

Best Approximation in Space $L_{p,\alpha}[I]$ $\alpha > 0 . 0 < p < 1$, $I = [a, b]$ By Means of Modulus of Smoothness.

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Abstract

The aim of this work is to estimate the degree of best approximation for unbounded function $f \in L_{p,\alpha}[I], \alpha > 0,\ 0 < p < 1$ and n a positive integer, by using discrete prove, where the polynomial $P \in \prod_{n-1}$ such that $\prod_{n-1}$ be the set of all polynomials of degree $(n-1)$.

Keywords: weight function, modulus of smoothness , Hardy inequality

1. Introduction and Definitions

For $p \in (0,1)$ the space $L_{p,\alpha}, \alpha > 0$, is defined to be the class of all unbounded functions f. In this paper we denote by $\|.\|_{L_{p,\alpha}(0,1)}$, the quasi norm on the interval $(0,1)$.

A further minor difference is that $\|.\|_{p,\alpha}$ does not satisfy the triangle inequality. See[2] The space $L_{p,\alpha}, \alpha > 0, (0 < p < 1)$ equipped with the distance[1]

$$d(f, g) = \int_a^b |f(x) - g(x)|^p dx.$$

Definition 1. [1]

An integrable function w is called a weight function on the interval $[a, b]$, if $w(x) \geq 0$ for all $x \in [a, b]$.

For example $w(x) = e^{\alpha x}$, $\alpha > 0$.

Consider $L_{p,\alpha}(0,1), 0 < p < 1$ the space of all unbounded functions f on X such that $|f(x)| \leq Me^{\alpha x}$, where

M is positive real number , which are defined the following quasi norm

$$\|f\|_{p,\alpha}^p = \left(\int_X \left|\frac{f(x)}{e^{\alpha x}}\right|^p dx\right)^{\frac{1}{p}} < \infty \qquad (1)$$

The modulus of smoothness of order n of the function $f \in L_p(X)$ is given by[4]

$$\omega_n(f;\delta)_p = \underbrace{sup}_{|h|<\delta}\{\|\Delta_h^n f(.)\|_p\}, \delta > 0 \tag{2}$$

Where

$$\Delta_h^n f(x) = \sum_{i=0}^n (-1)^{n-i}\binom{n}{i} f(x+ih) \tag{3}$$

We may written the modulus of smoothness of order n of the function $f \in L_{p,\alpha}(X)$ by:

$$\omega_n(f;\delta)_{p,\alpha} = \underbrace{sup}_{|h|<\delta}\{\|\Delta_h^n f(.)\|_{p,\alpha}\}, \delta > 0 \tag{4}$$

The degree of best approximation of unbounded function $f \in L_{p,\alpha}$is defined by :[5]

$$E(f)_{p,\alpha} = \underbrace{inf}_{P\in P_{n-1}} \|f-P\|_{p,\alpha},\ \alpha > 0, (0 < p < 1) \quad . \tag{5}$$

The inequalities of Hardy for $f(x) \geq 0, x \in (0,\infty)$, $\alpha > 0$ and $(0 < p < 1)$, is given as [3]

$$\left(\int_0^\infty \left(t^{-\alpha}\int_0^t f(u)\frac{du}{u}\right)^p \frac{dt}{t}\right)^{\frac{1}{p}} \leq \frac{1}{\alpha}\left(\int_0^\infty (t^{-\alpha}f(t))^p\frac{dt}{t}\right)^{\frac{1}{p}} \tag{6}$$

$$\left(\int_0^\infty \left(t^{+\alpha}\int_t^\infty f(u)\frac{du}{u}\right)^p \frac{dt}{t}\right)^{\frac{1}{p}} \leq \frac{1}{\alpha}\left(\int_0^\infty (t^{+\alpha}f(t))^p\frac{dt}{t}\right)^{\frac{1}{p}} \tag{7}$$

The following results are required to prove our main theorem.

