Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.5, No.9, 2015

Homotopy Perturbation Method for the Strongly Nonlinear Darcy-Forscheimer Model

Saheed Ojo Akindeinde
Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Osun State, Nigeria

Abstract

We derive approximate analytical solution to the strongly nonlinear Darcy-Forscheimer model through the homotopy perturbation method. The approximate solutions compared well with numerical results computed via the bvp4c routine of Matlab as well as existing results in the literature.

Keywords: Semi-analytical Solution, Homotopy Perturbation, Variational Iteration, Darcy-Forscheimer Model, bvp4c

1. Introduction

It is a general knowledge that closed form solutions to differential equations are difficult to come by [14]. In some cases, the need to investigate the effects of certain parameters of the model necessitates approximate analytical or numerical solutions. Well-known methods in this regards include the Homotopy Perturbation Method (HPM), the Adomian Decomposition Method (ADM) and the Variational Iteration Method (VIM) and their several variations. He in 1998 introduced the Homotopy Perturbation Method (HPM) through his work [1] and later popularized through his subsequent applications to different problems [2, 3, 9, 13, 10, 11, 12]. HPM is an analytical technique for solving both nonlinear and linear problems. The method produces an infinite series solution of the underlying problem. The popularity of the method has soared in the last decade. It has been applied by several authors to problems in different facet of science [22, 23, 24, 25, 26]. It is worth mentioning that the method has been established to be a special case of the Homotopy Analysis Method (HAM) [21]. On the other hand, the VIM was also introduced by He in [4, 5]. This method has equally been applied to both linear and nonlinear problems [15, 27, 28] and the references therein.

In this paper, we shall apply the variational iteration method and the homotopy perturbation method to approximate the solution of the strongly nonlinear Darcy-Forscheimer model. The darcy-Forscheimer model has been studied extensively in the literature [16, 17, 18, 19]. However, to the best knowledge of the author, construction of analytical solution to the model has not been attempted and not previously reported in the literature. It is worth mentioning that in [20] a heuristic idea was used to obtain a closed form solution of the extended Brinkman-Forscheimer model. In the absence of a closed form solution of the present problem and to validate our result, numerical solution shall be computed via the Matlab routine bvp4c as reference solution.

2. Mathematical Formulation of the Darcy-Forscheimer Model

The steady state and pressure driven unidirectional flow of fluid through a horizontal channel filled with porous media is often modeled by the Brinkman-Forscheimer momentum equations represented by the second-order nonlinear differential equation

$$\mu_{\text{eff}}\frac{du^*}{dy^{*2}} = \frac{\mu}{K}u^* + \frac{\rho C_f}{\sqrt{K}}u^{*2} + G \tag{2.1}$$

where the variables denote respectively the drag coefficient, the pressure gradient, the permeability, the fluid density, the fluid viscosity and the effective fluid viscosity inside the porous medium [6]. The impermeable walls of the channel are assumed to be at the positions $y^* = \pm h.$ By introducing the similarity variables

$$x = \frac{x^*}{P_e H}, \quad y = \frac{y^*}{h}, \quad u = \frac{Gh^2 u^*}{\mu}$$

the dimensionless form of the model equation is obtained as

$$\frac{d^2u}{dy^2} - s^2u - Fsu^2 + \frac{1}{M} = 0 \quad (2.2)$$

subject to the boundary conditions

$$u(\pm 1) = 0.$$

In the above, the variables

$$M = \frac{\mu_{\text{eff}}}{\mu}, \quad F = \frac{C_f \rho G H^3}{\mu_{\text{eff}}\mu}, \quad Da = \frac{K}{H^2}, \quad s = \left(\frac{1}{MDa}\right)^{\frac{1}{2}}$$

where P_e, Da denote respectively the Peclet and Darcy numbers.

3. Semi-Analytical Solution Methods.

In the sequel, we shall apply the variational iteration method [4, 5] and the homotopy perturbation method to solve the model problem of Section 2 and report on the results obtained.

3.1 Variational Iteration Method.

To illustrate the general idea behind this method, let us consider a general nonlinear problem

$$Lu(y) + Nu(y) = f(y)$$

where L, N, and f denote a linear operator, nonlinear operator and a forcing term respectively. The main technique of the variational iteration method applied to the above problem is to
is to construct a correction function

$$u_{n+1} = u_n + \int_0^x \lambda(Lu_n(t) + N\tilde{u}_n(t) - f(t))dt$$

for the nonlinear system. In the above, l represents a Lagrange multiplier, which can be identified optimally through variational theory, u_n denotes the nth approximate solution and denotes a $\tilde{u}_n$ restricted variation in the sense that holds true. The series $\delta\tilde{u}_n = 0$ solution of the problem is then obtained from the approximate solutions $u_1, u_2, \ldots$ as

$$u(y) = \lim_{n\to\infty} u_n.$$

3.2 Applications to the Model Problem

In an attempt to solve (2.2) by the variational iteration technique, the Lagrange multiplier l has to be identified. By defining the correction functional associated with (2.2) as

