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HIGHER ORDER FORMS OF SOME ASYMMETRIC KERNELS

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ABSTRACT

This paper is to investigate the properties of higher-order form of some asymmetric kernels and consider its relevance to human life. The higher – order forms of some asymmetric kernels were derived from their asymmetric counterpart based on the formula by Jones and Foster (1993). The performances of these asymmetric kernels and their higher – order counterpart were considered in terms of their mean integrated squared error, asymptotic mean integrated squared error (AMISE) and their optimal window width h. The results of these asymmetric kernels and that of their higher – order asymmetric counterparts are compared. An appreciable error reduction was achieved in the higher-order asymmetric kernel when compared to their asymmetric counterparts. The error propagation also drops considerably as h increases. This method can be used to forecast performances of stock exchange and insurance claim.

Key words: asymmetric kernels, higher – order forms of asymmetric kernels, exponential and gamma kernels, asymptotic mean integrated squared error (AMISE) and the optimal window width.

1.1 INTRODUCTION

The univariate kernel density estimator is generally given by

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} k\left(\frac{x - X_i}{h}\right) \qquad 1.1$$

where the kernel function k (.) is a probability density function (pdf) which integrates to one, n is the sample size, h is the window width and $X_1, X_2, X_3, ..., X_n$ is a univariate set of data drawn from a continuous distribution function f. Many existing literature suggest that the choice of the window width is crucial Silverman (1986) and Sheather (2004).

The method behind the proposed higher -order form of asymmetric kernel lies in the use of the formula by Jones and Foster (1993). In their work, they highlighted that if $k_{(i)}(t)$ denotes an i^{th} order kernel, then any higher –order kernel is given by

$$k_{(i+2)}(t) = \frac{3}{2}k_{(i)}(t) + \frac{1}{2}tk^1_{(i)}(t) \qquad 1.2$$

We shall use (1.2) to generate the higher – order asymmetric kernels. To investigate this, we took the exponential and the gamma kernels as our asymmetric kernels.

The derivation of the schemes for AMISE lies mainly in the application of Taylor's Series expansion of f(x) up to order 2m (where m is positive integer). These ideas are applied to the proposed higher –order asymmetric kernels of choice which are the exponential and the gamma kernels.

This work is necessitated by the presence of asymmetric densities in most disciplines of Engineering, Sciences and the Social Sciences. It has been observed in real life practice that some useful probability functions do not really exhibit the symmetric character. The exponential, the gamma and the beta distributions are some examples of asymmetric distribution Rohatgi (1984). The families of the distribution above provide probability models that are very useful in Engineering and Science disciplines Devore (1991) and Mugdadi and Lahrech (2004). For instance, the memory-less property of most pieces of equipment, as well as the waiting time a customer spends in a restaurant servicing point before being served are some distinctive uses of the exponential distributions Rohatgi (1984).
All asymmetric kernels share the property that the shape of the kernel changes according to the value of data (x). A good example of this asymmetric kernel according to Hagmann and Scaillet (2003) is the gamma kernel.

2.1 ASYMMETRIC KERNEL OF ORDER ONE.

As for the asymmetric kernel, suppose we define it with the following asymmetric conditions:

$$\left.\begin{array}{ll}(1) & \int k(t)\,dt = 1 \\ (2) & V^2 = \int t^2 k(t)^2\,dt < \infty\end{array}\right\} \qquad 1.3$$

We also assume f^{1} and f^{11} are not only continuous but also square integrable with $\lim_{n\to\infty} h = 0$ and $\lim_{n\to\infty} nh = \infty$

3.1 HIGHER ORDER ASYMMETRIC KERNELS.

Improving the conditions of (1.3) and modify it to read

$$\left.\begin{array}{ll}(i) & \int k(t)dt = 1 \\ (ii) & \int t^2\, k(t)dt = \int t^4 k(t)\; dt = \ldots = \int t^{2m-2} k(t)dt = 0 \\ . & \\ . & \\ . & \\ (iii) & \int t^{2m-1}\, k(t)dt = J_{2m-1} \neq 0 \forall m = 1(1)r\end{array}\right\} \qquad 1.4$$

