Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.5, No.7, 2015

A common fixed point theorem for fuzzy 2-metric space using common E.A. like property

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Abstract:

In this paper, we are proving a common fixed point theorem for mappings satisfying common E.A. like property in fuzzy 2-metric spaces, which improves and generalize result of Yadav and Thakur [5].

Keywords: Fuzzy 2-metric spaces, weakly compatible mapping, common E.A. like property.

1. Introduction:

Zadeh [6] introduced the concept of fuzzy set in 1965. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, mathematical programming, engineering sciences, medical sciences (medical genetics nervous system), image processing control theory, communication etc. Kramosil and Michalek [1] introduced the concept of fuzzy metric space in 1975, which opened an avenue for further development of analysis in such spaces. Pant [2] introduced the notation of R-weakly commutativity of mappings in metric spaces and proved some common fixed point theorem. Vasuki [3] proved the fuzzy version of Pant's Theorem. Recently, Yadav and Thakur [5] generalized the result of Vasuki [3] for Fuzzy 2-Metric Spaces. On the other hand, Wadhwa et al. [4] introduced the notion of common E.A. like property and proved some common fixed point theorems in fuzzy metric spaces.

In this paper we prove a common fixed point theorem for mappings satisfying common E.A. like property in fuzzy 2-metric spaces, which improves and generalize the result of [5].

2. Preliminaries:

Definition 2.1 [5]: An operation $*$: $[0.1]^3 \rightarrow [0, 1]$ is called a t-norm of $\{[0.1], *\}$ is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2$ whenever $a_1 \leq a_2$, $b_1 \leq b_2$, $c_1 \leq c_2$, $\forall$ $a_1, b_1, c_1, a_2, b_2, c_2 \in [0, 1]$.

Definition 2.2 [5]: A 3-taple (X, M, *) is said to be a fuzzy 2- metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^3 \times [0, \infty)$ satisfying the following conditions; $\forall$ $x, y, z \in X$, $s, t > 0$

1) $M(x, y, z, 0) = 0$,

2) $M(x, y, z, t) = 1$ for all $t > 0$ if and only if at least two of three points are equal,

3) $M(x, y, z, t) = M(y, x, z, t) = M(z, x, y, t)$ symmetry about three variables,

4) $M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3) \leq M(x, y, z, t_1+t_2+t_3)$ $\forall$ $x, y, z, u \in X$ and $t_1, t_2, t_3 > 0$,

5) $M(x, y, z, \cdot): [0, \infty) \rightarrow [0, 1]$ is left continuous,

6) $\lim_{t \rightarrow \infty} M(x, y, z, t) = 1$.

The function value M(x, y, z, t) may be interpreted as the probability that the area of triangle is less than t.

Definition 2.3: A pair of self mapping {f, g} of a fuzzy 2-metric space (X, M, *) is said to be weakly compatible if they commute at the coincidence point i.e., If $fu = gu$ for some $u \in X$, then $fgu = gfu$.

Definition 2.4: Let f and g be two self-maps of a fuzzy metric space (X, M, *) then they are said to satisfy E.A property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\rightarrow\infty} fx_n = \lim_{n\rightarrow\infty} gx_n = z \text{ for some } z \in X.$$

Now E.A. property in fuzzy 2-metric spaces defined as follow:

Definition 2.5: A pair of self-mapping {f, g} of a fuzzy 2-metric spaces (X, M, *) is said to satisfy E.A property, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} M(fx_n, gx_n, a, t) = t \text{ for some } t\in X$$

Definition 2.6 [4]: Let A, B, S and T be self maps of a fuzzy metric space (X, M, ∗), then the pairs (A, S) and (B, T) said to satisfy common E. A. Like property if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z,$$

where $z \in S(X) \cap T(X)$ or $z \in A(X) \cap B(X)$.