$$u_{n+1} = u_n + \int_0^y \lambda(y,\epsilon)\Big(u_n''(\epsilon) - s^2u_n(\epsilon) - Fs\,\tilde{u}_n^2(\epsilon) + \frac{1}{M}\Big)d\epsilon \quad (3.2)$$

the Lagrange multiplier can then be obtained as the solution of the following stationary conditions:

$$\begin{aligned} \delta u_n(\epsilon):&\quad \lambda'' - s^2\lambda(y,\epsilon) = 0, \\ \delta u_n(y):&\quad 1 - \lambda'(\epsilon)\Big|_{\epsilon=y} = 0, \\ \delta u_n'(y):&\quad \lambda(\epsilon)\Big|_{\epsilon=y} = 0. \end{aligned} \quad (3.3)$$

Now solving the above stationary conditions yield

$$\lambda(y,\epsilon) = \frac{1}{2s}\Big(e^{s(\epsilon-y)} - e^{-s(\epsilon-y)}\Big) = \frac{1}{2s} sinh(s(\epsilon-y)). \qquad (3.4)$$

Substituting (3.4) into (3.2), we obtain the iterative scheme

$$u_{n+1} = u_n + \frac{1}{2s}\int_0^y sinh(s(\epsilon-y))\Big(u_n''(\epsilon) - s^2 u_n(\epsilon) - Fs\, u_n^2(\epsilon) + \frac{1}{M}\Big)d\epsilon. \qquad (3.5)$$

Remark: As hinted in [5] the Lagrange multiplier for nonlinear problem is difficult to identify in general. In the immediate previous discussion, this parameter has been identified using the idea of restricted variation. It is well known that inexact identification of the Lagrange multiplier this way results in slow convergence of the iteration (3.5).

Typically, the first iteration of variational iteration method gives a very accurate approximation [5]. However, for this present problem, the second iteration of (3.5) with initial solution

$$u_0(y) = u_0 = \tfrac{y^2-1}{3}$$

that obviously satisfies the boundary conditions $u(\pm 1) = 0$ yields an inaccurate approximation involving almost 160 terms! The iterative procedure therefore become unmanageable coupled with a very slow convergence.

In what follows, we shall consider another related technique called the Homotopy Perturbation method whose computation does not involve the Lagrange multiplier and with better convergence property for this nonlinear problem.

3.3 Homotopy Perturbation Method (HPM)

We shall illustrate the homotopy perturbation method by the following general nonlinear equation

$$L(u) + N(u) - f(y) = 0, \quad y \in \Omega \qquad (3.8)$$

with the boundary conditions

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad y \in \partial\Omega.$$

We construct a homotopy

$$v(y,p) : \Omega \times [0,1] \to \Re$$

which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(y)] = 0,$$

and implies

$$H(v,p) = [L(v) - L(u_0)] + p[L(u_0) + N(v) - f(y)] = 0 \qquad (3.9)$$

where $p \in [0,1]$ is called the homotopy parameter and u_0 is an initial approximation of (3.8). When $p = 0$ and $p = 1$, we obtain

$H(v,0) = L(u) - L(u_0) = 0$ and $H(v,1) = L(v) + N(v) - f(y) = 0.$

In the interval $0 < p < 1$ the homotopy $H(p,v)$ deforms from $L(u) - L(u_0)$ to $L(v) + N(v) - f(y)$.

Thus, the solution of (3.9) may be expressed as

$$v = v_0 + pv_1 + p^2v_2 + p^3v_3 + \dots$$

Eventually at p=1, the system takes the original form of the equation and the final stage of deformation gives the desired solution. Thus taking limit

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \dots$$

3.3.1 Application to the Model Problem

We shall now apply HPM to (2.2). Noting that $Lu = \frac{d^2u}{dy^2}, \quad Lu_0 = \frac{d^2u_0}{dy^2}$

we apply the convex homotopy perturbation method to the problem to obtain

$$H(U,p) = (1-p)\left(\frac{d^2U}{dy^2} - \frac{d^2u_0}{dy^2}\right) + p\left(\frac{d^2U}{dy^2} - s^2U - FsU^2 + \frac{1}{M}\right) = 0$$

which implies that

$$\left(\frac{d^2U}{dy^2} - \frac{d^2u_0}{dy^2}\right) + p\left(\frac{d^2u_0}{dy^2} - s^2U - FsU^2 + \frac{1}{M}\right) = 0. \qquad (3.10)$$

Now assume the solution U of (3.10) can be written in the form

$$U = U_0 + pU_1 + p^2U_2 + p^3U_3 + \dots$$

then on substituting this series solution into (3.10), collecting terms in like powers of p and comparing coefficients we obtain the linear equations

$$\begin{aligned}
p^0 &: \quad \frac{d^2U_0}{dy^2} - \frac{d^2u_0}{dy^2} = 0 \\
p^1 &: \quad \frac{d^2U_1}{dy^2} + \frac{d^2u_0}{dy^2} - s^2U_0 - FsU_0^2 + \frac{1}{M} = 0 \\
p^2 &: \quad \frac{d^2U_2}{dy^2} - s^2U_1 - 2FsU_0U_1 = 0 \\
p^3 &: \quad \frac{d^2U_3}{dy^2} - s^2U_2 - 2FsU_0U_2 - FsU_1^2 = 0 \\
&\vdots
\end{aligned}$$