Then the optimal window width and the global error for (1.4) are respectively:

$$MISE\;\; \hat{f}(x) \;\approx \frac{1}{\left((2m-1)!\right)^2}\int h^{4m-2} J^2{}_{2m-1}\, f^{(2m-1)}(x)^2\, dx + \frac{1}{nh}\int k(t)^2\, dt \qquad 1.5$$

and

$$h^{4m-1} = \frac{\left((2m-1)!\right)^2 \int k(t)^2\, dt}{J^2_{2m-1}(4m-2)n \int f^{(2m-1)}(x)^2\, dx}$$

$$h_{opt} \approx \left[\frac{((2m-1)!)^2 \int k(t)^2\, dt}{J_{2m-1}^2 (4m-2)\, n \int f^{(2m-1)}(x)^2\, dx} \right]^{\frac{1}{4m-1}} \quad 1.6$$

This is the generalized optimal window width corresponding to conditions in (1.4).
Substituting (1.6) into (1.5) gives

$$MISE\ f(x) = \frac{4m-1}{2(2m-1)} \left\{ \frac{2(2m-1)}{((2m-1)!)^2} \right\}^{\frac{1}{4m-1}} J_{2m-1}^{\frac{2}{4m-1}} . . \left\{ \int k(t)^2\, dt \right\}^{\frac{4m-2}{4m-1}} \left\{ \int f^{(2m-1)}(x)^2\, dx \right\}^{\frac{1}{4m-1}} n^{\frac{4m-2}{4m-1}}$$

1.7

where m is the order, J_{2m-1} is as defined in (1.4)

Equations (1.6) and (1.7) are the asymptotic expressions for the generalized optimal window width h and the MISE terms corresponding to (1.3) and (1.4) respectively. For details see Osemwenkhae and Izevbizua (2005).

4.1 CONSTRUCTING HIGHER- ORDER KERNEL

There are several rules for constructing higher-order kernels, some of which are: If kernels with several vanishing moments are considered, define $\mu_j(k) = \int x^j k(x) dx$ to be the j^{th} moment of the kernel K. Then we will say that K is a k^{th}- order kernel Jones and Foster (1993).

4.1.1 THE EXPONENTIAL KERNEL.

The exponential kernel is given by:

$$k(t) = e^{-t}, -\infty < t < \infty \quad 1.8$$

On differentiation, we have

$$k^1(t) = -e^{-t} \quad 1.9$$

Now substitute equation (1.8) and (1.9) into equation (1.2), we have

$$k_{ex}(t) = \frac{3}{2} e^{-t} - \frac{1}{2} t e^{-t} = \frac{1}{2}(3-t) e^{-t} \quad 2.0$$

Thus, equation (2.0) is the higher -order version of the exponential kernel.

4.1.2 THE GAMMA KERNEL

The Gamma kernel is given by

$$k(t) = \frac{\beta^\alpha t^{\alpha-1}}{\Gamma(\alpha)} e^{-t\beta}, 0 < t < \infty \quad 2.1$$

On differentiation, we have

$$k^1(t) = (\alpha-1) \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-2} e^{-\beta t} + (-\beta) \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\beta t}$$

$$= e^{-\beta t} \frac{\beta^\alpha}{\Gamma(\alpha)} \left[(\alpha-1) t^{\alpha-2} - \beta t^{\alpha-1} \right] \quad 2.2$$

On substituting equation (2.1) and (2.2) into equation (1.2) we have

$$k(t)=\frac{\beta^{\alpha}}{2\Gamma(\alpha)}e^{-\beta t}\left[(2+a)t^{\alpha-1}-\beta t^{\alpha}\right],0<t<\infty \qquad 2.3$$

Thus, equation (2.3) is the higher -order version of the gamma kernel.

