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# A NOTE ON SPECIAL GEODESIC MAPPINGS ON RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we have studied some properties of special geodesic mappings on Riemannian manifolds. We have shown that if $f : (M^n, g) \to (\bar{M}^n, \bar{g})$ is a special geodesic mapping of a flat Riemannian manifold $(M^n, g)$ onto a Riemannian manifold $(\bar{M}^n, \bar{g})$, then $\bar{M}^n$ is of constant curvature.

## 1. Introduction

During last 20 years the theory of geodesic mapping of affine - connected, Riemannian and Kahlerian spaces has been attractive field of investigation and many new and interesting results have appeared. The geodesic problems was first posed by E. Beltrami [2, 3]. Significant contributions to the investigation of the general laws of this theory were made by T. Levi- Civita [6], T. Y. Thomas[16], A. S. Solodovnikov[14, 15] G. I. Kruchkovich[5], N. S. Sinyukov[13], L. P. Einsenhart[4], A. Z. Petrov[12], A. P. Norden[11] and others.

## 2. Preliminaries

Let $M^n$ and $\bar{M}^n$ be a $n$-dimensional Riemannian manifold with metric $g$ and $\bar{g}$ and Levi-Civita connections $\nabla$ and $\bar{\nabla}$ respectively. A diffeomorphism $f : M^n \to \bar{M}^n$ is called a geodesic mapping of $M^n$ and $\bar{M}^n$ if $f$ maps any geodesic in $M^n$ onto a geodesic in $\bar{M}^n$. It is known that [7, 8] a manifold $M^n$ admits a geodesic mapping onto $\bar{M}^n$ if and only if the Levi-Civita equations

$$\bar{\nabla}_X Y = \nabla_X Y + \pi(Y)X + \pi(X)Y \tag{2.1}$$

holds for any tangent vectors $X$, $Y$ and $\pi$ is a differential 1-form. In local form, we may write (2.1) as

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \pi_j \delta^h_i + \pi_i \delta^h_j, \tag{2.2}$$

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where $\bar{\Gamma}^h_{ij}$ and $\Gamma^h_{ij}$ are the Christoffel symbols of $\bar{M}^n$ and $M^n$, $\pi_i$ are components of $\pi$ and $\delta^h_i$ is the Kronecker delta.
The condition (2.2) is equivalent to the following Levi-Civita equation [7]

$$g_{jk,i} = 2\pi_i\, g_{jk} + \pi_j\, g_{ik} + \pi_k\, g_{ij}, \tag{2.3}$$

where ","denotes the covariant differentiation in $M^n$ and $\pi_i$ is some gradient like vector i.e., $\pi_i = \pi_{,i}$. It is known that[7]

$$\pi_i = \partial_i \pi, \quad \pi_i = \frac{1}{2n+1}.\frac{\partial}{\partial x^i}\log|\frac{det\,\bar{g}}{det\,g}|, \quad \partial_i = \frac{\partial}{\partial x^i}.$$

If $\pi_i \neq 0$ holds, then the mapping is called non-trivial geodesic mapping; otherwise trivial of affine.

The Curvature tensor of a Riemannian manifold $M^n$ is given by [10]

$$R^h_{ijk} = \frac{\partial}{\partial x^i}\Gamma^h_{jk} - \frac{\partial}{\partial x^j}\Gamma^h_{ik} - \Gamma^h_{im}\Gamma^m_{jk} - \Gamma^h_{jm}\Gamma^m_{ik}. \tag{2.4}$$

If a mapping $f : M^n \longrightarrow \bar{M}^n$ is geodesic then from (2.2) and (2.4), we obtain the following relation:

$$\bar{R}^h_{ijk} = R^h_{ijk} - \psi_{jk}\delta^h_i + \psi_{ik}\delta^h_j, \tag{2.5}$$

where $R^h_{ijk}$ and $\bar{R}^h_{ijk}$ are the Riemannian curvature tensors of the manifold $M^n$ and $\bar{M}^n$ respectively and $\psi_{ij}$ is given by

$$\psi_{ij} = \pi_{i,j} - \pi_i\pi_j. \tag{2.6}$$

In index free notation (2.5) and (2.6) can be written as

$$\bar{R}(X,Y,Z) = R(X,Y,Z) - \psi(Y,Z)X + \psi(X,Z)Y \tag{2.7}$$

and

$$\psi(Y,Z) = (\nabla_Y \pi)Z - \pi(Y)\pi(Z). \tag{2.8}$$

Contracting $X$ in the equation (2.7), we get the following relation between Ricci tensors $\bar{R}ic(Y,Z)$ and $Ric(Y,Z)$ of manifolds $\bar{M}^n$ and $M^n$ respectively

