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LAGRANGE DUALITY IN CONVEX OPTIMIZATION: INDUSTRIAL APPLICATION IN ABAKALIKI BAKERY FACTORY

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Abstract

The convex optimization problems of the form,

$$minimize\ \{f(x)|x \in X\}$$

can be transformed to their dual problems, called Lagrange dual problems, which help to solve the main problem. First, with the dual problem, one can determine lower bounds for the optimal value of the original problem. Again, under certain conditions, the solutions of both problems are equal (strong duality). In this case the dual problem often offers an easier and clear analytical approach to the solution. In this paper, we focus on the mechanics of Lagrange duality, its relation to primal and dual problems. A technique of proving strong duality via slater's constraint qualification is presented with particular application to baking factory in Abakaliki.

1. Introduction

In constrained optimization,

$$\begin{aligned} &\text{Minimize} \quad f(x) \\ &\text{Subject to } g_i(x) \leq 0,\ i = 1, \cdots, m \\ &\qquad\qquad h_i(x) = 0,\ i = 1, \cdots, p \end{aligned} \tag{1}$$

It is often possible to convert the original form of optimization problem (1) known as the primal problem to a dual form which is termed a dual problem. Duality is a powerful and widely employed tool in applied mathematics for a number of reasons. First, the dual program is always convex even if the primal is not. Second, the number of variables in the dual is equal to the number of constraints in the primal which is often less than the number of variables in the primal program. Third, the maximum value achieved by the dual problem is often equal to the minimum of the primal. Usually dual problem refers to the Lagrangian dual problem. The Lagrangian dual problem is obtained by using nonnegative Lagrange multipliers to add to the constraints to the cost function, and the solving for some primal variable values that minimize the lagrangian. This solution gives the primal variables as function of the Lagrange multipliers, which are called dual variables, so that the new problem is to maximize the objective function with respect to the dual variables under the derived constraints on the dual variables.

In general the optimal values of the primal and dual problems need not be equal the difference is called duality gap. For convex optimization problems the duality gap is zero under constraint qualification condition. Thus a solution to the dual provides a bound on the value of the solution to the primal problem. Essentially, the Lagrangian function is a linear combination of cost function and constraint function. Thus, the minimization problem (1) is of the structure,

$$L(x, \lambda, v) = f(x) + \sum_{i=1}^{m} \lambda_i\, g_i(x) + \sum_{i=1}^{p} v_i\, h_i(x) \tag{2}$$

Thereby reducing a constrained problem to an unconstrained problem.

1.1 Literature Review

Duality is one of the oldest and most fruitful ideas in mathematics. Survey of its history has shown how it has constantly been generalized. Duality in mathematics is not a theorem, but a "principle". It has a simple origin, it is very powerful and useful, and has a long history going back hundreds of years. Over time it has been adopted and modified and so can still use it in novel situations. It appears in many subjects in mathematics (geometry, algebra and analysis).Fundamentally, duality gives two different points of view of looking at the same object (Frenkel *et al*, 2001; Pólik, and Terlaky, 2007; Shapiro and Nemirovskii, 2003; Ramana,M.V. 1997; Shapiro and Scheinberg, 2000).

George *et al* (1997) maintained that Linear programming (LP or linear optimization) is a mathematical method for determining a way to achieve the best outcome (such as maximum profit or lowest cost) in a given mathematical model for some list of requirements represented as linear relationships. More formally, linear programming is a technique for the optimization of a linear objective subject to linear equality and linear inequality constraints. The feasible set of LP is called a convex polyhedron, which is a set defined as the intersection of finitely many half spaces (Ben-Israel *et al*, 1981).

According to Stephen Wright (1997), Alizadeh *et al* (1998) and Gonzalez-Lima *et al* (2009), primal-Dual interior- points methods, given any linear program, there is another related linear program called the dual, which provides an upper bound to the optimal value of the primal problem. In a matrix form, we can express the primal problem as

Maximize $c^T x$ subject to $Ax \leq b$, $x \geq 0$ with the corresponding symmetric dual problem,

Minimize $b^T y$ subject to $A^T y \geq y$, $y \geq 0$.

However, Michael J. Toad (2002) in his survey paper "The many facets of linear programming" says there are two ideas fundamental to duality theory. According to him, one is the fact that (for the symmetric dual) the dual of a dual linear program is the original primal linear program. Secondly, every feasible solution for a linear program gives a bound on the optimal value of the objective of its dual. Both Verma A. P. (2006) and, Vandenberghe and Boyd (1995) in their respective works state that if the primal or dual has a finite optimal solution, then the other one also possesses the same with equal optimal values of the objective functions.