5.1 THE OPTIMAL WINDOW WIDTH h_{opt} AND THE ASYMPTOTIC MEAN INTREGRATER SQUARED ERROR (AMISE).

The optimal window width formulae for the asymmetric kernels as in Osemwenkhae and Izevbizua (2005) is given as

$$h_{opt} \approx \left[\frac{((2m-1)!)^2\int k(t)^2\,dt}{J_{2m-1}^2(4m-2)n\int f^{(2m-1)}(x)^2\,dx}\right]^{\frac{1}{4m-1}}, \quad 0<m<\infty \qquad 2.4$$

The asymptotic mean integrated squared error (AMISE) formulae as in Osemwenkhae and Izevbizua (2005) is given as

$$\text{AMISE} \approx \frac{1}{((2m-1)\,!)^2}\left[\frac{(2\text{m-1})!)^2\int \text{k}(\text{t})^2\text{dt}}{\text{J}_{2\text{m-1}}^{\ 2}(4\text{m-2})\ \text{n}\int f^{\ 2\text{m-1}}_{\ (\text{x})^2\text{dx}}}\right]^{\frac{4m-2}{4m-1}} \text{J}_{2\text{m-1}}^{\ 2}\int f^{\ 2m-1}_{\ (\text{x})^2\text{dx}}$$

$$+\ \frac{1}{n}\left[\frac{\text{J}_{2\text{m-1}}(4\text{m-2}))n\int f^{2m-1}(\text{x})^2\text{dx}}{\text{J}_{2\text{m-1}}^{\ 2}(4\text{m-1})!\int k(t)^2\ dt}\right]\int k(t)^2\ dt, \quad 0<\text{m}<\infty \qquad 2.5$$

From equations (2.4) and (2.5) we shall obtain the optimal window width formula for the higher order exponential and the gamma kernels. We also obtain the asymptotic mean integrated squared error (AMISE) formula for the higher order exponential and the gamma kernels.

To get the optimal window width formula for the exponential kernel, we substitute equation (2.0) into equation (2.4) which gives

$$h_{opt} \approx \left[\frac{((2m-1)!)^2\int\left[\frac{1}{2}(3-t)e^{-t}\right]^2 dt}{\int t\left[\frac{1}{2}(3-t)e^{-t}\right]^2 dt\,(4m-2)n\int f^{(2m-1)}(x)^2\,dx}\right]^{\frac{1}{4m-1}} \qquad 2.6$$

As for the optimal window width formula for the gamma kernel, we substitute equation (2.3) into equation (2.4) which gives

$$h_{opt} \approx \left[\frac{((2m-1)!)^2\int\left[\frac{\beta^{\alpha}}{\Gamma(\alpha)}\left[\left(1+\frac{\alpha}{2}\right)t^{\alpha-1}-\frac{1}{2}\beta t^{\alpha}\right]e^{-\beta t}\right]^2 dt}{\int t\left[\frac{\beta^{\alpha}}{\Gamma(\alpha)}\left[\left(1+\frac{\alpha}{2}\right)t^{\alpha-1}-\frac{1}{2}\beta t^{\alpha}\right]e^{-\beta t}\right]^2 dt\,(4m-2)n\int f^{(2m-1)}(x)^2\,dx}\right]^{\frac{1}{4m-1}} \qquad 2.7$$

$\alpha > 0$ *is the shape parameter and* $\beta > 0$ *is the rate parameter.*

For the asymptotic mean integrated squared error (AMISE) formula for the exponential kernel, we substitute equation (2.0) into equation (2.5) which gives

$$AMISE \approx \frac{1}{((2m-1)\;!)^2}\left[\frac{(2m-1)!)^2 \int \left[\frac{1}{2}(3-t)e^{-t}\right]^2 dt}{\int t\left[\frac{1}{2}(3-t)e^{-t}\right]^2 dt\;(4m-2)\;n\int f^{2m-1}_{(x)^2 dx}}\right]^{\frac{4m-2}{4m-1}} \int t\left[\frac{1}{2}(3-t)e^{-t}\right]^2 dt \int f^{2m-1}_{(x)^2 dx}$$

$$\frac{1}{n}\left[\frac{\int t\left[\frac{1}{2}(3-t)e^{-t}\right]^2 dt(4m-2)\Big)n\int f^{2m-1}(x)^2dx}{\int t\left[\frac{1}{2}(3-t)e^{-t}\right]^2 dt(4m-1)!\int\left[\frac{1}{2}(3-t)e^{-t}\right]^2 dt}\right]\int\left[\frac{1}{2}(3-t)e^{-t}\right]^2 dt \qquad 2.8$$

