2] defined the complex-valued metric space (X,d) in the following way;

Definition 1.1. Let X be a non-empty set. Suppose that the mapping $\mathrm{d}: \mathrm{X} \times \mathrm{X} \to \mathbb{C}$ satisfies the following conditions:

(C1) $0 \leq \mathrm{d}(\mathrm{x}, \mathrm{y})$ for all $\mathrm{x}, \mathrm{y} \in \mathrm{X}$ and $d(x, y) = 0$ if and only if $x = y$;

(C2) $d(x,y) = d(y,x)$ $for\ all\ x,y \in X$.
$(C3)$ $d(x,y) \preccurlyeq d(x,z) + d(z,y)$ $for\ all\ x,y,z \in X$.

Then d is called a complex-valued metric on X, and (X,d) is called a complex-valued metric space. point $x \in X$ is called an interior point of $A \subseteq X$ if there exists $r \in \mathbb{C}$, where $0 \prec r$, such that

$$B(x,r) = \{y \in X : d(x,y) \prec r\} \subseteq A.$$

A point $x \in X$ is called a limit point of $A \subseteq X$, if for every $o \prec r \in \mathbb{C}$,

$$B(x,r) \cap (A - X) \neq \phi.$$

The set A is called open whenever each element of A is an interior point of A.A subset B is called closed whenever each limit point of B belongs to B.

The family $\mathcal{F} \coloneqq \{B(x,r): x \in X, 0 \prec r\}$ is a sub-basis for a Hausdroff topology τ on X.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}, with\ 0 \prec c$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0, d(x_n, x) \prec c, then\ \{x_n\}$ is called convergent. Also, $\{x_n\}$ converges to x (written as, $x_n \to x\ or\ \lim_{n\to\infty} x_n = x$); and x is a limit point uof$\{x_n\}$. The sequence $\{x_n\}$ converges to x if ana only if $\lim_{n\to\infty} |d(x_n, x)| = 0$.

If for every $c \epsilon \mathbb{C}$, with $0 \prec c$ there exists $n_0 \epsilon \mathbb{N}$ such that for all $n > n_{0,}\ d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called Cauchy sequence in (X,d). If every Cauchy sequence converges in X, then X is called a complete complex-valued metric space. The sequence $\{x_n\}$ is called Cauchy if and only if $\lim_{n\to\infty} |d(x_n, x_{n+m})| = 0$.

Definition1.2 ([3]). A pair of self-mappings $A, S: X \to X$ is called weakly-compatiblecif they commute at their coincidence points. That is, if there be a point $u \epsilon X$ such that $Au = Su, then\ ASu = SAu$, for each $u \epsilon X$.

Definition1.3 [7] Let f and g be a self-maps on a set X. If $w = fx = gx$ for some $x\ in\ X$, then x is called a coincidence point of f and g , and w is called a point of coincidence f and g.

Definition1.4 [4]. Let $A, S: X \to X$ be two self-maps of a complex-valued metric space (X,d). The pair (A,S) is said to satisfy property (E.A), if there exists a sequence $\{x_n\}$ In X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \epsilon X$.

IMPLICIT RELATION

In our results,we deal with implicit relation . Let ϕ be the set of all real continuous functions $\phi: (\mathbb{R}^+)^3 \to \mathbb{R}$,non-decreasing in the first argument and satisfying the following conditions:

(ϕ_1) For $t \geq 0, \phi(0,0,t) < t\ or\ \phi(0,t,t) < t$

MAIN RESULTS

Theorem 2.1. Let (X,d) be a complex-valued metric space and $A, B, S, T: X \to X$ be four self mapping satisfying:

(i) $A(X) \subseteq T(X), B(X) \subseteq S(X)$,
(ii) $d(Ax, By) \leq k\phi[d(By, Sx), d(By, Ty), d(Ax, Sx)]$
(iii) The pairs (A,S) and (B,T) are weakly compatible.
(iv) One of the pair (A,S) or (B,T) satisfy property (E.A).

Where $k \epsilon (0,1)\ and\ \phi: [0,1]^3 \to [0,1]$, $\phi(t) \leq t$.

If the range of one of the mappings S(X) or T(X) is a complete subspace of X then mapping A,B,S and T have a unique common fixed point in X.

Proof:- First suppose that the pair (B,T) satisfy property (E.A). Then by definition, there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = t$. For some $t\epsilon X$.

Further $B(X) \subseteq S(X)$, there exist a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence $\lim_{n\to\infty} Sy_n = t$. We claim that $\lim_{n\to\infty} Ay_n = t$. If not, then putting $x = y_{n,} y = x_n$ in condition (ii), we have

$$d(Ay_n, Bx_n) \leq K\phi[d(Bx_n, Sy_n), d(Bx_n, Tx_n), d(Ay_n, Sy_n)]$$

$$\leq K\phi[d(t,t), d(t,t), d(Ay_n, t)]$$

$$\leq K\phi[0,0, d(Ay_n, t)]$$

$$d(Ay_n, t) \leq Kd(Ay_n, t)$$

$$d(Ay_n, t) \leq 0.$$

Which is a contradiction. Thus $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = t$.