Boyd and Vandenberghe (2009) in their textbook defined convex optimization as a special class of mathematical optimizing problems, which includes least-squares and linear programming problems. While the mathematics of convex optimization has been studied for about a century, several related recent developments have stimulated new interest in the topic. The first is the recognition that interior-points methods, developed in 1980 to solve linear programming problems; can be used to solve convex optimization problems. The second development is the discovery that convex optimization problems were more prevalent in practice than was previously thought (Jeyakumar and Li, 2009). Since 1990 many applications have been discovered in areas such as automatic control system, electric circuit design, estimation and signal processing, communications and networks, data analysis and modeling, statistics, and finance. There are great advantages to recognizing or formulating a problem as convex optimization problem. The most basic advantage is that the problem can be solved, very reliably and efficiently using interior-point methods or other special methods for convex optimizations. However, Herhenson *et al* found that solution methods are reliable enough to be embedded in a computer-aided design or analysis tool, or even a real-time reactive or automatic control system. However, Bertsekas (1999) said they are also theoretical or conceptual advantages of formulating a problem as a convex optimization problem. The associated dual problem, for example, often has an interesting interpretation in terms of the original problem and sometimes leads to an efficient or distributed method of solving it.

According to Ben-Tac and Nemirovski (2001), mathematical programming deals with optimization programs of the form; Minimize $f(x)$Subject to $g_{i(x)} \leq 0, i = 1, \dots, m \;\; X \subseteq \mathbb{R}^{\mathrm{n}}$ which essentially was born in 1948, where George Dantzig inverted linear programming as far as numerical processing of programs (P) is concerned, there exists a "solvable case" – the one of the convex optimization programs, where the objective f and constraints g_i are convex functions. Under minimal additional "computability assumptions" (which are satisfied in basically all applications) a convex optimization program is computationally "tractable" "moderately" with the dimensions of the problem and the required number of accuracy digits.

According to Dimitri *et al* (2002) in their paper Min Common/Max Crossing Duality, duality in optimization is often considered to be a manifestation of a fundamental dual description of a closed convex set, as the closure of the union of all line segments connecting the points of the set and the intersection of all closed halfspaces containing the set.This is largely true but it is also somewhat misleading, because the strongest duality theorems in optimization require assumptions such as the Slater condition and other constraint qualifications, whose connection to the dual description of closed convex sets is not readily apparent (Pataki G., 2007). As a result, one often observes a dichotomy in various developments of optimization duality theory found except perhaps in the eyes of a skilled mathematician. For example, the proof of the main duality theorem of linear programming is often developed based on Farkas' lemma (Jeyakumar and Lee, 2008) whose relation with the preceding dual closed convex set description is not readily apparent, and in other cases it is developed based on the termination properties of the simplex method (Koberstein and Suhl, 2007), with hardly any geometrical insight resulting.

2. Convex Sets

A set C is convex if, for any $x, y \in C$ and $\lambda \in \mathbb{R}$ with $0 \leq \lambda \leq 1$,

$$\lambda x + (1 - \lambda)y \in C. \quad (3)$$

Intuitively, this means that if we take any two elements in C, and draw a line segment between these two elements, then every point on that line segment also belongs to C.

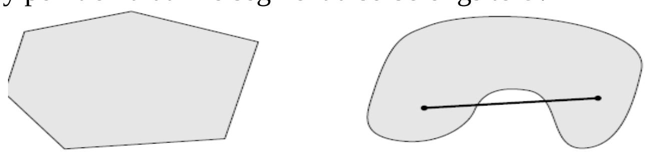

Fig.1: Shows an example of one convex and one non-convex set

2.1 Convex Function

Convex optimization is centred on the notion of convex function. A function $f: \mathbb{R}^n \longrightarrow R$ is convex if its domain (denoted $D(f)$) is a convex set, and for all $x, y \in D(f)$ and $\alpha \in \mathbb{R},\ 0 \leq \alpha \leq 1$,

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) \quad (4)$$

The way to think about this definition is that if we pick any two points on the graph of a convex function and draw a straight line between them, then the portion of the function between these two points will lie below this straight line.

3. Convexity and Differentiability

3.1 First Order condition for convexity

Theorem1 : Suppose S is a non-empty open convex set and $f(x): S \longrightarrow \mathbb{R}$ is differentiable (that is, the gradient $\nabla_x f(x)$ exists at all points x in the domain of f. Then $f(x)$ is convex function if and only if $f(x)$ satisfy the following gradient inequality

$$f(y) \geq f(x) + \nabla_x f(x)^T (y - x). \quad \text{for all } x, y \epsilon S \quad (5)$$

The function $f(x) + \nabla_x f(x)^T (y - x)$ is called first-order approximation to the function f at the point x. This can be thought of as approximating f with its tangent line at the point x. The first order condition for convexity says that f is convex if and only if the tangent line is a global under-estimator of the function f. In other words, if we take our function and draw a tangent line at any point, then every point on this line will lie below the corresponding point on f.