Also for the asymptotic mean integrated squared error (AMISE) formula for gamma kernel, we substitute equation (2.3) into equation (2.5) which gives

$$AMISE \approx \frac{1}{((2m-1)\;!)^2}\left[\frac{(2m-1)!)^2 \int \left[\frac{\beta^\alpha}{\Gamma(\alpha)}\left[\left(1+\frac{\alpha}{2}\right)t^{\alpha-1}-\frac{1}{2}\beta t^\alpha\right]e^{-\beta t}\right]^2 dt}{\int t\left[\frac{\beta^\alpha}{\Gamma(\alpha)}\left[\left(1+\frac{\alpha}{2}\right)t^{\alpha-1}-\frac{1}{2}\beta t^\alpha\right]e^{-\beta t}\right]^2 dt\;(4m-2)\;n\int f^{2m-1}_{(x)^2 dx}}\right]^{\frac{4m-2}{4m-1}}$$

$$\int t\left[\frac{\beta^\alpha}{\Gamma(\alpha)}\left[\left(1+\frac{\alpha}{2}\right)t^{\alpha-1}-\frac{1}{2}\beta t^\alpha\right]e^{-\beta t}\right]^2 dt\int f^{2m-1}_{(x)^2 dx}$$

$$\frac{1}{n}\left[\frac{\int t\left[\frac{\beta^\alpha}{\Gamma(\alpha)}\left[\left(1+\frac{\alpha}{2}\right)t^{\alpha-1}-\frac{1}{2}\beta t^\alpha\right]e^{-\beta t}\right]^2 dt(4m-2)\Big)n\int f^{2m-1}(x)^2dx}{\int t\left[\frac{\beta^\alpha}{\Gamma(\alpha)}\left[\left(1+\frac{\alpha}{2}\right)t^{\alpha-1}-\frac{1}{2}\beta t^\alpha\right]e^{-\beta t}\right]^2 dt(4m-1)!\int\left[\frac{\beta^\alpha}{\Gamma(\alpha)}\left[\left(1+\frac{\alpha}{2}\right)t^{\alpha-1}-\frac{1}{2}\beta t^\alpha\right]e^{-\beta t}\right]^2 dt}\right]$$

$$\int\left[\frac{\beta^\alpha}{\Gamma(\alpha)}\left[\left(1+\frac{\alpha}{2}\right)t^{\alpha-1}-\frac{1}{2}\beta t^\alpha\right]e^{-\beta t}\right]^2 dt \qquad 2.9$$

6.1 CALCULATIONS OF OPTIMAL WINDOW WIDTH h_{opt} AND THE ASYMPTOTIC MEAN INTREGRATER SQUARED ERROR (AMISE).

To demonstrate this, the data obtained from Osemwenkhae and Orhionkpaiyo (2006) will be used. This data describes some standard test on two hundred (200) fairly used car reams before they failed some specific gauge test. In their work they obtained $\sigma = 5, n = 200$,

$$\int f^{1}(x)^{2}\,dx = \frac{\sqrt{\pi}}{4}\pi\,\sigma^{3}$$

Using the data on equations (2.6), (2.7), (2.8) and (2.9) we obtain tables 1, 2 and figures 1, 2 .

7.1 DISCUSSION OF RESULTS

In this work, we compared the h-optimal of the asymmetry kernels, as well as the higher -order version of the asymmetry kernels. We also examined the AMISE of the asymmetry kernel and their higher -order counterpart.

From Table 1, we observed that the h- optimal (h_{opt}) values for asymmetry kernel, as well as the higher -order counterpart increase as the values of the order (m) increases. It was also noticed that the h- optimal (h_{opt}) values of the asymmetry kernels are higher when compared to that of the higher-order asymmetry kernels.

The graphs of the exponential as well as that of the higher-order exponential kernels were drawn (see figure 1) to show their performance. We noticed a decrease in the value of the AMISE term in the higher-order form of the exponential kernel than that of the exponential kernel examined from order one to four (1 to 4), between four and beyond, there was known noticeable difference between the higher-order form of the exponential kernel and the exponential kernel examined.

