

Mupad Models for Splines

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Abstract

The aim of this paper is to reconstruct relevant & well elaborated models for Splines as well as their interpolation via Mupad Numeric & Graphic Functions , these models have to include linear piecewise model, quadratic piecewise model, cubic piecewise model, certain models for Runge's phenomenon will find their places in between, these models aiming to represent computerized resolution for Runge's phenomenon. Also it will refer to advantages & disadvantage of interpolation methods as Lagrange polynomials, Neville's, Divided differences and Cubic spline, giving more attention to the last one in accordance to its significance place among other methods. Here is direct verification for Weierstrass theorem, also the author will come across the specific theoretical approaches to verify his results as well as the reconstructed models.

Keywords: Rainbow spline, linear, quadratic, cubic, natural, clamped splines, Newton, Neville, Lagrange interpolations, Chebyshev knots, Runge's phenomenon, Weierstrass theorem.

1.Introduction

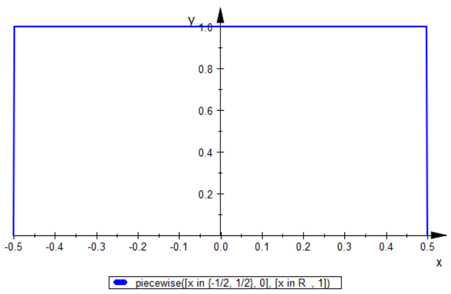
At the preface of this paper, it seems suitable to refer to transparent conceptual overlapping between function interpolation & function approximation , here the first tends to construct a polynomial coming across a set of given points (data), while the second is a process of finding a polynomial or other functions haven't to be polynomial but other as Fourier series, i.e. the 2nd include the 1st but not vice versa . Although one is here interesting in piecewise interpolation for its strong relation to splines construction, the author would like to introduce the fundamental concepts of splines, also the methods of their interpolation referring to their advantages as well as disadvantage. The word spline in language means a thin flexible strip of wood or metal that was used in ships or carriages industry, then naturally it has to be twisted or/and curved to be fitted in accordance to the desired design, frequently their shapes take a fashion of polynomials within 1st, 2nd or 3d degree, in one or some combinations of them. Nowadays splines play significance role in automobile & other tools manufacturing. Here as it at other pivots of our modern life, mathematics is coming to contribute in the process of models creation, moreover the super technical language MATLAB & its integrating part MUPAD play its considered role as the newest & strongest tools in mathematical as well as technical modeling. Henceforward in engineering practice whereas used to fix special nails or pins at a board accordingly to their coordinate values, then to figurate the suitable strip among them to create the aimed design, it very much likes that or it is so much look likes in mathematics computation, while it is possible to construct a model for a spline or its interpolate using MUPAD numeric & graphics Functions, as one will see between the lines of this paper. Therefore, it seems relevant to remain negative as well as positive aspects of Newton, Lagrange ,Neville's &Cubic spline interpolation[1][5][7], so one can keep an imagination about them and when they can be applied, also what's the best choice for certain cases as well as what's the cost of each method. With respect to Lagrange interpolation, although it needs not equidistant knots nor solution of multiple linear equations but any increasing or decreasing in knots' number embrace one to repeat his calculations of functions' basis, so it's computationally expensive at the case of high degree interpolation, also it makes abrupt jumps through by its path. while Newton interpolations so called divided differences is distinguished by simplicity of coefficients determination & none effort-consuming at the case of increasing the polynomial interpolation degree, these methods extensively used before the era of new computer generations, At last the Neville's one depends on recursively generating of interpolation polynomials, at some cases iteration is needed, as it on the n+1 distinct points to successively generate the polynomials themselves[15][16][4][14].Therefore due to that all the interpolation methods at certain cases care of oscillations while by they create humps or fast & sharp frequencies at their edges or path deviations which don't fit or even approximate the function has to be interpolated, I.J.Shoenberg in the four ties of the preceding Century suggested in his paper "Contribution to the problem of approximation of equidistant Data by Analytic Functions"_ an effective method for overcoming those oscillations & it helps to construct a smooth curve by connecting suitably required polynomial pieces which creates visibly smooth continuous function, it is called spline interpolation or piecewise spline interpolation, it has another advantage that it is easily applicable in computer languages, in accordance to the author experience, it seemed that MUPAD numeric functions & graphics tools (as an integrated part of MATLAB) can be considered as the best computational apparatus for creating the required models for splines & their interpolation, due to that and strongly related within its context, one found it good occasion to construct odd & even order interpolation models adjustable as certain resolution for Runge's phenomenon.