Proof: Suppose $f(x)$ is convex, then $\forall\ \alpha \in [0,1]$,

$f(\alpha y + (1 - \alpha)x) \leq \alpha f(y) + (1 - \alpha)f(x)$ this implies

$$\frac{f(x + \alpha(y - x) - f(x)}{\alpha} \leq f(y) - f(x)$$

Letting $\alpha \to 0$ $\nabla f(x)^t(y-x) \leq f(y) - f(x)$

$$\therefore \nabla f(x)^t(y-x) + f(x) \leq f(y)$$ establishes the ''only if '' part.

Now suppose that the gradient inequality holds for all $x, y \in S$. Let W and Z be any two points in S. Let $\alpha \in [0,1]$, and set $x = \alpha w + (1-\alpha)z$.

Then, $f(w) \geq f(x) + \nabla f(x)^t(w-x)$ and $f(z) \geq f(x) + \nabla f(x)^t(z-x)$

Taking the convex combination of the above inequalities we obtain

$$\alpha f(w) + (1-\alpha)f(z) \geq f(x) + \nabla f(x)^t(\alpha(w-x) + (1-\alpha)(z-x)$$

Expanding the coefficient of $\nabla f(x)^t$, we have

$$\alpha w - \alpha x + z - x - \alpha z + \alpha x$$

$$= \alpha w + z - x - \alpha z.$$ Recall that $x = \alpha w + (1-\alpha)z$.

We have therefore, $\alpha w + z - \alpha w - z + \alpha z - \alpha z = 0$

$$\therefore \alpha f(w) + (1-\alpha)f(z) \geq f(x) + \nabla f(x)^t 0$$

$$= f(x)$$

$$= f(\alpha w + (1-\alpha)z)$$

Which shows that $f(x)$ is convex.

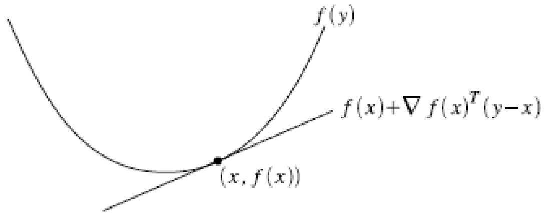

Fig 3: Illustration of first-order condition for convexity

3.2 Second Order Condition for Convexity

Theorem 2: Suppose C is a non-empty open convex set and $f(x): \mathrm{C} \longrightarrow \mathbb{R}$ is twice differentiable (that is, the Hessian $\nabla_x^2 f(x)$ is defined for all point x in the domain of $f(x)$). Let $H(x)$ denote the Hessian of $f(x)$. Then $f(x)$ is convex if and only if $H(x)$ is positive semidefinite for all $x \epsilon C$. That is,

$$\nabla_x^2 f(x) \geq 0 \text{ for all } x\epsilon C$$

For a function on $\mathbb{R}$, (i.e. in one dimension) this reduces to the simple condition $f''(x) \geq 0$ (and dom$f$ convex) which means that the derivative is non-decreasing. The condition $\nabla_x^2 f(x) \geq 0$ can be interpreted geometrically as the requirement that the graph of the function have positive (upward) curvature at x. Observe that;

(a) If $\nabla^2 f(x)$ is positive semi-definite for all $x \in C$, then f is convex over C.

(b) If $\nabla^2 f(x)$is positive definite for all $x \in C$, then f is strictly convex over C.

(c) If C is open and f is is convex over C, then $\nabla^2 f(x)$ is positive semi-definite for all $x \in C$

Proof: suppose $f(x)$ is convex. Let $\bar{x} \in C$ and d be any direction. Then for $\lambda \geq 0$ sufficiently small $\bar{x} + \lambda d \in C$. We have

$$f(\bar{x}+\lambda d) = f(\bar{x}) + \nabla f(\bar{x})^t(\lambda d) + \frac{1}{2}(\lambda d)^t \mathrm{H}(\bar{x})(\lambda d) + \|\lambda d\|^2 \alpha(\bar{x},\lambda d),$$

Where $\alpha(\bar{x},\lambda d) \to 0$ as $y \to 0$

Using gradient inequality, we obtain

$$\lambda^2(\frac{1}{2}d^t \mathrm{H}(\bar{x})d + \|d\|^2 \alpha(\bar{x},\lambda d) \geq 0$$

Dividing by $\lambda^2 \geq 0$, and letting $\lambda \to 0$, we obtain

$d^t \mathrm{H}(\bar{x})d \geq 0$ proving the ''only if ''part.

Conversely, suppose that $\mathrm{H}(z)$ is positive definite for all $z \in C$. Let $x, y \in C$ be arbitrary. Invoking the second-order version of the Taylor's theorem, we have:

$$f(y) = f(x) + \nabla f(x)^T(y-x) + \frac{1}{2}(y-x)^T H(z)(y-x)$$

For some z which is a convex combination of x and y (and hence $z \in C$). Since $\mathrm{H}(z)$ is positive semidefinite, the gradient inequality holds, and hence f is convex.

4. Continuity of Convex Functions

Theorem 4: Let $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$. Suppose f is convex on D where D is non-empty, open and convex then f is continuous on D.