Now,suppose first that S(X) is a complete subspace of X, then $t = Su$ for some $u\epsilon X$.Subsequently, we have

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = t = Su \ldots\ldots\ldots\ldots\ldots.(2.1)$$

We claim that Au=Su. For putting $x = u \ and \ y = x_n$ in (ii) we have

$$d(Au, Bx_n) \leq K\phi[d(Bx_n, Su), d(Bx_n, Tx_n), d(Au, Su)]$$

Letting $n \to \infty$ and using eq.(2.10, we have

$$d(Au, t) \leq K\phi[d(t,t), d(t,t), d(Au, t)]$$

$$\leq K\phi[0,0, d(Au, t)]$$

Hence $d(Au, t) = 0$

Whence Au=t=Su. Hence u is a coincidence point of (A,S). Now, the weak compatibility of pair (A,S) implies that $ASu = SAu, \ or \ At = St$.

On the other hand, since $A(X) \subseteq T(X)$, there exist v in X such that $Au = Tv$. Thus $Au = Su = Tv = t$. Let us show that v is a coincidence point of (B,T), i.e, $Bv = Tv = t$. if not, then putting $x = u, y = v$ in (ii), we have

$$d(Au, Bv) \leq K\phi[d(Bv, Su), d(Bv, Tv), d(Au, Su)]$$

Or

$$d(t, Bv) \leq K\phi[d(Bv, t), d(Bv, t), d(t,t)]$$

$$d(t, Bv) \leq K\phi[d(Bv, t), d(Bv, t), 0]$$

$$d(Bv, t) \leq Kd(Bv, t)$$

$$d(Bv, t) \leq 0$$

Which is a contradiction. Thus$Bv = t$. Hence, $Bv = Tv = t$, and v is coincidence point of B and T. Further the weak compatibility of pair (B,T) implies that $BTv = TBv \ or \ Bt = Tt$. Therefore t is a common coincidence point of A,B,S and T.

In order to show that t is a common fixed point, let us put $x = u \ and \ y = t$ in (ii) we have

$$d(t, Bt) = d(Au, Bt) \leq K\phi[d(Bt, Su), d(Bt, Tt), d(Au, Su)]$$

$$\leq K\phi[d(Bt, t), d(Bt, t), d(t,t)]$$

$$\leq K\phi[d(Bt,t),d(Bt,t),0]$$

$$d(t,Bt) \leq Kd(t,Bt)$$

$$(1-K)d(t,Bt) \leq 0$$

$$d(t,Bt) \leq 0$$

Which is a contradiction. Thus $Bt = t. Hence\ At = Bt = St = Tt = t.$

Similar argument arises if we assume that T(X) is a complete subspace of X. Similarly, the property (E.A) of the pair (A,S) will give the similar result.

For uniqueness of common fixed point, let us assume that ω be another common fixed point of A, B, S, T. Then putting $x = \omega, y = t$ in (ii) we have

$$d(\omega,t) = d(A\omega,Bt) \leq K\phi[d(Bt,S\omega),d(Bt,Tt),d(A\omega,S\omega)]$$

$$\leq K\phi[d(t,\omega),d(t,t),d(\omega,\omega)]$$

$$d(\omega,t) \leq K\phi[d(\omega,t),0,0]$$

$$(1-K)d(\omega,t) \leq 0$$

$$d(\omega,t) < 0$$

Which is a contradiction. Thus $\omega = t$. Hence $At = Bt = St = Tt = t$. and t is th e unique common fixed point of A,B,S,T.This Completes the proof.

If A=B *and* S=T in Theorem 2.1,we have the following result:

Corollary 2.2. Let (X,d) be a complex-valued metric space and $A,S: X \to X$ be two self mapping satisfying:

(i) $A(X) \subseteq S(X)$,

(ii) $d(Ax,Ay) \leq k\phi[d(Ay,Sx),d(Ay,Sy),d(Ax,Sx)]$

(iii) The pairs (A,S) is weakly compatible.

(iv) One of the pair (A,S) satisfy property (E.A).

Where $k\epsilon(0,1)$, $\phi: [0,1]^3 \to [0,1]$ *and* $\phi(t) \leq t.$

If the range S(X) is a complete subspace of X then mapping A and S have a unique common fixed point in X.

References

[1] S. Banach, S ̈ur les op ́erations dans les ensembles abstraits et leur application aux ˜equations int ́egrales, Fund. Math. 3 (1922) 133–181.
[2] A. Azam, B. Fisher, M. Khan, Common fixed point theorems in complex valued metric spaces, Numer. Funct. Anal. Optim. 3 (3) (2011) 243–253.
[3] S. Bhatt, S, Chaukiyal, R.C. Dimri, A common fixed point theorem for weakly compatible maps in complex-valued metric spaces, Int. J. Math. Sci. Appl. 1(3) (2011) 1385–1389.
[4] M. Aamri, D.El. Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 27 (1) (2002) 181–188.
[5] H.K. Pathak, R.R. Lop ́ez, R.K. Verma, A common fixed point theorem of integral type using implicit relation, Nonlinear Funct. Anal. Appl. 15 (1)(2009) 1–12.
[6] W. Sintunavarat, P. Kumam, Generalized common fixed point theorems in complex valued metric spaces and application, Journal of Inequalities and Applications 2012, 2012:84.
[7] G.Jungck and B.E.Rhoades, Fixed points for set valued functions without continuity, Indian.J.Pure Appl.Math.,29 (1998), 227-238.