In the same way, when the graph of the gamma kernel was drawn against the higher-order gamma kernel (see figure 2) we observed that the higher-order gamma kernel performed better than the gamma kernel between order 1 and 4. Beyond this, the difference is not pronounced in the graph.

8.1 CONCLUSION

From Tables 1 and 2 an appreciable error reduction was achieved when we compare higher -order version results of asymmetric kernels with its asymmetric counterpart. The error propagation drops considerably as h increases. When the asymmetry kernel results are compared with that of the higher -order asymmetric kernels in terms of their AMISE, the results showed an improvement in the higher -order asymmetric kernel.

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APPENDIX

Table1. Values of h_{AMISE} for the asymmetry kernels: the exponential and the gamma as well as their higher –order versions.

m	h_{AMISE}			
	Exponential $K(t)=e^{-t}$, $0<t<\infty$	Higher–order exponential $k(t)=\frac{1}{2}(3-t)e^{-t}$ $0<t<\infty$	Gamma $K(t)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}t^{\alpha-1}e^{-\beta t}$, $0<t<\infty$	Higher–order gamma $K(t)=\frac{\beta^{\alpha}}{2\Gamma(\alpha)}e^{-\beta t}\left[(2+\alpha)t^{\alpha-1}-\beta t^{\alpha}\right]$, $0<t<\infty$
1	1.6425	1.5456	1.6425	1.2065
2	1.7641	1.7187	1.7641	1.5457
3	2.3619	2.3231	2.3619	2.1713
4	3.0229	2.9863	3.0229	2.8420
5	3.7072	3.6717	3.7072	3.5309
6	4.4035	4.3687	4.4035	4.2299
7	5.1073	5.0729	5.1073	5.3773
8	5.8160	5.7819	5.8160	5.6450
9	6.5284	6.4945	6.5284	6.3580
10	7.2435	7..2097	7.2435	7.0736

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Table2. Values of AMISE for the asymmetry kernels: the exponential and the gamma as well as their higher –order versions

m	AMISE			
	Exponential $K(t)=e^{-t}$, $-\infty<t<\infty$	Higher–order exponential $k(t)=\frac{1}{2}(3-t)e^{-t}$ $-\infty<t<\infty$	Gamma $K(t)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}t^{\alpha-1}e^{-\beta t}$, $0<t<\infty$	Higher–order gamma $K(t)=\frac{\beta^{\alpha}}{2\Gamma(\alpha)}e^{-\beta t}\left[(2+\alpha)t^{\alpha-1}-\beta t^{\alpha}\right]$, $0<t<\infty$
1	1.3252 E – 04	1.3931 E – 03	1.3252 E – 04	1.3734 E – O3
2	1.1132 E – 04	2.8020 E – 04	2.8320 E – 04	4.5556 E – 04
3	1.0640 E – 04	1.0112 E – 04	1.0604 E – 04	2.0782 E – 04
4	1.0471 E – 04	5.6065 E – 05	1.0471 E – 04	1.1946 E – 04
5	1.9280 E – 05	3.5466 E – 05	1.9280 E – 05	7.7258 E – 05
6	1.9006 E – 05	2.4388 E – 05	1.9006 E – 05	5.3877 E – 05
7	1.8827 E – 05	1.7772 E – 05	1.8827 E – 05	3.9675 E – 05
8	1.4328 E – 05	1.5513 E – 05	1.4328 E – 05	3.0396 E – 05
9	1.1174 E – 05	1.0615 E – 05	1.1174 E – 05	2.4017 E – 05
10	9.0826 E – 06	8.5557 E – 06	9.0826 E – 06	1.9446 E – 05

Table 3. DURATION OF 200 FAIRLY USED CAR REAMS ON THE ROAD

The duration (in months) of 200 fairly used car reams on the road before they failed some specific gauge test is displayed below. The data was obtained from a mechanic workshop and spare part station in an urban city in Nigeria.