2.Spline'Models

A spline is defined[18] as a piecewise-polynomial real function $S : [a , b] \to R$, on an interval $[a , b]$ composed of k subintervals $[t_{i-1} , t_i]$ with $a = t_0 < \ldots < t_{k-1} < t_k = b$. The restriction of S to an interval is a polynomial $P_i=[t_{i-1},t_i]\to R$, so that : $S(t) = P_1(t)$, $t_0 \le t \le t1$, $S(t) = P_2(t)$, $t_1 \le t \le t2$, …, $S(t) = P_k(t)$, $t_{k-1} \le t \le t_k$. in accordance with the definition above, the simplest spline $B_{0(x)}$ has a value of 1 if $-1/2 \le x \le 1/2$ & takes zero value everywhere else, its piecewise function can be formulated as below:

$$\mathbf{B_o(x)} = \begin{cases} \mathbf{1} & \textbf{if} \quad \mathbf{x \ge -1/2} \\ \mathbf{1} & \textbf{if} \quad \mathbf{x \le 1/2} \\ \mathbf{0} & \textbf{if} \quad \textbf{otherwise} \end{cases}$$

Mupad Graphics Function plot(piecewise…) is so easy way to create an elegance model for the piecewise linear spline as in the following model[11][2]: plot(piecewise([x >= -1/2, 1], [x <= 1/2, 1],[x=-1/2,0],[x=1/2,0]), #x=-1/2..1/2, #Legend)

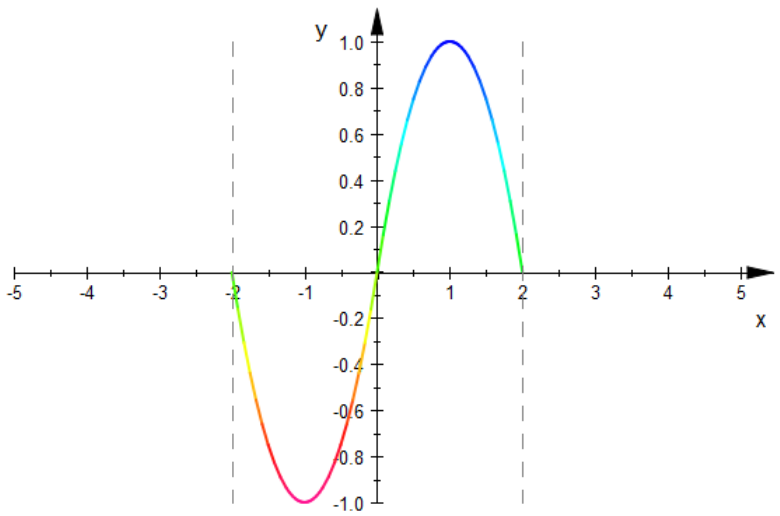


In spite of model's simplicity above but it's useful from informative as well as mathematical viewpoint, while it creates new approach to spline modeling via Mupad instructions , so it is a first step at new environment of such models construction in the spline researches, also it's benefit able at some cases while higher piecewise spline construction. As it seems at the linear model, the spline isn't smooth at the knots while the 1st derivatives of its pieces at these points (knots) is discontinuous, though it leads one to create somewhat advanced model-say- quadratic spline as further step likes :

$$S(x) = \begin{cases} x^2 - 2x & -2 \le x \le 0 \\ 2x - x^2 & 0 \le x \le 2 \end{cases}$$