Lemma 2.7: Let (X, M, *) be a fuzzy 2-metric space. If there exists $k\in(0, 1)$ such that

$M(x, y, z, kt) \geq M(x, y, z, t)$ for all $x, y, z\in X$ with $z\neq x$, $z\neq y$ and $t>0$, then $x=y$.

D.S. Yadav and S.S Thakur [5] proved following result:

Theorem 2.8 [5]: Let (X, M, *) be a complete fuzzy 2- metric space and let f and g be R- weakly commuting self mappings of X satisfying the conditions:

$$M(fx, fy, w, t) \geq r(M(gx, gy, w, t)),$$

Where r: [0, 1] → [0, 1] is a continuous function such that r (t) > t for each $0 < t < 1$. The sequence $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n\to x$, $y_n\to y$, $t > 0$ implies

$$M(x_n, y_n, w, t)\to M(x, y, w, t),$$

If the range of g contains the range of f and if either f or g is continuous, then f and g have unique common fixed point.

We use following in our results:

(2.9.1) Let Φ be the set of all real continuous functions F: $[0, 1]^5 \to [0, 1]$ non decreasing in each coordinate variable and such that

$$F(t, 1, 1, 1, 1) \geq t,\ F(1, t, 1, 1, 1) \geq t,\ F(t, t, t, t, 1) \geq t;\ \ \forall\, t\in[0, 1].$$

Now we give our main result:

3. Main Results:

Theorem 3.1: Let A, B, S and T be self maps of a fuzzy 2-metric spaces (X, M, *) satisfying the following condition:

(3.1.1) For some $F\in\Phi$ there exists a constant $k\in(0, 1)$ such that

$\forall x, y, z\in X$ and $t > 0$, $\alpha \geq 0$,

$$M(Ax, By, z, kt) \geq F\left\{\begin{matrix} M(Ax, Ty, z, t), M(Sx, By, z, t), M(Ty, Sx, z, t), \\ \dfrac{\alpha M(Ty, Ax, z, t) + M(Ty, Sx, z, t)}{1 + \alpha M(Ax, Sx, z, t)}, \dfrac{M(By, Sx, z, t) + M(Ax, Sx, z, t)}{1 + M(Ax, By, z, t)} \end{matrix}\right\};$$

(3.1.2) Pairs (A, S) and (B, T) satisfy common E.A. like property.

(3.1.3) Pairs (A, S) and (B, T) are weakly compatible.

Then A, B, S and T have a unique common fixed point in X.

Proof: Since (A, S) and (B, T) satisfy common E. A. Like property therefore there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z_1$$

where $z_1 \in S(X)\cap T(X)$ or $z_1 \in A(X)\cap B(X)$.

Suppose $z_1 \in S(X)\cap T(X)$, now we have $\lim_{n\to\infty} Ax_n = z_1 \in S(X)$ then $z_1 = Su$ for some $u \in X$.

No we claim that Au = Su, from (3.1.1) we have,

$$M(Au, By_n, z, kt) \geq F\left\{\begin{matrix} M(Au, Ty_n, z, t), M(Su, By_n, z, t), M(Ty_n, Su, z, t), \\ \frac{\alpha M(Ty_n, Au, z, t) + M(Ty_n, Su, z, t)}{1+\alpha M(Au, Su, z, t)}, \frac{M(By_n, Su, z, t) + M(Au, Su, z, t)}{1+M(Au, By_n, z, t)} \end{matrix}\right\};$$

Taking limit $n \to \infty$, we get

$$M(Au, z_1, z, kt) \geq F\left\{\begin{matrix} M(Au, z_1, z, t), M(z_1, z_1, z, t), M(z_1, z_1, z, t), \\ \frac{\alpha M(z_1, Au, z, t) + M(z_1, z_1, z, t)}{1+\alpha M(Au, z_1, z, t)}, \frac{M(z_1, z_1, z, t) + M(Au, z_1, z, t)}{1+M(Au, z_1, z, t)} \end{matrix}\right\};$$

$$M(Au, z_1, z, kt) \geq F\left\{\begin{matrix} M(Au, z_1, z, t), 1, 1, \\ \frac{\alpha M(z_1, Au, z, t) + 1}{1+\alpha M(Au, z_1, z, t)}, \frac{1+M(Au, z_1, z, t)}{1+M(Au, z_1, z, t)} \end{matrix}\right\} = F\{M(Au, z_1, z, t), 1, 1, 1, 1\};$$