Proof: The set D is open $\Rightarrow \forall\, x \in D\ \exists\, r > 0$: $B(x,r) \subset D$.

Suppose that f is convex, we show that $\forall\ (x_n) \subset (D)$: the sequence $x_n \longrightarrow x$ and $f(x_{n)} \longrightarrow f(x)$

Define $A = \{Z | \|z - x\| = \alpha\}$. Then $A \subset B(x,r)$.

$\forall\ n \geq N\ \exists z_n \in A,\ \ni\ \ \lambda_n \in (0,1)$, $1-\lambda_n \to 0$ and $\lambda_n \to 1$

We have that,

$$x_n = \lambda_n + (1-\lambda_n)z_n$$

By convexity of f,

$$f(x_n) \leq \lambda_n f(x) + (1-\lambda_n) f(z_n)$$

Taking the limit, as $n \to \infty$ and $\lambda_n \to 1$

$$\limsup_{n\to\infty} f(x_n) \leq f(x) \qquad (1)$$

Similarly, $\forall\ n \geq N\ \exists w_n \in A,\ \ni\ \lambda_n \in (0,1)$, $1-\lambda_n \to 0$ and $\lambda_n \to 1$ such that

$x \leq \lambda_n x_n + (1-\lambda_n) w_n$

$$f(x) \leq \lambda_n f(x_n) + (1-\lambda_n) f(w_n)$$

Since f is convex

$$f(x) \leq \liminf_{n\to\infty} f(x_n) \qquad (2)$$

Combining equation (1) and (2)

$$\limsup_{n\to\infty} f(x_n) \leq f(x) \leq \liminf_{n\to\infty} f(x_n)$$

So $f(x_n) \longrightarrow f(x)$; $\therefore$ f is continuous $\forall\ x \in D$ hence the proof.

5. Convex Optimization Problems

A convex optimization problem is the study of mathematical optimization problems of the form ;

Minimize $f(x)$

Subject to $x \in C$ (5)

where $f: R^n \to R$ is convex function, C is a convex set, and $x \in R^n$ is the optimization variable. However, since this can be a little vague, we will often write it as

minimize$f(x)$

subject to $g_i(x) \leq 0,\ i = 1, \cdots, m$

$h_i(x) = 0,\ i = 1, \cdots, p$

where f is a convex function, g_i are convex functions, and h_i are affine functions, and x is the optimization variable.

6. The Lagrangian

The basis of Lagrange duality theory is an artificial construct called the lagrangian. Given a convex constrained minimization problem of the form;

$(P) \min_{x \in \mathbb{R}^n} (fx)$

Subject to $g_i(x) \leq 0, \quad i = 1, \cdots, m$ (3.1)

$h_i(x) = 0, \quad i = 1, \cdots, p$

where $x \in \mathbb{R}^n$ is the optimization variable, $f: \mathbb{R}^n \to \mathbb{R}$ and $g_i: \mathbb{R}^n \to \mathbb{R}$ are differentiable convex function and $h_i: \mathbb{R}^n \to \mathbb{R}$ are affine functions: The lagrangian of the above problem (3.1) is the function $L: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ defined by

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i\, g_i(x) + \sum_{i=1}^{p} \mu_i\, h_i(x) \qquad (3.2)$$

The first argument of the lagrangian is a vector $x \in \mathbb{R}^n$, whose dimensionality matches that of the optimization variable in the original optimization problem. Hence x is referred as the primal variables of the lagrangian. The second argument $\lambda \in \mathbb{R}^m$ with one variable λ_i for each of the m convex inequality constraints in the original optimization problem the final argument of the lagrangian is a vector $\mu \in \mathbb{R}^p$, with are variable μ_i for each of the affine equality constraints in the original optimization problem. These elements λ and μ are collectively known as the dual variables of the lagrangian or Lagrange multipliers. The lagrangian can be seen as the costs associated with violating different constraints.

6.1 Langrange Dual Function

The Lagrange dual function is defined as the infimum of the lagrangian over x: $q: \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}, \quad q(\lambda, \mu) = \inf L(x, \lambda, \mu)$

$q(\lambda, \mu) = \inf\left(f(x) + \sum_{i=1}^{m} \lambda_i\, g_i(x) + \sum_{i=1}^{p} \mu_i\, h_i(x)\right)$ (3.3)

Lemma 1. If (λ, μ) are dual feasible, then $q(\lambda, \mu) \leq p^*$. (3.4)

Where p^* denotes the primal optimal value.The dual function provides lower bounds on the optimal value p^* of the original problem when $\lambda \geq 0$.