% Mupad model for piecewise quadratic spline

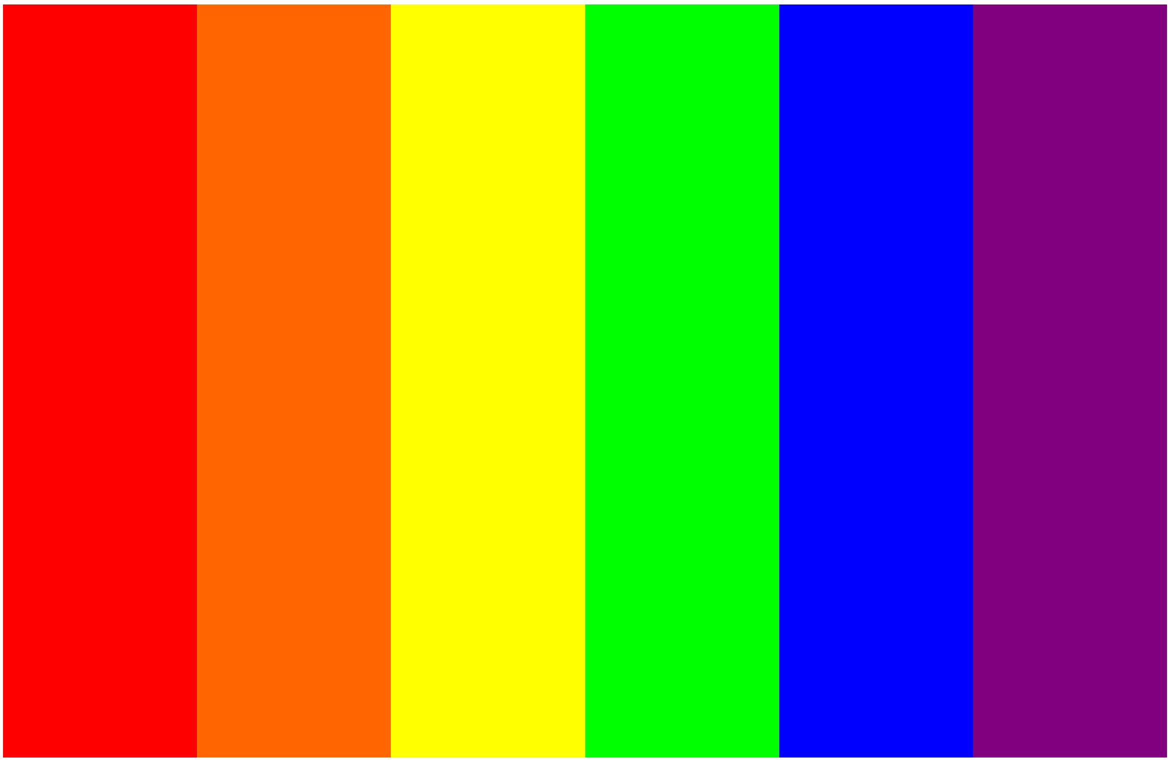
plot(piecewise([-2<=x < 0,x^2+2*x], [0 <= x and <=2,2*x-x^2]),LineColorType=Rainbow):



Therefore here is a short digression ,who don't like rainbow ? may be only those who haven't seen. Here also rainbow is viewed as a symbol of wellness in myths & legends, but the most interesting and related within this thesis, that this lovely natural phenomena is considered as one of the most splendid masterpieces and eternal gifts of GOD for humankind, it's an aesthetical two dimensional spline that consist of six pieces whereas every one of them is bilinear piece in distinct basic color. moreover it naturally illustrates the spline idea in unique coincidence. for mater of consistency, its interpretation in physics says that the water drops **split** the white ray of the sun to create so called the sun spectrum, may be more surprising that rainbow sometimes appears in arcs or circular shape[19][9]. Here below a Mupad model for bi-linear spline:

% Mupad model for Spectrum Bilinear Spline via Mupad Advanced graphics[13][9][16]

```
plot(plot::Rectangle(0..1, 1..3, Filled = TRUE,
        FillPattern = Solid,
        FillColor = RGB::Red),
  plot::Rectangle(1..2, 1..3, Filled = TRUE,
        FillPattern = Solid,
        FillColor = RGB::Orange),
  plot::Rectangle(2..3, 1..3, Filled = TRUE,
        FillPattern = Solid,
        FillColor = RGB::Yellow),
  plot::Rectangle(3..4, 1..3, Filled = TRUE,
        FillPattern = Solid,
        FillColor = RGB::Green),
  plot::Rectangle(4..5, 1..3, Filled = TRUE,
        FillPattern = Solid,
  plot::Rectangle(5..6, 1..3, Filled = TRUE,
        FillPattern = Solid,
        FillColor = RGB::Violet),
        AxesVisible = FALSE,LinesVisible=FALSE))
```