Using (2.9.1) we have,

$$M(Au, z_1, z, kt) \geq M(Au, z_1, z, t);$$

Lemma 2.7 implies that $Au = z_1 = Su$.

Since the pair (A, S) is weak compatible, therefore $Az_1 = ASu = SAu = Sz_1$.

Again, $\lim_{n\to\infty} By_n = z_1 \in T(X)$ then $z_1 = Tv$ for some $v \in X$.

No we claim that $Tv = Bv$, from (3.1.1) we have,

$$M(Ax_n, Bv, z, kt) \geq F\left\{\begin{matrix} M(Ax_n, Tv, z, t), M(Sx_n, Bv, z, t), M(Tv, Sx_n, z, t), \\ \frac{\alpha M(Tv, Ax_n, z, t) + M(Tv, Sx_n, z, t)}{1+\alpha M(Ax_n, Sx_n, z, t)}, \frac{M(Bv, Sx_n, z, t) + M(Ax_n, Sx_n, z, t)}{1+M(Ax_n, Bv, z, t)} \end{matrix}\right\};$$

Taking limit $n \to \infty$, we get,

$$M(z_1, Bv, z, kt) \geq F\left\{\begin{matrix} M(z_1, z_1, z, t), M(z_1, Bv, z, t), M(z_1, z_1, z, t), \\ \frac{\alpha M(z_1, z_1, z, t) + M(z_1, z_1, z, t)}{1+\alpha M(z_1, z_1, z, t)}, \frac{M(Bv, z_1, z, t) + M(z_1, z_1, z, t)}{1+M(z_1, Bv, z, t)} \end{matrix}\right\} = F\left\{\begin{matrix} 1, M(z_1, Bv, z, t), \\ 1,1,1 \end{matrix}\right\};$$

Using (2.9.1) we have, $M(z_1, Bv, z, kt) \geq M(z_1, Bv, z, t)$;

Lemma 2.7 implies that $Bv = z_1 = Tv$.

Since the pair (B, T) is weak compatible, therefore $Tz_1 = TBv = BTv = Bz_1$.

Now we show that $Az_1 = z_1$, from (3.1.1) we have,

$$M(Az_1, By_n, z, kt) \geq F\left\{\begin{matrix} M(Az_1, Ty_n, z, t), M(Sz_1, By_n, z, t), M(Ty_n, Sz_1, z, t), \\ \frac{\alpha M(Ty_n, Az_1, z, t) + M(Ty_n, Sz_1, z, t)}{1+\alpha M(Az_1, Sz_1, z, t)}, \frac{M(By_n, Su, z, t) + M(Az_1, Sz_1, z, t)}{1+M(Az_1, By_n, z, t)} \end{matrix}\right\};$$

Taking limit $n \to \infty$, we get

$$M(Az_1, z_1, z, kt) \geq F\left\{\begin{matrix} M(Az_1, z_1, z, t), M(z_1, z_1, z, t), M(z_1, z_1, z, t), \\ \frac{\alpha M(z_1, Az_1, z, t) + M(z_1, z_1, z, t)}{1+\alpha M(Az_1, z_1, z, t)}, \frac{M(z_1, z_1, z, t) + M(Az_1, z_1, z, t)}{1+M(Az_1, z_1, z, t)} \end{matrix}\right\};$$

$$M(Az_1, z_1, z, kt) \geq F\left\{\begin{matrix} M(Az_1, z_1, z, t), 1, 1, \\ \frac{\alpha M(z_1, Az_1, z, t) + 1}{1+\alpha M(Az_1, z_1, z, t)}, \frac{1+M(Az_1, z_1, z, t)}{1+M(Az_1, z_1, z, t)} \end{matrix}\right\} = F\{M(Az_1, z_1, z, t), 1, 1, 1, 1\};$$

Using (2.9.1) we have,

$$M(Az_1, z_1, z, kt) \geq M(Az_1, z_1, z, t);$$

Lemma 2.7 implies that $Az_1 = z_1$.