Lemma 2. [3]

For bounded function $f \in L_p[0,1], 0 < p < 1, k \geq 1$ and $0 \leq \delta \leq 1/(k+1)$. then

$$\omega_k(f;\delta)_p^p \leq C\delta^k\left\{\int_\delta^{1/(k+1)} t^{-kp}\omega_{k+1}(f;t)_p^p\frac{dt}{t} + \|f\|_p^p\right\} \tag{8}$$

Lemma 3.

For unbounded functions $f \in L_{p,\alpha}[0,1]$, $\alpha > 0$, $(0 < p < 1)$, $k \geq 1$, and $0 \leq \delta \leq 1/(k+1)$. then

$$\omega_k(f;\delta)_{p,\alpha}^p \leq C\delta^k\left\{\int_\delta^{1/(k+1)} t^{-kp}\omega_{k+1}(f;t)_{p,\alpha}^p\frac{dt}{t} + \|f\|_{p,\alpha}^p\right\} \tag{9}$$

PROOF:

$$\omega_k(f;\delta)_{p,\alpha}^p = \underbrace{sup}_{|h|<\delta}\left\{\left\|\Delta_h^k f(.)\right\|_{p,\alpha}^p\right\} \quad \delta > 0, \alpha > 0$$

$$= \underbrace{sup}_{|h|<\delta}\left\{\left(\int_0^1\left|\Delta_h^k\frac{f(.)}{e^{\alpha x}}\right|^p dx\right)^{\frac{1}{p}}\right\}$$

$$= \omega_k(fe^{-\alpha x};\delta)_p^p \tag{10}$$

then in view of lemma (2.) we get

$$\omega_k(fe^{-\alpha x};\delta)_p^p \leq C\delta^k\left\{\int_\delta^{1/(k+1)} t^{-kp}\omega_{k+1}(fe^{-\alpha x};t)_p^p\frac{dt}{t} + \|fe^{-\alpha x}\|_p^p\right\} \tag{11}$$

Since

$$\omega_{k+1}(fe^{-\alpha x};t)_p^p = \underbrace{sup}_{|h|<t}\left\{\left(\int_0^1\left|\Delta_h^{k+1}\frac{f(.)}{e^{\alpha x}}\right|^p dx\right)^{\frac{1}{p}}\right\}$$

$$= \underbrace{sup}_{|h|<t}\left\{\left\|\Delta_h^k f(.)\right\|_{p,\alpha}^p\right\}$$

$$= \omega_{k+1}(f;t)_{p,\alpha}^p \tag{12}$$

Moreover,

$$\|fe^{-\alpha x}\|_p^p = \left\{\left(\int_0^1 \left|\Delta_h^k \frac{f(.)}{e^{\alpha x}}\right|^p dx\right)^{\frac{1}{p}}\right\}$$

$$= \|f\|_{p,\alpha}^p \qquad (13)$$

Therefore, equation (10), (12) and (13) implies

$$\omega_k(f;\delta)_{p,\alpha}^p \le C\delta^k \left\{\int_\delta^{1/(k+1)} t^{-kp}\omega_{k+1}(f;t)_{p,\alpha}^p \frac{dt}{t} + \|f\|_{p,\alpha}^p\right\} \qquad \blacksquare$$

Corollary 4. For unbounded function $f \in L_{p,\alpha}[0,1]$, $\alpha > 0$, $(0 < p < 1), 1 \le k < n$ and $0 \le \delta \le 1$. Then

$$\omega_k(f;\delta)_{p,\alpha}^p \le C\delta^{kp}\left\{\int_\delta^1 t^{-kp}\omega_n(f;t)_{p,\alpha}^p \frac{dt}{t} + \|f\|_{p,\alpha}^p\right\} \qquad (14)$$

PROOF: we prove the inequality (14) by induction for m, the inequality (14) is true for k=1, suppose that this induction is true for k=m. Then we take k=m+1, by lemma (3.) and the Hardy inequality (7) we obtain

$$\omega_k(f;\delta)_{p,\alpha}^p \le C\delta^{kp}\left\{\int_\delta^1 \left(t^{np-kp-1}\int_t^1 v^{-np-1}\,\omega_{k+1}(f;v)_{p,\alpha}^p dv\right)dt + \int_\delta^1 t^{np-kp}\|f\|_{p,\alpha}^p dt + \|f\|_{p,\alpha}^p\right\}$$

$$\le C\delta^{kp}\left\{\int_\delta^1 t^{-kp}\,\omega_{n+1}(f;t)_{p,\alpha}^p \frac{dt}{t} + \|f\|_{p,\alpha}^p\right\}. \qquad \blacksquare$$