18.5	18.4	12.3	23.5	31.0	17.4	25.2	20.9	26.8	21.3
13.6	18.1	15.8	21.5	27.2	24.2	20.7	22.7	28.4	27.4
21.2	26.7	12.4	15.3	26.5	22.6	25.7	18.8	21.2	20.4
26.4	19.6	18.2	18.8	20.6	17.0	19.3	13.9	20.7	14.4
26.0	19.1	19.8	20.7	20.0	26.5	16.1	26.3	29.3	13.1
28.7	17.4	20.1	22.8	22.3	11.2	25.4	18.6	19.8	21.0
9.1	19.9	18.4	20.7	19.9	22.8	17.1	13.5	16.7	24.8
18.8	24.3	31.0	15.4	14.7	19.4	22.7	23.8	24.5	12.5
25.5	31.9	11.3	29.4	11.1	20.2	22.7	23.9	20.0	17.0
14.6	16.7	16.3	22.4	24.1	16.7	18.4	22.1	25.2	22.6
16.5	28.3	7.1	20.4	22.2	17.3	17.8	22.0	22.2	18.9
11.5	11.9	27.2	24.1	23.1	24.2	13.2	16.8	27.3	18.3
10.8	22.7	13.6	24.3	21.1	24.0	30.0	23.6	19.4	19.2
15.1	24.5	16.7	16.8	14.9	22.3	17.2	22.6	15.2	17.8
16.1	29.6	23.8	15.4	26.2	23.5	20.4	25.4	12.3	16.0
9.4	19.6	22.3	25.6	18.4	28.2	18.8	6.2	7.2	23.2
17.2	17.4	24.4	14.0	15.8	21.5	34.2	22.3	20.5	20.3
18.0	23.4	23.0	12.2	15.9	22.9	26.3	27.3	21.5	26.3
20.7	18.1	13.1	23.6	17.9	29.3	24.4	.11.4	19.9	21.3
18.2	23.8	14.4	23.2	17.7	18.3	26.7	20.2	21.0	20.3

Source: Global Journal of Mathematical Sciences Vol.6.No.2. 2006.

Figure1: The graph of order (m) and AMISE of the exponential and higher order exponential kernel

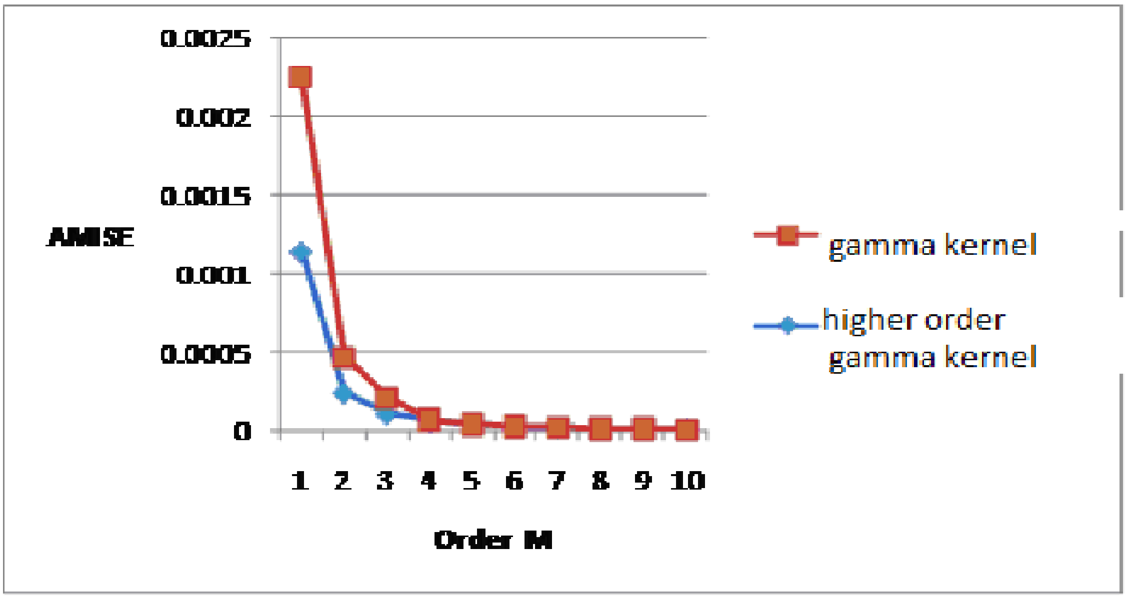

Figure 2: The graph of order (m) and AMISE of the gamma and higher order gamma kernels.