Proof:

Observe that, $q(\lambda, \mu) = \inf_{x \epsilon X} L(x, \lambda, \mu) \leq L(x^*, \lambda, \mu)$

$$= f(x^*) + \sum_{i=1}^{m} \lambda_i\, g_i(x^*) + \sum_{i=1}^{p} \mu_i\, h_i(x^*)$$

$$\leq f(x^*) = p^*$$

The lemma shows that that given any dual feasible(λ, ν), the dual objective $g(\lambda, \nu)$, provides a lower bound on the optimal value p^* of the primal problem. Since the dual problem involves maximizing the dual objective over the space of all dual feasible(λ, ν), it follows that the dual problem can be seen as a search for the tightest possible lower bound on p^*. This gives rise to a property of any primal and dual optimization problem pairs known as weak duality: However, the lower bound from Lagrange dual function depends on (λ, μ); therefore the best lower bound that can be obtained from Lagrange dual function is

$(D) =$ maximize $\mathrm{q}(\lambda, \mu)$ (3.5)

Subject $\lambda \geq 0$

Equation (3.5) is called Lagrange dual problem with dual variables (λ, μ). Observe that (3.5) again is convex optimization (maximaztion of concave functions over linear constraints and Lagrange dual problem associated with (3.1). The optimal value of (3.5) is denoted by d^*.If $g(\lambda, \mu) = -\infty$ the dual function gives a non-trivial lower bound on p^*. This is possible only where $\lambda \geq 0$ and $(\lambda, \mu) \in Dg(\lambda, \mu) > -\infty$. The pair (λ, μ) is therefore referred to as dual feasible. The pair (λ^*, μ^*) is called dual optimal or optimal Lagrange multipliers.

Theorem 5: Weak Duality

Let x^* be any feasible solution to the primal problem (3.1) and (λ^*, u^*) be a feasible solution to the dual problem (3.5) . Then, the objective function of D evaluated at (λ^*, u^*) is less or equal to the objective function P evaluated at x^* that is $q(\lambda^*, \mu^*) \leq f(x^*)$.

Proof: Let x^* be any feasible point meaning that $g(x^*) \leq 0$, and $h(x^*) = 0$

Then we have, for any μ and $\lambda \geq 0$.

$$\sum_{i=1}^{m} \lambda_i\, g_i(x^*) + \sum_{i=1}^{p} \mu_i\, h_i(x^*) \leq 0,$$

$$\Rightarrow L(x^*, \lambda, \mu) = f(x^*) + \sum_{i=1}^{m} \lambda_i\, g_i(x^*) + \sum_{i=1}^{p} \mu_i\, h_i(x^*) \leq f(x^*)$$

$$\Rightarrow q(\lambda, \mu) = \inf L(x, \lambda, \mu) \leq L(x^*, \lambda, \mu) \leq f(x^*),\ \forall x^* \quad (3.7)$$

Clearly, weak duality is a consequence of Lemma 1 using (λ^*, u^*) as the dual feasible point. For some primal/dual optimization problems, an even stronger result holds, known as strong duality:

Theorem 6: (Strong Duality). For any pair of primal and dual problems which satisfy certain technical conditions called constraint qualifications, both optimization problems (3.1) and (3.5) has the same solution that is,

$$d^* = p^*. \tag{3.8}$$

The proof of strong duality shall be x-rayed in the next chapter using two techniques namely strong duality via slaters'constraint qualification and convex theorem on alternatives.

Example 1: Linear optimization duality.

Consider

$$\text{Minimize}\{C^T x \mid Ax \geq \text{b}\}$$

The lagrangian is

$$\text{L}(\lambda, v) =: C^T x + \lambda^T(b - Ax)$$

The Lagrange dual function is

$$g(\lambda) = \min_x C^T x + \lambda^T(b - Ax) = \begin{cases} \lambda^T b \; if \; C^T = \lambda^T A \\ -\infty \; otherwise \end{cases}$$

So the dual is

$$max\{g(\lambda) \mid \lambda \geq 0\} = \max\{\lambda^T b \mid c^T = \lambda^T A, \lambda \geq 0\}$$

We have strong duality.

7. Strong Duality via Slater's Constraint Qualification

Strong duality is a condition that holds when the optimality gap is zero, which is equation (3.8). This case is possible if the optimization problem is convex and satisfies slater's constraint qualification.

Theorem 9: Consider the convex optimization problem of the form;

$$\begin{aligned} &\text{minmize } f(x) \\ &\text{Subject to} \quad g_i(x) \leq 0 \;\; i = 1, \cdots, m \\ &\qquad\qquad\quad Ax = b \end{aligned} \tag{4.1}$$

With the assumption that $f, g_i, \cdots, g_m$ are convex $A \in \mathbb{R}^{p,n}$, $b \in \mathbb{R}^p$ then there exists an $x \in relint\mathcal{F}$ such that

$$g_i(x^*) < 0, \qquad i = 1, \cdots, m, \;\; Ax^* = b \tag{4.2}$$

Implying existence of strictly feasible point.

Proof: We begin by defining the set.

$$\mathcal{A} = \{(u, v, t)\colon \exists x \in \mathcal{F}, g_i(x) \leq \mu, \;\; h_i(x) = v, f(x) \leq t\}:$$

This is convex since each of the $g_i(x)$ are convex functions.