Lemma 5.[1] Let $f \in L_p[a,b]$, $\alpha > 0$, $(0 < p < \infty)$. Then there exist a constant c such that

$$\|f - c\|_p^p \le \frac{1}{b-a}\int_a^b\int_a^b |f(x) - f(y)|^p dxdy$$

$$= \frac{2}{b-a}\int_0^{b-a}\int_a^{b-t} |f(x+t) - f(x)|^p dxdt \le 2\omega_1(f;b-a)_p^p \;. \qquad (15)$$

Lemma 6. For unbounded function $f \in L_{p,\alpha}[a,b]$, $\alpha > 0$, $(0 < p < 1)$. Then there exist a constant M such that

$$\|f - M\|_{p,\alpha}^p \le \frac{1}{b-a}\int_a^b\int_a^b \left|\frac{f(x)}{e^{\alpha x}} - \frac{f(y)}{e^{\alpha y}}\right|^p dxdy$$

$$= \frac{2}{b-a}\int_0^{b-a}\int_a^{b-t} |(f(x+t) - f(x))e^{-\alpha x}|^p dxdt \le 2\omega_1(f;b-a)_{p,\alpha}^p, \qquad (16)$$

Where the constant $c = (b-a)^{-1}\int_a^b f(t)dt$

PROOF: Suppose the function

$$\varphi(y) = \int_a^b \left|\frac{f(x)}{e^{\alpha x}} - \frac{f(y)}{e^{\alpha y}}\right|^p dx\,, \quad x,y \in [a,b].$$

We can find $y_1 \in [a,b]$ such that

$$\varphi(y_1) \le \frac{1}{b-a}\int_a^b \left|\frac{f(x)}{e^{\alpha x}} - \frac{f(y)}{e^{\alpha y}}\right|^p dy$$

setting $M = f(y_1)$ to get

$$\int_a^b \left|\frac{f(x)}{e^{\alpha x}} - M\right|^p dx \le \frac{1}{b-a}\int_a^b\int_a^b \left|\frac{f(x)}{e^{\alpha x}} - \frac{f(y)}{e^{\alpha y}}\right|^p dxdy$$

(17)

By (lemma 5.) we have

$$\int_a^b \int_a^b \left|\frac{f(x)}{e^{\alpha x}} - \frac{f(y)}{e^{\alpha y}}\right|^p dxdy = \int_0^{b-a} \int_a^b \left|\frac{f(x)}{e^{\alpha x}} - \frac{f(y)}{e^{\alpha y}}\right|^p dxdy \quad (18)$$

Therefore from (17) and (18) we obtain (16). ■

Lemma 7. For $f \in L_{p,\alpha}[a,b]$, $\alpha > 0$, $(0 < p < 1)$. Then for every $n \in N, n \geq 1$ there exists a step-function ϑ_m such that

$$\|f - \vartheta_m\|_{p,\alpha}^p \leq 2n \int_0^{1/n} \int_0^{1-t} |(f(x+t) - f(x))e^{-\alpha x}|^p dxdt \leq 2\omega_1 \left(f; \frac{1}{n}\right)_{p,\alpha}^p \quad (19)$$

PROOF: suppose $F = fe^{-\alpha x}$

from lemma (6.) there exists a constants $M_i, i = 1,2,3, \dots n,$ such that

$$\int_{x_{i-1}}^{x_i} |F(x) - M|^P \leq 2n \int_0^{1/n} \int_{x_{i-1}}^{x_i} |F(x-t) - F(x)|^P dxdt \quad , i = 1,2,3, \dots n.$$

We set the step function $\vartheta_m(x) = M_i$ where $x \in (x_{i-1}, x_i), i = 1,2, \dots n$, that is implies(19).

Main result

Theorem(8.)