Returning back to the quadratic spline, it ensures the spline continuity at the expected knots, here also the function values of the connected polynomials have the same value at the knots, what means that spline pieces are going on smoothly along its path. Before coming on details in construction of cubic spline pieces(polynomials), let's assume a set within n+1 points that equals $\{x_m, y_m\}$, $0 \le m \le n$, $b = x_0 < x_1 < x_2 < x_3 < \ldots < x_n = c$,, then one has to construct cubic piecewise polynomials say $S_m(x)$ on $[x_m, x_{m+1}]$ that satisfy any piecewise polynomial within its 1st & 2nd derivatives are continuous on $[x_0, x_n]$. Except that, one has to define a polynomial p on [b, c] within knots set $b = x_0 < x_1 < x_2 < x_3 < \ldots < x_n = c$, then the cubic interpolate S(x) that satisfies p has to verify the conditions below:

1st: $S(x) = S_{m,0}(x) + S_{m,1}(x)(x-x_m) + S_{m,2}(x)(x-x_m)^2 + S_{m,3}(x)(x-x_m)^3$; $x \in [x_m, x_{m+1}]$,

$m = 0,1,2,\ldots,n$

This condition is frequently written as following:

$$S(x) = \begin{cases} S_0(x) & ,x_0 \le x \le x_1 \\ S_1(x) & ,x_1 \le x \le x_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ S_n(x) & ,x_{n-1} \le x \le x_n \end{cases}$$

2nd: $S_m(x_m) = P(x_m)$; $m = 0,1,\ldots,n-1$

$S_m(x_{m+1}) = P(x_{m+1})$; $m = 0,1,\ldots,n-1$

3rd: $S_m(x_{m+1}) = S_{m+1}(x_{m+1})$; $m = 0,1,\ldots,n-2$

4th: $S'_m(x_{m+1}) = S'_{m+1}(x_{m+1})$; $m = 0,1,\ldots,n-2$

5th: $S''_m(x_{m+1}) = S''_{m+1}(x_{m+1})$; $m = 0,1,\ldots,n-2$

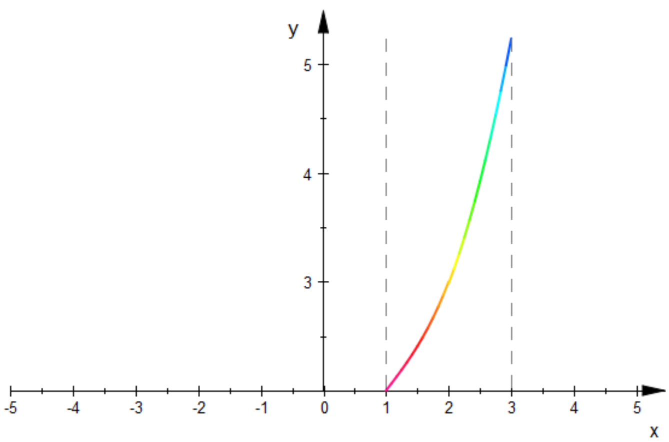
6th: The following boundary conditions are satisfied as below:

a) $S''(x_0) = S''(x_n)$ at the case of natural or free boundary

b) $S'(x_0) = P'(x_0)$ & $S'(x_n) = P'(x_n)$ at the case of clamped boundary [15,16].

Therefore here is Mupad model for the condition (a)

```
plot(piecewise([1<=x <= 2,2+3/4*(x-1)+1/4*(x-1)^3],[2<= x <= 3,3+(3/2)*(x-2)+3/4*(x-2)^2-1/4(x-2)^3]),LineColorType=Rainbow)
```



Hereafter to interpret ate those condition more clearly, one can say that 1st condition indicate to that S(x) is constructed by piecewise cubic polynomials i.e. every one of its pieces (polynomials) is of third degree, while the 2nd assigns to that these polynomials verify the given points set, whether the 3rd & 4th say that spline pieces are smoothly continuous whereas the first derivative of every two consequent polynomials are of equal value at their knot, while the 5th condition ensure that the second derivative of the spline is continuous, therefore the 6th specify the natural (free) & clamped spline as one will see at the coming paragraphs.[15][4][6].

Hence one can verify the model above in accordance to the boundary condition whereas the rest conditions are transparently satisfied, then one here needs the second derivatives of $S(x_0)$, $S(x_n)$,then $S''(x_0) = 3x/2 - 3/2$, for that its value at $x = 1$ equal zero & $S''(x_n) = 9/2 - 3x/2$, which also it takes zero value at $x = 3$, then the model satisfy the natural (free) spline. Also the clamped one easily can be verified through by the conditions above.