The optimal solution to our problem is

$$p^* = \inf\{t\colon (0,0,t) \in \mathcal{F}\}.$$

In order to evaluate the dual function at a point (λ, μ) with $\lambda \geq 0$, we can minimize the affine function.

$$(\lambda, \mu, 1)^T(u, v, t) \text{ over } \mathcal{A}$$

Next, if $\lambda \geq 0$

$$g(\lambda, v) = \inf\{(u, v, t)(\lambda, \mu, 1)^T\}: (u, v, t) \in \mathcal{A}$$

If the infimum is finite then

$$(\lambda, v, 1)^T (u, v, t) \geq g(\lambda, v) \tag{4.3}$$

Defines a nonvertical supporting hyperplane to $\mathcal{A}$. In particular since

$(0,0, p^*) \in \text{bd}\ \mathcal{A}$,we have

$$p^* = (\lambda, v, I)^T (0,0, p^*) \geq g(\lambda, v\ ) \tag{4.4}$$

Strong duality holds if an only if we have equality in (4.4).

7.1 Slater's Constraint Qualification

Definition: (Slater's condition) There exist $x^* \in \mathbf{relint}\mathcal{F}$ with $g_i(x) < 0$ for $i = 1, \cdots, m$ and $Ax^* = b$ (existence of strickly feasible point).

Theorem 10:- Slater's condition implies strong duality

Proof: We shall make two additional assumption before the proof: first that $\mathcal{F}$ has a nonempty interior, hence $\text{relint}\mathcal{F} = \text{int}\mathcal{F}$, and second, that rank $A = p$. We assume that p^* is finite (since there is a feasible point, we can only have $p^* = -\infty$ or p^* finite; if $p^* = -\infty$, then $d^* = -\infty$ by weak duality.

Consider the following convex set

$\beta = \{(0,0, S) \in \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}: S < p^*\}$, which is obviously disjoint from $\mathcal{A}$ by separating hyperplane theorem, there exists $(\lambda^*, v^*, \mu) \neq 0$ and α such that

$$(u, v, t) \in \mathcal{A} \Rightarrow U\lambda^{*T} + Vv^{*T} + t\mu \geq \alpha \tag{4.5}$$

And $$(u, v, t) \in B \Rightarrow \lambda^{*T}\mu + V^{*T}v + \mu t \leq \alpha \tag{4.6}$$

This implies that $\lambda^* \geq 0$, by equation (4.5) (since $\mathcal{A}$ is closed under u getting larger), and $\mu \geq 0$, by the second which says that $\mu t \leq \alpha$. From the inequality (4.5) for any $x \in \mathcal{F}$, we have,

$$\sum_{i=1}^{m} \lambda^*{}_i\, g_i(x) + v^{*T}(Ax - b) + \mu f(x) \geq \alpha \geq \mu p^* \tag{4.7}$$

We proceed in two cases

Case I: $\mu > 0$ dividing (4.7) by μ, we obtain

$$L\left(x, {}^{\lambda^*}\!/_{\mu}, {}^{v^*}\!/_{\mu}\right) \geq p^*$$

For all $x \in \mathcal{F}$. Thus minimizing over x it follows that $g(\lambda, v) \geq p^*$ where $\lambda = {}^{\lambda^*}\!/_{\mu}$ and $v = {}^{v^*}\!/_{\mu}$. By weak duality, we have

$g(\lambda, v) \leq p^*$, so infact $g(\lambda, v) = p^*$. This shows that strong duality holds and that the dual optimum is attained.

Case II: $\mu = 0$; Using (4.6) it follows that for x^* satisfying slater's condition, we have

$$\sum \lambda_i^* g_i(x^*) \geq 0$$

Therefore $\lambda^* = 0$ since all $g_i(x^*) < 0$ and $\lambda^* \geq 0$. From $(\lambda^*, v^*, \mu^*) \neq 0$ and $\lambda^* = \mu = 0$, we conclude that $v^* \neq 0$. Thus (4.6) implies that

$$V^*(Ax - b) \geq 0.$$

By assumption, $V^*(Ax - b) = 0$, since $x^* \in \text{in}\, t\,\mathcal{F}$. It follows that there exists points in $\mathcal{F}$ with $V^{*T}(Ax - b) < 0$ unless $A^T v^* = 0$, thus contradicting our assumption that rank $A = p$

8. Application of Duality

A baking factory produces bons[1] and chinchin[2]. Each product requires raw material resource and labour. Production times required for the product are measured at different times respectively. Total labour hours per week are only 96 hours. Raw material required for Bons and chinchin are 2 and 3 units respectively. Total supply of raw material is 60 units per week. The net profit per units of products, resource requirements of the product and labour are summarized below in Table 5.1(a).

	Baking time (hours)		
Products	**Bons**	**Chinchi**	**Total per week**
Raw material constraint	2	3	60
Labour hours constraint	4	3	96
Profit /unit (Naira)	40	35	

Table 5.1(a): Net profit per units of products, resource requirements of the product and labour.