Now we show that $Bz_1 = z_1$, from (3.1.1) we have,

$$M(Ax_n, Bz_1, z, kt) \geq F\left\{\begin{matrix} M(Ax_n, Tz_1, z, t), M(Sx_n, Bz_1, z, t), M(Tz_1, Sx_n, z, t), \\ \frac{\alpha M(Tz_1, Ax_n, z, t) + M(Tz_1, Sx_n, z, t)}{1+\alpha M(Ax_n, Sx_n, z, t)}, \frac{M(Bz_1, Sx_n, z, t) + M(Ax_n, Sx_n, z, t)}{1+M(Ax_n, Bz_1, z, t)} \end{matrix}\right\};$$

Taking limit $n \to \infty$, we get,

$$M(z_1, Bz_1, z, kt) \geq F\left\{\begin{matrix} M(z_1, z_1, z, t), M(z_1, Bz_1, z, t), M(z_1, z_1, z, t), \\ \frac{\alpha M(z_1,z_1,z,t)+M(z_1,z_1,z,t)}{1+\alpha M(z_1,z_1,z,t)}, \frac{M(Bz_1,z_1,z,t)+M(z_1,z_1,z,t)}{1+M(z_1,Bz_1,z,t)} \end{matrix}\right\};$$

$$M(z_1, Bz_1, z, kt) \geq F\{1, M(z_1, Bz_1, z, t), 1,1,1\};$$

Using (2.9.1) we have,

$$M(z_1, Bz_1, z, kt) \geq M(z_1, Bz_1, z, t);$$

Lemma 2.7 implies that $Bz_1 = z_1$.

Hence, $Az_1 = Sz_1 = Bz_1 = Tz_1 = z_1$.

Thus z_1 is common fixed point of A, B, S and T.

To prove uniqueness we suppose that p and q are two common fixed point of A, B, S and T such that $p \neq q$, then from (3.1.1) we have,

$$M(Ap, Bq, z, kt) \geq F\left\{\begin{matrix} M(Ap, Tq, z, t), M(Sp, Bq, z, t), M(Tq, Sp, z, t), \\ \frac{\alpha M(Tq,Ap,z,t)+M(Tq,Sp,z,t)}{1+\alpha M(Ap,Sp,z,t)}, \frac{M(Bq,Sp,z,t)+M(Ap,Sp,z,t)}{1+M(Ap,Bq,z,t)} \end{matrix}\right\};$$

$$M(p, q, z, kt) \geq F\left\{\begin{matrix} M(p, q, z, t), M(p, q, z, t), M(q, p, z, t), \\ \frac{\alpha M(q,p,z,t)+M(q,p,z,t)}{1+\alpha M(p,p,z,t)}, \frac{M(q,p,z,t)+M(p,p,z,t)}{1+M(p,q,z,t)} \end{matrix}\right\};$$

$$M(p, q, z, kt) \geq F\left\{\begin{matrix} M(p, q, z, t), M(p, q, z, t), M(q, p, z, t), \\ M(q, p, z, t), 1 \end{matrix}\right\};$$

Using (2.9.1) we have,

$$M(p, q, z, kt) \geq M(p, q, z, t);$$

Lemma 2.7 implies that $p = q$. This completes the proof of the theorem.

Remark 3.2: Theorem 3.1 never requires the containments of ranges, completeness of the space and not involves continuity of the mappings.

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