3. Mupad models for Runge's phenomenon

Runge's phenomenon is a certain contradiction within Weierstrass approximation theorem that states for every continuous function f(x) defined on [a, b], there exists a set of polynomial functions of degree n likes $P_n(x)$ for $n = 0,1,2,\ldots$,that approximate f(x) through by uniform convergence over its interval whereas n tends to infinity, in other words when n tends to infinity & $a \le x \le b$,the maximum of the absolute value $(f(x) - P_n(x))$ equals zero.

Therefore in accordance to this theorem, higher degree polynomials make the best approximation then best interpolation for f(x), but it seemed for Carl David Tolme Runge (1901), that it doesn't satisfy for certain functions likes $f(x) = 1/1+25x^2$, Runge discovered that an oscillation occurs at the edges of the function interval while polynomial interpolation by high degree over a set of equidistant points[17].

There are many researches specifically computerized ones have been done aiming at reasonable resolution of the assigned phenomenon, for instance[3] concludes in between that equispaced polynomial interpolation of $f(x) = 1/1+ 16x^2$ over [-1,1], as n increases, there is spectral convergence for $abs(x) < 0.7942$ and exponential divergence otherwise. Another one[8] makes reasonable comparison between interpolation of $f(x) =1/1+25x^2$ by uniformly spaced nodes & Chebyshev nodes, except that there are many approaches for oscillation minimizing as so called Change of interpolation points , it's going to use the densely distributed nodes close to the edges of [-1,1] , through one divided by square root of $(1-x^2)$, as it the Chebyshev nodes, moreover the constrained minimizations that coming on fitting a polynomial of n^2 degree to interpolating polynomial p whose p' or p'' has minimal L^2 norm[18], therefore the best approach which evidently comes in accordance with Mupad Function so named Numeric::cubicSpline(op(data))[12], that is using of piecewise polynomials (spline curves) to come across & treat oscillations of Runge's function, that is done through by increasing the number of polynomial pieces to construct the target spline & not to increasing the polynomial degree. Here below a set of Mupad models for odd & even order interpolating polynomial that exhibits a concrete resolution for Runge's Phenomenon:

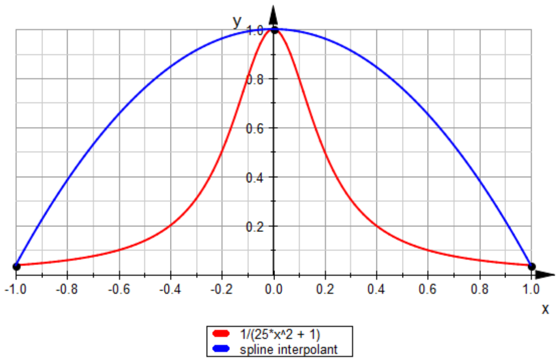
These models are constructed via the Mupad programme below within certain modification at the data line :

% Mupad programme generating models for Runge's phenomenon [8][12][13][16][1][17]

```
f:=x->1/(1+25*x^2):
data:=[[2*i/4,f(2*i/4)] $ i = -2..2]:
S:=numeric::cubicSpline(op(data)):
plot(plot::Function2d(f(x),x=-1..1,Color= RGB::Red,
               LegendText = expr2text(f(x))),
 plot::PointList2d(data, Color = RGB::Black),
 plot::Function2d(S(x), x = -1..1, Color = RGB::Blue,
               LegendText = "spline interpolant"),
 GridVisible = TRUE, SubgridVisible = TRUE,
 LegendVisible = TRUE):
```