The objective is to determine how many units of bons and chinchin that should be produced per week to maximize the profit.

The working model for the factory's problem is

Maximize $Z = 40x_1 + 35x_2$

Subject to $2x_1 + 3x_2 \leq 60$ *raw material constrait*

$4x_1 + 3x_2 \leq 96$ *labour hour constrait*

$x_1, x_2 \geq 0$

In canonical form

Maximize $Z = 40x_1 + 35x_2 + 0s_1 + 0s_2$

Subject to $2x_1 + 3x_2 + S_1 = 60$

$4x_1 + 3x_2 + S_2 = 96$

$x_1, x_2, S_1 \text{and } S_2 \geq 0$

The model formulated above is the primal and its equivalent to

$$AX \leq b \text{ and } Z = CX,\ X \geq 0$$

Where a_{ij} is called the technological matrix

b = resource availability.

$$Z = [40, 35]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A = \begin{bmatrix} 2 & 3 \\ 4 & 3 \end{bmatrix}, b = \begin{bmatrix} 6 & 0 \\ 9 & 6 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

[1] Locally made round-shaped bakery product in Nigeria

[2] Nuts-like bakery products locally made in Nigerian bakery factories

$$b = \begin{bmatrix} 6 & 0 \\ 9 & 6 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

x_1 and x_2 are the production volumes of bons and chinchin.

8.1 Dual of the Problem

Let y_1, y_2 be the dual variables. The problem is to determine y as to minimize $f(y)$; ie.

$$[60, 96] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = f(y)(minimize)$$

$$\begin{bmatrix} 2 & 4 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq [40, 35]$$

We find $f(y)$ to

Minimize $60\, y_1 + 96\, y_2$

$subject\ to\ 2y_1 + 4y_2 \geq 40$

$3y_1 + 3y_2 \geq 35$

$y_1, y_2 \geq 0$

By converting the dual form into standard form; we now have

$$maximize\ \ Z^* = -60y_1 - 96y_2 + OS_1 + OS_2$$

$$subject\ to\ 2y_1 + 4y_2 - S_1 = 40$$

$$3y_1 + 3y_2 - s_1 = 35$$

Introducing the artificial variable M we have

$$maximize\ \ Z^* = -60y_1 - 96y_2 + OS_1 + OS_2 - MR_1 - MR_2$$

$$subject\ to \qquad 2y_1 + 4y_2 - S_1 + R_1 = 40$$

$$3y_1 + 3y_2 - S_2 + R_2 = \quad 35$$

$$y_1, y_{2,} s_1, s_{2,} R_1, R_2 \geq 0$$

Tableau 2

	C_j	-60	-96	0	0	$-M$	$-M$	
c_i	basis	y_1	y_2	S_1	S_2	R_1	R_2	b
-96 -60	y_2 y_1	0 1	1 0	$\frac{-1}{2}$ $1/2$	$1/3$ $-\frac{2}{3}$	$-1/3$ $-2/3$	$-\frac{1}{3}$ $2/3$	25/3 $\frac{10}{3}$
Z_j $C_j - Z_j$		-60 0	-96 0	18 -18	8 -8	-18 $-(M-18)$	-8 $-(M-8)$	- 1000

Table 5.2(b): Final Table of the dual problem

Optimality conditions: since all $(c_j - Z_j) \leq 0$. The optimality is reached; hence, the optimal solution is $y_2 = {}^{25}/_3$, $y_1 = {}^{10}/_3$

$$f(y)_{min} = ₦1000$$

The optimal solution is to make 18 pieces of bons and 8 packs of chinchin per week. With this optimal strategy, the net profit is ₦ 1000.00. Hence, the optimal value is the same. Therefore, there is no duality gap.

9. Economic Interpretation of the Dual

Suppose the factory wishes to buy insurance form the net profit. Let y_1 be the amount (₦) payable to the factory for every labour hour cost due to accident and let y_2 be the amount (₦) payable to the factory for every raw material unit cost due to late delivery. The insurance officer therefore tries to minimize the total amount ₦($60y_1 + 96y_2$) payable to the factory. The factory however insists that the insurance company covers all his loss that is his net profit since he cannot make the products. The company's problem is

$$minimize \;\; Z = 60y_1 + 96y_2$$

$$subject\ to \qquad 2y_1 + 4y_2 \geq 40 \;\; Net\ profit\ from\ bons$$

$$3y_1 + 3y_2 \geq 35 \; Net\ profit\ from\ chinchin$$

$$y_1, y_2 \geq 0$$

The dual solution provides an interesting economic interpretation such as the shadow price (marginal elements of RHS element.) The shadow price can be seen as the improvement in the objective value per unit increase in the right hand side. The shadow prices are the solution to the dual problem. The shadow prices (imputed price) can as well be seen as the opportunity costs associated with the insurance policy. The unit measure of the shadow price is $y_1 =$ ₦ ${}^{10}/_3$ per unit of raw material and $y_2 = {}^{25}/_3$ per hour for the labour.

