Derivation of an Implicit Runge – Kutta Method for First Order Initial Value Problem in Ordinary Differential Equation using Hermite, Laguerre and Legendre Polynomials.

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Abstract

In this paper, three Implicit Runge – Kutta methods are derived using Hermite, Laguerre and Legendre polynomials for the direct solution of general first order initial value problems of ordinary differential equations with constant step size. The analysis of the properties of the developed methods were investigated and found to be consistent, convergent and A – stable. The efficiency of the methods were tested on some numerical examples and are found to give better approximations than the existing methods.

Keywords: Implicit Runge – Kutta shemes, Collocation, Interpolation, canonical polynomial and A – stable.

1. Introduction

Considering the initial value problem of the form

$$y'(x) = f(x, y), y(a) = y(0), \qquad a \le x \le b \tag{1}$$

The general Implicit Runge – Kutta method with *v* slope for general first order initial value problem in ordinary differential equation of the form (1) as defined by Jain (1987) is

$$K_i = hf\left(t_n + c_i h, y_n + \sum_{j=1}^{v} a_{ij} k_j\right), \; i = 1,2,3,\ldots,v$$

and

$$y_{n+1} = y_n + \sum_{j=1}^{v} w_i k_i,$$

where

$$c_i = \sum_{j=1}^{v} a_{ij}, \; i = 1,2,3,\ldots,v$$

and

$a_{ij}, 1 \le ij \le v, w_1, w_2, w_3, \ldots, w_v$, are arbitrary.

The general solution for the differential (1) is approximated by calculating the solution of a related first order differential equation. The general single – step is defined

$$y_{n+1} = y_n + h\phi(x_n, y_n, h), \qquad n = 0,1,2,3,\ldots, N-1 \tag{2}$$

where ϕ(x,y,h) is a function of the augmented x, y, h and it depends on right – hand side of (1), and ϕ(x, y, h) is the increment function. If y_{n+1} can be obtained simply by evaluating the right – hand side of (2), then the single – step method is called explicit else it called implicit (Jain, 1987). The true value y(xn) will satisfy

$y(x_{n+1}) = y_n + h\phi(x_n, y(x_n)) + \tau_n, \quad n = 0,1,2,3,\ldots, n-1$ where τ_n is the truncation error.

Assuming that (1) has unique solution.

$y \in R^m$, $f \in R^m$ and $a = x_0 < x_1 < x_2 < ---- < x_i < x_{i+1} < ---- < x_n = b$, where the number of subintervals is specified by $N = (b-a)/h$

If we further assume a constant step size $h = x_{i+1} - x_i$ and adopt the notation $y(x_0) = y_0$, $y(x_{i+j}) = y_{j+1}$, where j is a positive real constant (not necessarily an integer). (Lambert, 1973).

The schemes are generated by collocation using transformed Hermite, Laguerre and Legendre polynomials of degree one.

2. Derivation of the schemes

Suppose that equation (1) has a unique solution $y(x)$ which can be approximated as accurately as possible by (Scheld, 1989)

$$y_n(x) = \sum_{j=0}^{n} a_j Q_j(x), \; x \in [x_j, x_{j+1}] \tag{3}$$

where $Q_j(x),\; j = 0(1)n,$ are certain canonical polynomials and a's are constants to be determined.

That is:

$$y_1(x) = a_0 Q_0(x) + a_1 Q_1(x) \tag{4}$$

By defining an operator

$L = \frac{d}{dx} + 1$ to derive the basis $Q_j(x)$ as

$$LQ_j = Lx^j$$

$$Lx^j = \left(\tfrac{d}{dx} + 1\right)x^j$$

$$= jx^{j-1} + x^j$$

Assume that the inverse of L exist, that is $LL^{-1} = 1$.

Then

$$LL^{-1}x^j = jLL^{-1}Q_{j-1} + LL^{-1}Q_j(x)$$

$$x^j = jQ_{j-1}(x) + Q_j(x)$$

This gives

$Q_0(x) = 1$

$Q_1(x) = x - 1$

$Q_2(x) = x^2 - 2x + 2$, etc.

Considering $y_n(x)$ as an exact polynomial solution of the perturbed equation

$$y'_n = f(x,y) + \tau H_n(x) \quad (5)$$

$$y_n(x_i) = y_i \quad (6)$$

Collocating (5) at point $x \in [x_i, x_{i+1}]$ and interpolate (3).