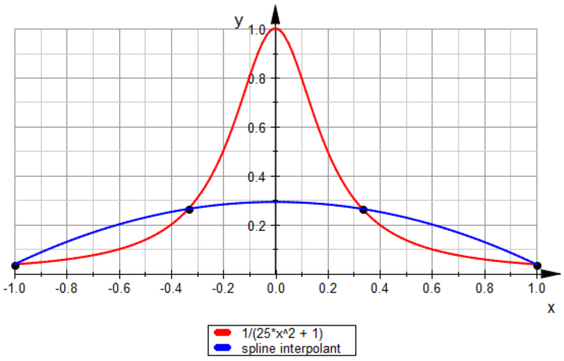
Mupad models via odd ordered pp

pp of 3rd order

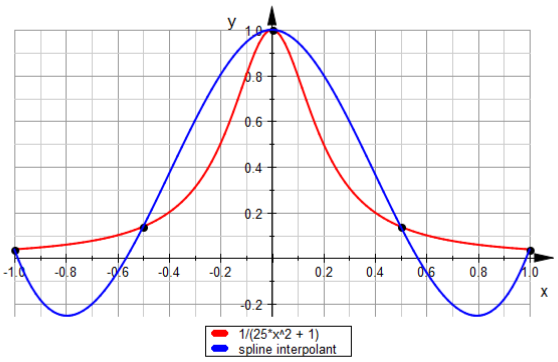


Mupad models via even ordered pp

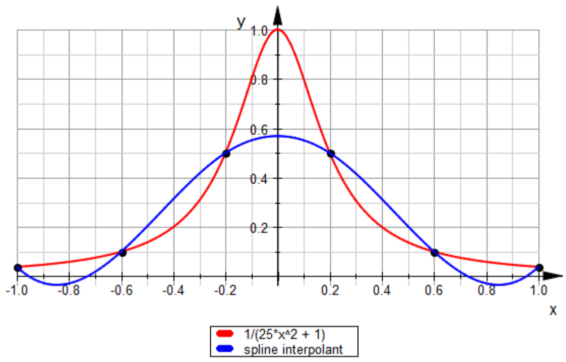
pp of 4th order



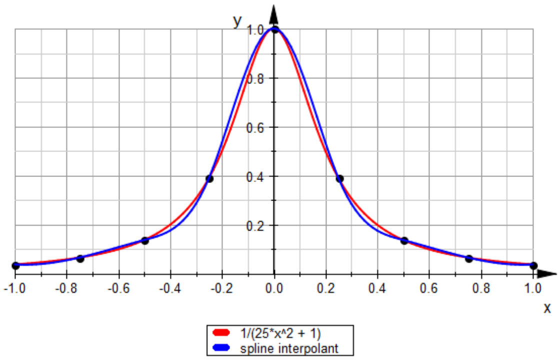
pp of 5th order



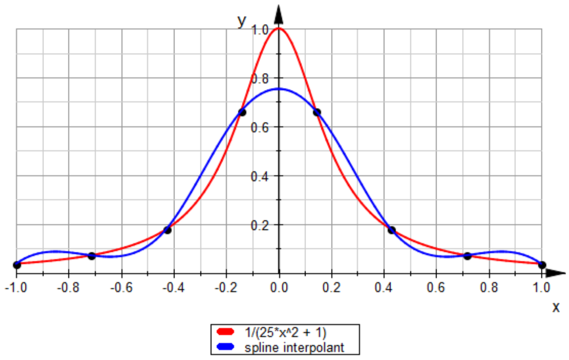
pp of 6th order



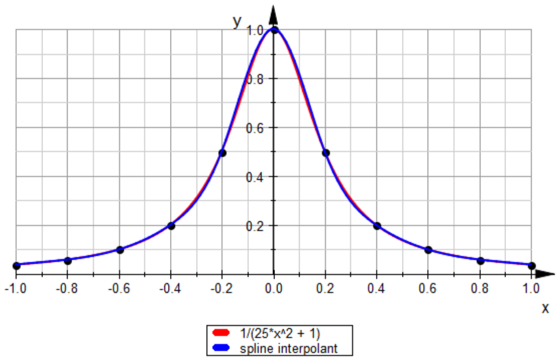
pp of 9th order



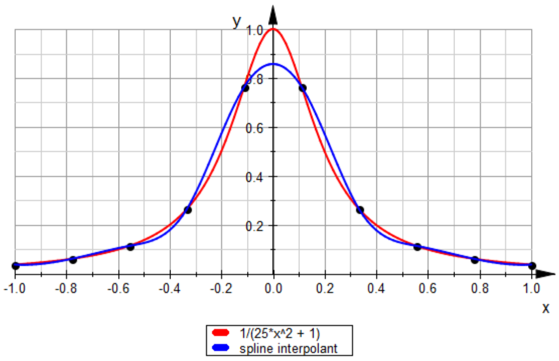
pp of 8th order



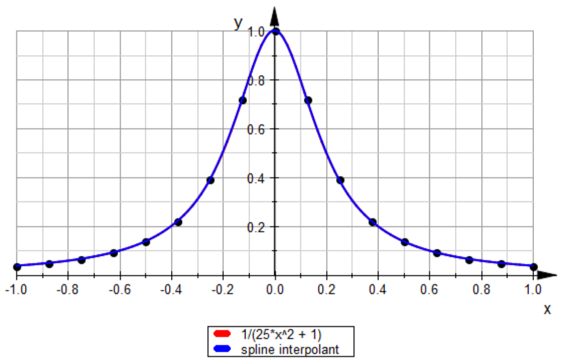
pp of 11th order



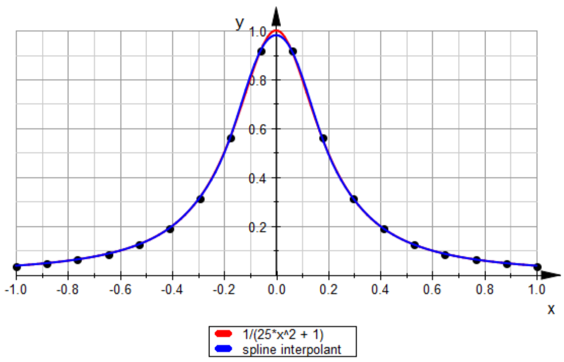