Suppose the insurance company agrees with the factory. The imputed cost for every 1 unit of bons lost is 2 unit of raw material x ₦ ${}^{10}/_3$ per unit material + 4 hours of labour x ₦ ${}^{25}/_3$ per hour

$$\frac{20}{3} + \frac{100}{3} = ₦\ 40$$

Again for every 1 unit of chinchin lost, 3 unit of raw material X ₦ $^{10}/_{3}$ per unit material +3 unit of labour X ₦$^{25}/_{3}$/hr

= 10 + 25 = ₦35

. If the factory's net profit is valued at the shadow prices, we find that

$Z_{min} = 60\ X\ ^{10}/_{3} + 96\ X\ ^{25}/_{3} =$ ₦ 1000.00 is exactly equal to the optimal value of the objective function of the factory's decision problem.

With the optimal value of ₦ 1000.00 (the amount the factory expects to receive). The business can be managed smoothly except for premium that the insurance company will charge.

Observe that, the shadow prices of the dual gives the decision variables of the primal, hence when the factory bakes 18 pieces of bons ($x_1 = 18$) and sells making a profit of ₦40.00 for each and for 8 packs of chinchin ($x_2 = 8$) sold and making a profit of ₦35.00. The total profit turns out to be ₦1000.00 Moreover, the elements of the dual solution called lagrargian multiplers provides a tight bound on the optimal value of the primal. The dual solution of the factory's problem can be used to find a lower tight bound for the optimal value. This can be computed by multiplying each constraint by its corresponded dual solution and then sum them up;

$$^{10}/_{3}\ [2x_1 + 3x_2 \leq 60]$$

$$^{25}/_{3}\ [4x_1 + 3x_2 \leq 96]$$

$$\overline{40x_1 + 35x_2 \leq 1000}$$

The result on the left side is the objective function of the primal and a lower tight bound.

10. Sensitivity Analysis

Sensitivity analysis is an essential tool used to determine how the optimal solution changes when the parameters of the models are changed. It investigates the effects of the uncertainty on the model recommendation.

10.1 Changes in the Objective Function

In business environment, the net profit is an uncontrollable factor. The fixed prices of ₦ ($40\ and\ 35$) are the uncontrollable inputs that are determined by the market. The range of the objective function coefficient C, say must be determined before introducing new product so that the optimality will not be affected.

Any product that will be introduced must be within the range of

$$₦\left(\left[35 - {^{10}/_{3}} + 40 - {^{25}/_{3}}\ \right] and \left[35 + {^{10}/_{3}} + 40 + {^{25}/_{3}}\ \right]\right)$$

38.33 + 48.33 = ₦86.66k

$$\text{Again } \left[35 - {^{10}/_{3}} + 40 - {^{25}/_{3}}\ \right] = 63.33k.$$

The range therefore is

$$₦\ [63.33 \leq C \leq 86.66\ ]$$

10.2 Introducing New Product

Suppose the factory decides to introduce meat pie with profit margin of ₦70.00. The labour hours required for the production is 3 hours and 6 units of raw material, what should the factory do? We compute as follows:

6 unit of raw material $X\ ^{10}/_{3} = 60$

2 hours of labour $X\ ^{25}/_{3} = 25$

Total is ₦ 85.00

Since ₦$[85 > 63.33]$ *and less than* ₦ $86.66k$ the factory can go on with the production.

10.3 Changes in the Right Hand Side Constant of Constraints

The shadow price for product 1 (Bons) is ₦ $^{10}/_3$ and that of 2 (chinchin) is ₦ $^{25}/_3$. The shadow prices are also called marginal profitability of the products making additional bons would increase the factory's net profit by ₦ $^{10}/_3$ while if one bons is removed, the profit will be lowered by that amount. Similarly, making one additional pack of chinchin would increase its profit by the same amount if it is reduced by one pack.

Suppose the right side constants of constraint 1 and constraint 2 are changed from 60 to 30 and 96 to 60 respectively. What is the new optimum value? The changes can be incorporated in the constraints by using the formula.

$$\begin{matrix} Basic\ variables \\ in\ optimal\ table \end{matrix} = \begin{bmatrix} values\ of\ the \\ slack\ variables \end{bmatrix} \begin{bmatrix} new\ RHS \\ constant \end{bmatrix}$$

$$\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} ^2/_3 & -^1/_3 \\ -^1/_2 & ^1/_2 \end{bmatrix} \begin{bmatrix} 30 \\ 6o \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

$x_1 = 15$ *and* $x_2 = 10$. These values are non-negative hence the revised solution is feasible and optimal. The corresponding optimal objective function value is ₦ 950.00.

11. Conclusion

This paper has essentially looked at the mechanics of lagrangian duality, previewed a technique of proving that under favourable circumstances, two optimal values are equal; we refer to this relation as strong duality and applied duality to a baking factory where the sensitivity analysis of the perturbed problem were performed.

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