Substituting the $Q_j(x)$'s in (4), we get

$$y_1(x) = a_0(1) + a_1(x-1)$$
$$= a_0 + a_1 x - a_1 \quad (7)$$

$$y'_1(x) = a_1 \quad (8)$$

Substituting (7) in to (5) gives

$$y'_1(x) = a_1 = f(x,y) + \tau H_n(x) \quad (9)$$

Collocating (9) at the points $x = x_{i+\frac{1}{4}}$, $x = x_{i+\frac{1}{2}}$, and $x = x_{i+\frac{3}{4}}$, we obtain by initial condition

$$y'_1\left(x_{i+\frac{1}{4}}\right) = a_1 = f_{i+\frac{1}{4}} + \tau H_1\left(x_{i+\frac{1}{4}}\right) \quad (10)$$

$$y'_1\left(x_{i+\frac{1}{2}}\right) = a_1 = f_{i+\frac{1}{2}} + \tau H_1\left(x_{i+\frac{1}{2}}\right) \quad (11)$$

$$y'_1\left(x_{i+\frac{3}{4}}\right) = a_1 = f_{i+\frac{3}{4}} + \tau H_1\left(x_{i+\frac{3}{4}}\right) \quad (12)$$

Evaluating $H_1(x)$ using Hermite polynomial of degree one. The Hermite polynomial is given as (Pang, 1997)

$$H_n(x) = \sum_{r=2}^{N} (-1)^r \frac{n!}{r!(n-2r)!}(2x)^{n-2r}, \text{ where } N = \tfrac{n}{2} \text{ if } n \text{ is even and } N = \tfrac{(n-1)}{2} \text{ if } n \text{ is odd.}$$

With recurrence relation as

$$H_{n+1}(x) = 2xH_n(x) - 2_n H_{n-1}$$

where

$$H_0(x) = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2 - 2,\ H_3(x) = 8x^3 - 12x, \text{ e.t.c}$$

with

$$H_1\left(x_{i+\frac{1}{4}}\right) = \frac{1}{2},\ H_1\left(x_{i+\frac{1}{2}}\right) = 1 \text{ and } H_1\left(x_{i+\frac{3}{4}}\right)$$

Substituting for $H_1(x)$ in to (10), (11) , (12) and adding, we get

$$a_1 = f_{i+\frac{1}{3}} + f_{i+\frac{1}{2}} + f_{i+\frac{3}{4}} \quad (13)$$

By the initial condition (6)

$$y_1 = a_0 + a_1(x_i - 1)$$
$$a_0 = y_i - a_1(x_i - 1) \quad (14)$$

Substituting equation (14) in to (7) we get

$$y_1(x) = y_i - a_1(x_i - 1) + a_1(x-1)$$
$$= y_i + a_1(x - x_i) \quad (15)$$

Substituting for a1 from (13) into (15) and interpolating at $x = x_{i+1}$, gives

$$y_{i+1}(x) = y_i + \left(f_{i+\frac{1}{4}} + f_{i+\frac{1}{2}} + f_{i+\frac{3}{4}}\right)(x_{i+1} - x_i) \quad (16)$$

Since it was assumed that $h = x_{i+1} - x_i$ and $y(x_{i+1}) = y_{i+1}$, therefore,

$$y_{i+1} = y_i + h\left(f_{i+\frac{1}{4}} + f_{i+\frac{1}{2}} + f_{i+\frac{3}{4}}\right) \quad (17)$$

where the following can define as

$$f_{i+\frac{1}{4}} = f\left(x_{i+\frac{1}{4}}, y\left(x_{i+\frac{1}{4}}\right)\right)$$
$$f_{i+\frac{1}{2}} = f\left(x_{i+\frac{1}{2}}, y\left(x_{i+\frac{1}{2}}\right)\right)$$
$$f_{i+\frac{3}{4}} = f\left(x_{i+\frac{3}{4}}, y\left(x_{i+\frac{3}{4}}\right)\right)$$

Let $K_1 = f_{i+\frac{1}{4}}$, $K_2 = f_{i+\frac{1}{2}}$ and $K_3 = f_{i+\frac{3}{4}}$, then (17) can be written as

$$y_{i+1} = y_i + h(K_1 + K_2 + K_3) \quad (18)$$

Applying the same technique to Laguerre and Legendre polynomials with recurrence relations respectively as. (Cheney and Kincaid, 1999).