For unbounded functions $f \in L_{p,\alpha}[I]$ $\alpha > 0 . 0 < p < 1$, $n \geq 1$ and $I = [0,1]$. There exists a polynomial $P \in \prod_{n-1}$, where $\prod_{n-1}$the space of all polynomials of degree less than or equal zero, such that

$$\|f - P\|_{p,\alpha}^p \leq c\omega_n \left(f; \frac{|I|}{n}\right)_{p,\alpha}^p. \quad (20)$$

Where $c = c(k, P)$ be a constant.

PROOF: To prove this theorem by contradiction, suppose that (20) does not hold. Then there exists a sequence of functions $\{f_u\}_{u=1}^{\infty}, f_u \in L_{p,\alpha}[0,1]$ such that

$$\underbrace{inf}_{P \in P_{n-1}} \|f_u - P\|_{p,\alpha(0,1)}^p > u\omega_n \left(f_u; \frac{1}{n}\right)_{p.\alpha}, u = 1,2, \dots \quad (21)$$

Since the set of all polynomials $P \in \prod_{n-1}$ such that $\|P\|_{p,\alpha(0,1)}^p < 1$ which is a compact set in space $L_{p,\alpha}$, then for each u there exists a polynomial $P_u \in P_{n-1}$ such that

$$\|f_u - P_u\|_{p,\alpha(0,1)}^p = \underbrace{inf}_{P \in P_{n-1}} \|f_u - P\|_{p,\alpha(0,1)}^p \quad (22)$$

Therefore,

$$\|f_u - P_u\|_{p,\alpha}^p > u\omega_n \left(f_u; \frac{1}{n}\right)_{p.\alpha}, u = 1,2, \dots \quad (23)$$

Assume that

$$g_u = \gamma_u(f_u - P_u), \qquad \gamma_u = \|f_u - P_u\|_{p,\alpha}^{-1} \quad (24)$$

By using (23) we obtain

$$\|g_u\|_{p,\alpha}^p = \underbrace{inf}_{P \in \prod_{n-1}} \|f_u - P\|_{p,\alpha}^p = 1 \quad (25)$$

and

$$\omega_n \left(g_u; \frac{1}{n}\right)_{p,\alpha}^p < \frac{1}{u}, u = 1,2, \dots \quad (26)$$

Since $L_{p,\alpha}[0,1]$ $\alpha > 0$, $0 < p < 1$, is a complete metric space, then there exists a sequence $\{g_u\}_{u=1}^{\infty}$ in $L_{p,\alpha}(0,1)$, (i,e) there exists a function $g \in L_{p,\alpha}$ and a subsequence $\{g_{u_i}\}_1^{\infty}$ such that $\|g_{u_i} - g\|_{p,\alpha}^p \to 0$ as

$i \to \infty$

Corollary (4.) with $n = 1, u = n$ and (25), (26) yield

$$\omega_1(g_u;\delta)_{p,\alpha}^p \le c\delta^p \left\{\int_\delta^1 t^{-p}\frac{1}{u}\frac{dt}{t} + 1\right\} \le c_1\left(\frac{1}{u} + \delta^p\right) \tag{27}$$

For $0 \le \delta \le 1$ and $u = 1,2, \dots$; and for each $\varepsilon > 0$ there exist $u_0 > 0$ and $\delta_0 > 0$ such that

$$\omega_1(g_u;\delta)_{p,\alpha}^p < \varepsilon \quad \text{for} \quad 0 \le \delta \le \delta_0 \quad \text{and} \quad u > u_0 \tag{28}$$

Fix a value of $v > \frac{1}{\delta_0}$, from lemma (7) and (28) we obtain, for each $u > u_0$ there exist a step function $\vartheta_{u,v}$ such that

$$\left\|g_u - \vartheta_{u,v}\right\|_{p,\alpha}^p \le 2\omega_1\left(g_u;\frac{1}{v}\right)_{p,\alpha}^p < 2\varepsilon \tag{29}$$

On the other side equation (25) and (29) gives

$$\left\|\vartheta_{u,v}\right\|_{p,\alpha}^p \le \|g_u\|_{p,\alpha}^p + \left\|g_u - \vartheta_{u,v}\right\|_{p,\alpha}^p < 1 + 2\varepsilon \tag{30}$$