$$\bar{R}ic(Y,Z) = Ric(Y,Z) - (n-1)\psi(Y,Z). \tag{2.9}$$

## 3. A Special Geodesic Mapping $f : M^n \to \bar{M}^n$

In this section, we consider a geodesic mapping $f : M^n \to \bar{M}^n$ whose associated $1-$ form $\pi$ satisfies

$$(\bar{\nabla}_X \pi)(Y) = \pi(X)\pi(Y), \tag{3.1}$$

i.e., $\pi$ is recurrent with respect to Levi-Civita connection $\bar{\nabla}$. We call such a geodesic mapping as special geodesic mapping. Now, we have

$$(\bar{\nabla}_X \pi)(Y) = X(\pi(Y)) - \pi(\bar{\nabla}_X Y).$$

Using (2.1) in above, we get

$$(\bar{\nabla}_X \pi)(Y) = (\nabla_X \pi)(Y) - 2\pi(X)\pi(Y), \tag{3.2}$$

which, on using (3.1), gives as

$$(\nabla_X \pi)(Y) = 3\pi(X)\pi(Y). \tag{3.3}$$

Again, in view of equation of (2.8), we get from above

$$\psi(X, Y) = 2\pi(X)\pi(Y). \tag{3.4}$$

Due to above equation, expressions for curvature tensor, Ricci tensor and scalar curvature tensor given by equations (2.7) and 2.9), takes the following forms,

$$\bar{R}(X, Y, Z) = R(X, Y, Z) - 2\pi(Y)\pi(Z)X + 2\pi(X)\pi(Z)Y \tag{3.5}$$

and

$$\bar{R}ic(Y, Z) = Ric(Y, Z) - 2(n-1)\pi(Y)\pi(Z). \tag{3.6}$$

Now, we prove following theorems:

**Theorem 3.1.** *Let $f : (M^n, g) \to (\bar{M}^n, \bar{g})$ be special geodesic mapping of a Riemannian manifold $(M^n, g)$ onto a Riemannian manifold $(\bar{M}^n, \bar{g})$. Then associated $2-$ form $\psi(Y, Z)$ of special geodesic mapping satisfies*

$$(\bar{\nabla}_X \psi)(Y, Z) = 2\pi(X)\psi(Y, Z).$$

**Proof:** We have from equation (3.4)

$$(\bar{\nabla}_X \psi)(Y, Z) = 2(\bar{\nabla}_X \pi)(Y)\pi(Z) + 2\pi(Y)(\bar{\nabla}_X \pi)(Z),$$

which, in view of equation (3.1), gives

$$(\bar{\nabla}_X \psi)(Y, Z) = 2\pi(X)2\pi(Y)\pi(Z), \tag{3.7}$$

which, due to equation (3.4), yields

$$(\bar{\nabla}_X \psi)(Y, Z) = 2\pi(X)\psi(Y, Z). \tag{3.8}$$

This is complete proof of the theorem.

**Theorem 3.2.** *Let $f : (M^n, g) \to (\bar{M}^n, \bar{g})$ be special geodesic mapping of a Riemannian manifold $(M^n, g)$ onto a Riemannian manifold $(\bar{M}^n, \bar{g})$. Then associated $2-$ form $\psi(Y, Z)$ of special geodesic mapping satisfies*

$$(\nabla_X \psi)(Y, Z) = 5\pi(X)\psi(Y, Z).$$

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**Proof:** We have

$$(\bar{\nabla}_X\psi)(Y,Z) = X(\psi(Y,Z)) - \psi(\bar{\nabla}_X Y, Z) - \psi(Y, \bar{\nabla}_X Z).$$

Using equation (2.1) in above, we get

$$\begin{aligned}(\bar{\nabla}_X\pi)(Y) =& X(\psi(Y,Z)) - \psi(\nabla_X Y + \pi(Y)X + \pi(X)Y, Z)\\ &-\psi(Y, \nabla_X Z + \pi(Z)X + \pi(X)Z, Z).\end{aligned}$$

After simplification and using equation (3.4), we arrive at

$$(\bar{\nabla}_X\psi)(Y,Z) = (\nabla_X\psi)(Y,Z) - 8\pi(X)\pi(Y)\pi(Z),$$

which in view of (3.7), gives

$$(\nabla_X\psi)(Y,Z) = 5\pi(X).2\pi(Y)\pi(Z). \tag{3.9}$$

Again using equation (3.4) in above, we get

$$(\nabla_X\psi)(Y,Z) = 5\pi(X)\psi(Y,Z). \tag{3.10}$$

This proves the statement.