$$L_n(x) = e^x \cdot \frac{d^n}{dx^n}(x^n e^{-x})$$

$L_0(x) = 1,\ L_1(x) = 1 - x,\ L_2(x) = x^2 - 4x + 2,\ L_3(x) = -x^3 + 9x^2 - 18x + 6$, e.t.c

And

$$P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}(x^2 - 1)^n$$

$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1)$, e.t.c

$$y_{i+1} = y_i + h(-K_1 + K_2 + K_3) \quad (19)$$

$$y_{i+1} = y_i + \frac{h}{3}(K_1 + K_2 + K_3) \quad (20)$$

Let

$$K_1 = f\left(x_{i+\frac{1}{4}}, y_{i1}\right) = f_{i1}$$
$$K_2 = f\left(x_{i+\frac{1}{2}}, y_{i2}\right) = f_{i2}$$
$$K_3 = f\left(x_{i+\frac{3}{4}}, y_{i3}\right) = f_{i3}$$

Considering the following modified trapezoidal schemes to obtain y_{i1}, y_{i2} and y_{i3} (R. Taparki and M. R. Odekunle, 2010)

$$y_{i1} - y_i = \left(\tfrac{1}{2}\right)\left(\tfrac{h}{4}\right)\left(f_{i1} + f_i\right) \quad (21)$$

$$y_{i2} - y_i = \left(\tfrac{1}{2}\right)\left(\tfrac{h}{2}\right)\left(f_{i2} + f_i\right) \quad (22)$$

$$y_{i2} - y_{i1} = \left(\tfrac{1}{2}\right)\left(\tfrac{h}{4}\right)\left(f_{i2} + f_{i1}\right) \tag{23}$$

$$y_{i3} - y_{i2} = \left(\tfrac{1}{2}\right)\left(\tfrac{h}{4}\right)\left(f_{i3} + f_{i2}\right) \tag{24}$$

Adding equations (21) and (22) we get

$$y_{i2} + y_{i1} - 2y_i = \tfrac{h}{8} f_{i1} + \tfrac{h}{4} f_{i2} + \tfrac{3h}{8} f_i$$

And multiplying by ⅓ we get

$$\tfrac{h}{8} f_i = \tfrac{1}{3} y_{i2} + \tfrac{1}{3} y_{i1} - \tfrac{2}{3} y_i - \tfrac{h}{24} f_{i1} - \tfrac{h}{12} f_{i2} \tag{25}$$

Substituting (25) in to (21), we have

$$\tfrac{2}{3} y_{i1} - \tfrac{1}{3} y_{i2} - \tfrac{1}{3} y_i = \tfrac{h}{12} f_{i1} - \tfrac{h}{12} f_{i2} \tag{26}$$

Multiplying (26) by 3 and adding to (23), we get

$$y_{i1} = y_i + \tfrac{3h}{8} f_{i1} - \tfrac{h}{8} f_{i2} \tag{27}$$

Substituting (27) into (23) and simplifying we get

$$y_{i2} = y_i + \tfrac{h}{2} f_{i1} \tag{28}$$

Substituting (28) into (24) gives

$$y_{i3} = y_i + \tfrac{h}{2} f_{i1} + \tfrac{h}{8} f_{i2} + \tfrac{h}{8} f_{i2} \tag{29}$$

Thus, for equations (18), (19) and (20)

$$K_1 = f\left(x_{i+\frac{h}{4}}, y_i + \tfrac{3h}{8} K_1 - \tfrac{h}{8} K_2\right)$$

$$K_2 = f\left(x_{i+\frac{1}{2}}, y_i + \tfrac{h}{2} K_1\right)$$

$$K_3 = f\left(x_{i+\frac{3h}{4}}, y_i + \tfrac{h}{2} K_1 + \tfrac{h}{8} K_2 + \tfrac{h}{8} K_3\right)$$

3. Error Analysis

Using the method of error estimate given in Scheld (1990), equations (18) and (19) are of order two with error constant -$\frac{1}{2}$, while (20) is of order three with error constant of $\frac{1}{48}$.

4. Consistency and Convergence

The three numerical schemes are consistence since the order p≥1, and since that is the necessary and sufficient condition for convergence, hence the schemes are convergent. (Jain, 1987).

5. Stability Analysis

A single step method is said to be A – Stable, when applied to the test equation $y' = \lambda y_n$ gives rational Pade's approximation to $e^{\lambda h}$ and is of the form $y = R_1^s(q) y_n$ (Jain, 1987).