For the constant function $\vartheta_{u,v}(x),\ x \in \left(\frac{i-1}{v}, \frac{i}{v}\right), i = 1, \dots, v$. The following inequality holds.

$$\left\|\vartheta_{u,v}\right\|_{L_\infty[0,1]} \le \left(v\int_0^1\left|\vartheta_{u,v}(x)\right|^p dx\right)^{1/p} < \left((1+2\varepsilon)v\right)^{1/p} = M \tag{31}$$

Consider the set Φ that consists of all step functions ϑ of the type

$$\vartheta(x) = r\varepsilon^{1/p},\ x \in \left(\frac{i-1}{v}, \frac{i}{v}\right), i = 1, \dots v,\ \ r = 0 + 1 + \cdots,\ \ \|\vartheta\|_{L_\infty[0,1]} \le M. \tag{32}$$

Then it is clear that

$$\underbrace{inf}_{\vartheta\in\Phi}\left\|\vartheta_{u,v} - \vartheta\right\|_{p,\alpha}^p \le \int_0^1\left|\varepsilon^{1/p}\right|^p dx = \varepsilon,$$

And Φ is finite for the sequence $\left\{\vartheta_{u,v}\right\}_{u=u_0+1}^\infty$. By this and (30) we have that Φ is finite for the sequence $\{g_u\}_{u=u_0+1}^\infty$. Hence for appropriate $\{g_u\}_{i=1}^\infty$ we obtain

$$\left\|g_{u_i} - g\right\| \to 0 \ \ as \ \ i \to \infty \ \ for\ some\ \ g \in L_{p,\alpha}. \tag{33}$$

Hence

$$\underbrace{inf}_{P\in\prod_{n-1}}\|g - P\|_{p,\alpha}^p = \underbrace{inf}_{P\in\prod_{n-1}}\left\|g - g_{u_i} + g_{u_i} - P\right\|_{p,\alpha}^p$$

$$\ge \underbrace{inf}_{P\in\prod_{n-1}}\left\|g_{u_i} - P\right\|_{p,\alpha}^p - \left\|g - g_{u_i}\right\|_{p,\alpha}^p$$

From (25) and (33) we have

$$\underbrace{inf}_{P\in\prod_{n-1}}\|g - P\|_{p,\alpha}^p \ge 1 - \left\|g - g_{u_i}\right\|_{p,\alpha}^p \to 1 \quad as \quad i \to \infty,$$

then

$$\underbrace{inf}_{P\in\prod_{n-1}}\|g - P\|_{p,\alpha}^p = 1 \tag{34}$$

On the other hand, from (26) we obtain

$$\omega_n\left(g_u;\frac{1}{n}\right)_{p,\alpha}^p \le \omega_n\left(g_{u_i};\frac{1}{n}\right)_{p,\alpha}^p + 2^{np}\left\|g - g_{u_i}\right\|_{p,\alpha}^p \to 0 \quad as \quad i \to \infty.$$

Hence $\omega_n\left(g_u;\frac{1}{n}\right)_{p,\alpha}^p = 0$. Whence $g = P$ for the same $P \in \prod_{n-1}$, which contradictions equation (34).

This contradict our assumption that means our assumption in equation (21) is false (i.e equation (20) is true).

References

1. Burden , R. L.(1997). Numerical Analysis. Youngstown State University , P:487-492.
2. DeVore, R. A., & Lorentes, G. G. (1992). Constructive Approximation. Springer-verlag, New York.
3. Petrushev, P. P.,& Popov, V. A. (1987) .Rational approximation real functions. Cambridge University Press 1987 (Encyclopedia of mathematics and its applications; v. 28).
4. Sendov, B., & Popov, V. A.,(1983), “The averaged modulus of smoothness ” ,Publ. nouse of Bulg. Acad. Of Sci. Science.
5. Song Li, (1999). K-Functional, Weighted Moduli of Smoothness, and Best Weighted Polynomial Approximation on a Simplex. Springer-verlag, Vol.15, No.3, p. 395-406