**Theorem 3.3.** *Let $f : (M^n, g) \to (\bar{M}^n, \bar{g})$ be special geodesic mapping of a Riemannian manifold $(M^n, g)$ onto a Riemannian manifold $(\bar{M}^n, \bar{g})$. Then the tensor $D(X,Y,Z)$ defined by*

$$D(X,Y,Z) = (\bar{\nabla}_X\psi)(Y,Z) - (\bar{\nabla}_Y\psi)(X,Z) \tag{3.11}$$

*vanishes.*

**Proof:** Proof follows from the equations (3.4), (3.8) and (3.11).

**Theorem 3.4.** *Let $f : (M^n, g) \to (\bar{M}^n, \bar{g})$ be special geodesic mapping of a Riemannian manifold $(M^n, g)$ onto a Riemannian manifold $(\bar{M}^n, \bar{g})$. Then the tensor $E(X,Y,Z)$ defined by*

$$E(X,Y,Z) = (\nabla_X\psi)(Y,Z) - (\nabla_Y\psi)(X,Z) \tag{3.12}$$

*vanishes.*

**Proof:** Proof follows from the equations (3.4), (3.10) and (3.12).

**Theorem 3.5.** *If $f : (M^n, g) \to (\bar{M}^n, \bar{g})$ is a special geodesic mapping of a Riemannian manifold $(M^n, g)$ onto a Riemannian manifold $(\bar{M}^n, \bar{g})$, then*

$$(\bar{\nabla}_U\bar{R})(X,Y,Z) = (\bar{\nabla}_U R)(X,Y,Z) - 2\pi(U)\psi(Y,Z)X + 2\pi(U)\psi(X,Z)Y. \tag{3.13}$$

**Proof:** Proof follows from the equations (2.7) and (3.8).

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**Theorem 3.6.** *If $f : (M^n, g) \to (\bar{M}^n, \bar{g})$ is a special geodesic mapping of a Riemannian manifold $(M^n, g)$ onto a Riemannian manifold $(\bar{M}^n, \bar{g})$, then*

$$(\bar{\nabla}_X \bar{R}ic)(Y, Z) = (\bar{\nabla}_X Ric)(Y, Z) - 4(n-1)\pi(X)\pi(Y)\pi(Z). \quad (3.14)$$

**Proof:** Proof follows from the equations (3.1) and (3.6).

**Theorem 3.7.** *Let $f : (M^n, g) \to (\bar{M}^n, \bar{g})$ be special geodesic mapping of a flat Riemannian manifold $(M^n, g)$ onto a Riemannian manifold $(\bar{M}^n, \bar{g})$. Then $\bar{M}^n$ is of constant curvature.*

**Proof:** If $M^n$ is flat, then from equations (3.5) and (3.6), we have

$$\bar{R}(X, Y, Z) = -2\pi(Y)\pi(Z)X + 2\pi(X)\pi(Z)Y \quad (3.15)$$

and

$$\bar{R}ic(Y, Z) = -2(n-1)\pi(Y)\pi(Z). \quad (3.16)$$

In view of equations (3.15) and (3.16), we have

$$\bar{R}(X, Y, Z) = \frac{1}{n-1}[\bar{R}ic)(Y, Z)X - \bar{R}ic)(X, Z)Y], \quad (3.17)$$

which proves the result.

**Corollary 3.1.** *Let $f : (M^n, g) \to (\bar{M}^n, \bar{g})$ be special geodesic mapping of a flat Riemannian manifold $(M^n, g)$ onto a Riemannian manifold $(\bar{M}^n, \bar{g})$. Then $\bar{M}^n$ is symmetric iff it is Ricci symmetric.*

**Proof:** Differentiating (3.17) covariantly, we get

$$(\bar{\nabla}_U \bar{R})(X, Y, Z) = \frac{1}{n-1}[(\bar{\nabla}_U \bar{R}ic)(Y, Z)X - (\bar{\nabla}_U \bar{R}ic)(X, Z)Y]. \quad (3.18)$$

From above we see that if $\bar{M}^n$ is Ricci symmetric, then $\bar{M}^n$ is symmetric. Also if $\bar{M}^n$ is symmetric, then $\bar{M}^n$ is Ricci symmetric always true. This complete the proof.