Applying the test equation to the three schemes, gives a Pade's approximation, as such the methods are A – Stable.

Example

Consider the equation

$$y' = -2xy,\ x \in [0,1],\ y_0 = 1,\ h = 0.05$$

The exact solution is

$$y = e^{-x^2}$$

6. Discussions

Though the results performed well and approximate the exact solution better as the step size goes very small, but the results of high step size (h = 0.05) is given for easy comparison to those that would work on the improved versions.

Table 1 is the result obtained by applying the scheme (18) from Hermit polynomial and table 2 is the result obtained from applying the scheme (20) from Legendre polynomial.

7. Conclusion

The new numerical schemes derived follows the techniques of implicit form of Runge – Kutta methods proposed by Oladele (1997). The second order differential equation of the form $y'' = (x, y)$ version using Legendre polynomial, was carried out by Taparki and Odekunle (2010). Oladele derived the schemes with two K's, (that is K_1 and K_2) while the new schemes are derived with three K's, (that is K_1, K_2 and K_3) and the results are better.

The new schemes are of accuracy for direct numerical solution of general first order ordinary differential equations. The steps to the derivation of the new schemes are presented in the methodology while the analysis of the schemes proved to be consistent, convergent and A – Stable, the results prove to be good estimate of the exact equations.

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PRESENTATION OF THE RESULTS

The results below are the exact and the approximate solution to problem presented.

Table 1: Results of the example above at $h = 0.05$ for equation (18).

X	Exact Solution $y(x_n)$	Approximate Solution y_n	Errors
0.00	1.000000000	1.000622000	6.22E-04
0.05	0.997503122	0.996242800	1.260322E-03
0.10	0.990049834	0.986915100	3.134734E-03
0.15	0.977751237	0.972765600	4.985637E-03
0.20	0.960789439	0.953992200	6.9856637E-03
0.25	0.939413063	0.930859100	8.553963E-03
0.30	0.913931185	0.903690700	1.0240485E-02
0.35	0.884705905	0.872864100	1.1841805E-02
0.40	0.852143789	0.838800200	1.3343589E-02
0.45	0.816686483	0.801954100	1.4732383E-02
0.50	0.778800783	0.762805200	1.5995583E-02
0.55	0.738968488	0.721846500	1.7121988E-02
0.60	0.697676326	0.679574300	1.8102026E-02
0.65	0.655406254	0.636478600	1.8927654E-02
0.70	0.612626394	0.593033300	1.95593094E-02
0.75	0.569782825	0.549688300	2.0094525E-02
0.80	0.527292424	0.506861900	2.0430524E-02
0.85	0.485536895	0.464934900	2.0601992E-02
0.90	0.444858066	0.424245600	2.0612466E-02
0.95	0.405554505	0.385086800	2.0467705E-02
1.00	0.367879441	0.347703800	2.0175641E-02

Table 2: Result of the example above at $h = 0.05$ for equation (20)

X	Exact Solutions $y(x_n)$	Approximate Solutions y_n	Errors
0.00	1.000000000	0.997710200	2.2898E-03
0.05	0.997503122	0.990462800	7.04322E-03
0.10	0.990049834	0.978369800	1.1680034E-02
0.15	0.977751237	0.961614100	1.6137137E-02
0.20	0.960789439	0.940444600	2.0344839E-02
0.25	0.939413063	0.915170400	2.4242663E-02
0.30	0.913931185	0.886153100	2.777805E-02
0.35	0.884705905	0.853798400	3.09075505E-02
0.40	0.852143789	0.818546500	3.3597289E-02
0.45	0.816686483	0.780862200	3.5824283E-02
0.50	0.778800783	0.741225300	3.7575483E-02
0.55	0.738968488	0.700119900	3.8848588E-02
0.60	0.697676326	0.658025300	3.9651026E-02
0.65	0.655406254	0.615406800	3.9999454E-02
0.70	0.612626394	0.572707800	3.9918594E-02
0.75	0.569782825	0.530407700	3.9375125E-02
0.80	0.527292424	0.488750900	3.854152E-02
0.85	0.485536895	0.448147600	3.7389295E-02
0.90	0.444858066	0.408895100	3.5962966E-02
0.95	0.405554505	0.371245800	3.4308705E-02
1.00	0.367879441	0.335406500	3.2472941E-02