pp of 10th order



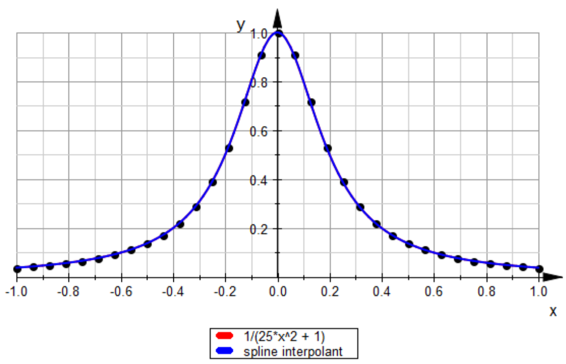
pp of 17th order



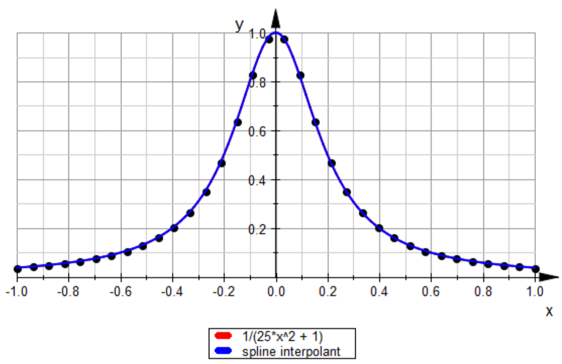
pp of 18th order



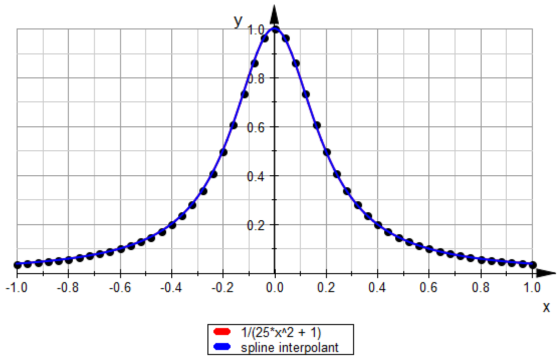
pp of 33rd order



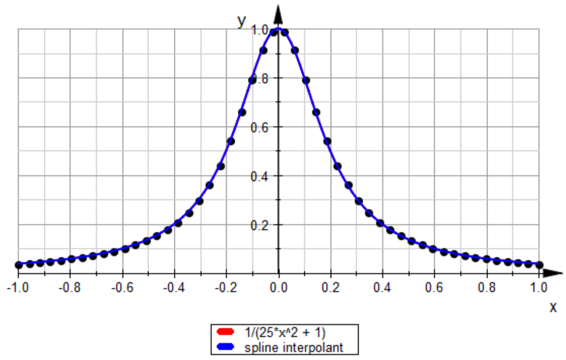
pp of 34th order



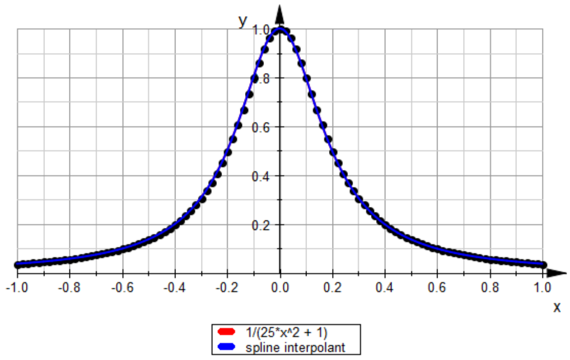
pp of 51st order



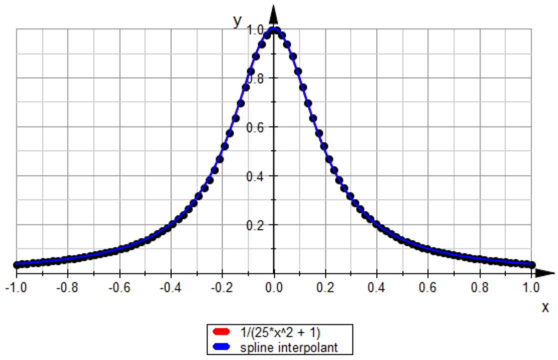
pp of 50th order



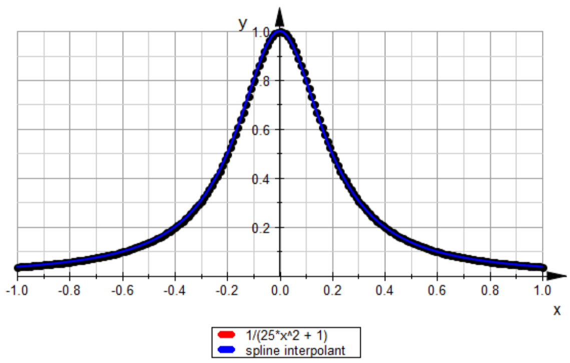
pp of 101st order



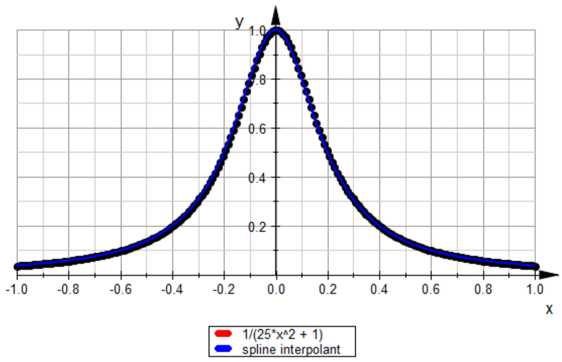
pp of 100 order



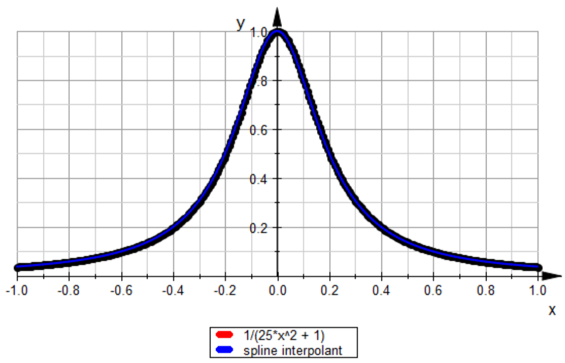
pp of 201st order



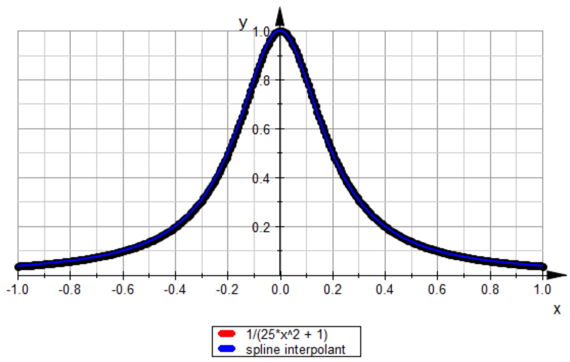
pp of 200th order



pp of 301st order



pp of 300th order



Therefore Runge's polynomial well interpolated as well as no oscillations have been appeared on the its edges, also there are no interweaving among the knots occurs after the 8th order of its interpolate, although the odd order interpolates exhibit better convergence than the even order ones, but pp interpolates behave in accordance to Weierstrass theorem.

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