**Theorem 3.8.** *Let $f : (M^n, g) \to (\bar{M}^n, \bar{g})$ be special geodesic mapping of a Riemannian manifold $(M^n, g)$ onto a Riemannian manifold $(\bar{M}^n, \bar{g})$. Then following relation holds:*

(1) *if $M^n$ is flat manifold , then $\bar{M}^n$ is divergence free and*
(2) *if $\bar{M}^n$ is flat, then $M^n$ is divergence free.*

**Proof:** (1) Divergence of the curvature tensor of a Riemannian manifold $\bar{M}^n$ is given by [1]

$$(div\bar{R})(X, Y, Z) = (\nabla_X \bar{R}ic)(Y, Z) - (\nabla_Y \bar{R}ic)(X, Z). \quad (3.19)$$

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Suppose $M^n$ is flat, i.e. $R(X,Y,Z) = 0$, therefore from equation (3.6), we have

$$\bar{R}ic(Y,Z) = -2(n-1)\pi(Y)\pi(Z).$$

From this, we get

$$(\bar{\nabla}_X \bar{R}ic)(Y,Z) = -2(n-1)[(\bar{\nabla}_X \pi)(Y)\pi(Z) + \pi(Y)(\bar{\nabla}_X \pi)(Z)]$$

which, due to the equation (3.1), gives

$$(\bar{\nabla}_X \bar{R}ic)(Y,Z) = -4(n-1)\pi(X)\pi)(Y)\pi(Z).$$

Clearly, we have

$$(\bar{\nabla}_X \bar{R}ic)(Y,Z) - (\bar{\nabla}_Y \bar{R}ic)(X,Z) = 0. \tag{3.20}$$

Using equation (3.20) in equation (3.19), we get

$$(div\bar{R})(X,Y,Z) = 0, \tag{3.21}$$

which shows that $\bar{M}^n$ is divergence free.
Similarly we can prove the other part (2).

**Theorem 3.9.** *Let $f : (M^n, g) \to (\bar{M}^n, \bar{g})$ be special geodesic mapping of a Riemannian manifold $(M^n, g)$ onto a Riemannian manifold $(\bar{M}^n, \bar{g})$. Then $\bar{M}^n$ is Ricci flat if and only if*

$$W(X,Y,Z) = \bar{R}(X,Y,Z). \tag{3.22}$$

.

**Proof:** The Weyl Projective curvature tensor $W$ of $M^n$ is given by [9]

$$W(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-1}[Ric(Y,Z)X - Ric(X,Z)Y]. \tag{3.23}$$

Suppose the equation (3.22) hold. Then from equation (3.23), we get

$$\bar{R}(X,Y,Z) = R(X,Y,Z) - \frac{1}{n-1}[Ric(Y,Z)X - Ric(X,Z)Y]. \tag{3.24}$$

Using equation (3.5) in above equation, we get

$$2\pi(X)\pi(Z)Y - 2\pi(Y)\pi(Z)X = \frac{1}{n-1}[Ric(X,Z)Y - Ric(Y,Z)X]. \tag{3.25}$$

In view of equation (3.6), the above equation, gives

$$\bar{R}ic(X,Z)Y = \bar{R}ic(Y,Z)X,$$

which on contraction, gives

$$\bar{R}ic(Y,Z) = 0.$$

This shows that $\bar{M}^n$ is Ricci flat manifold.

Conversely, suppose $\bar{M}^n$ is Ricci flat manifold. Then from equation (3.6), we have

$$Ric(Y, Z) = 2(n-1)\pi(Y)\pi(Z). \tag{3.26}$$

Using this in equation (3.23), we get

$$W(X,Y,Z) = R(X,Y,Z) + \frac{1}{n-1}[2(n-1)\pi(X)\pi(Z)Y - 2(n-1)\pi(Y)\pi(Z)X] \tag{3.27}$$

Now, equations (3.5) and (3.27), we get

$$W(X,Y,Z) = \bar{R}(X,Y,Z).$$

This completes the proof.

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