For simplicity, we chose the solution component U_0 and the initial guess of the exact solution u_0 to be the same. In turn, u_0 is chosen as $u_0(y) = U_0 = \frac{1-y^2}{3}$ which clearly satisfies the boundary conditions $U_i(\pm 1) = 0,\ i = 0, 1, 2, \ldots$

With these initial guesses and while respecting the stated boundary conditions, the linear equations above are solved analytically using Maple software, and in turn plugged into the series solution

$$u(y) = \sum_{i=1}^{n} U_i(y).$$

For $n = 2$ and different choices of the parameters F, s and M we obtained

1. $F = 1, s = 1, M = 1$

$$u(y) = -\frac{25849}{87480}y^2 - \frac{1543}{65610}y^4 - \frac{1127}{262440}y^6 + \frac{1}{29160}y^8 + \frac{1}{984150}y^{10} + \frac{1272551}{3936600}.$$

2. $F = 1.2, s = 1, M = 1$

$$u(y) = -0.2876985283\,y^2 - 0.02701775149\,y^4 - 0.003386385475\,y^6 + 0.00002067754731\,y^8 + 0.0000004855105447\,y^{10} + 0.3180815022.$$

3. $F = 1, s = 1, M = 2$

$$u(y) = -\frac{521}{3240}y^2 - \frac{11}{2430}y^4 - \frac{83}{19440}y^6 + \frac{1}{15120}y^8 + \frac{1}{291600}y^{10} + \frac{346043}{2041200}.$$

4. $F = 1, s = 2, M = 1$

$$u(y) = -0.1247933884\,y^2 - \frac{1505447033053243175 53}{3750000000000000000000}y^4 - \frac{30897821179374060857}{2812500000000000000000}y^6 + \frac{8371793493728936353}{7500000000000000000000}y^8 - \frac{333917689158527423}{18750000000000000000000}y^{10} + 0.1748261043.$$

4. Results and Discussion

Let us now compare the obtained semi-analytical results with numerical results. The results are shown in Table 4.1 below for the approximation of the solution $u(0)$

F	M	s	$u(0)$ by HPM	$u(0)$ by bvp4c	Percentage error
1	1	1	0.32326	0.32383	0.17%
1.2	1	1	0.31808	0.31915	0.3%
1	2	1	0.16952	0.16841	0.6%
1	1	2	0.17482	0.17445	0.2%

Table 4.1 Comparison of approximations to u_0 with those obtained through bvp4c for different values of parameters F, s and M.

It is clear from the above table that the approximate analytical solutions are in good agreement with those obtained through the Matlab routine bvp4c. Furthermore, the obtained results and the solution profiles shown below are in good agreement with results obtained through the Adomian Decomposition Method [7] and the Finite Difference Method [8] in the literature. A fairly accurate approximation has also been returned for the case $F = 1.2, s = 1, M = 1$ where the Adomian Decomposition Method has been reported not to converge for this choice of the parameters, see [7].

References

[1] J. H. He: Homotopy perturbation technique. *Computer Methods in Applied Mechanics and Engineering*. 178 (1999), 257-262.

[2] --, A coupling method of homotopy technique and perturbation technique for nonlinear problems. *International Journal of Non-Linear Mechanics.* 35(1) (2000), 37-43.

[3] --, Homotopy perturbation method: a new nonlinear analytical technique. *Applied Mathematics and Computation.* 135 (2003), 73-79.

[4] --, Variational iteration method a kind of non-linear analytical technique: some examples, *International Journal of Non-Linear Mechanics.* 34 (1999), 699-708.

[5] --, Variational iteration method: Some recent results and new interpretations, *Journal of Computational and Applied Mathematics.* 207 (2007), 3-17.

[6] K. Hooman, A perturbation solution for forced convection in a porous-saturated duct, *Journal of Computational and Applied Mathematics.* 211 (2008), 57-66.

[7] M. Madoda, The spectral Adomian decomposition method and its higher order based iterative schemes for solving highly nonlinear two-point boundary value problems, *M.sc., University of Johannesburg*, 2013.

[8] S.O. Akindeinde, Approximate Solutions of the Brinkman-Forscheimer Model, *Journal of Applied Mathematics and Bioinformatics*, 2015 (in press).

[9] J. H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities. *Applied Mathematics and Computation.* 151 (2004), 287-292.

[10] --, Application of homotopy perturbation method to nonlinear wave equations. *Chaos, Solitons and Fractals.* 26 (2005), 695-700.

[11] --, Homotopy perturbation method for solving boundary value problems. *Physics Letters A 350* (2006), 87-88.

[12] --, Limit cycle and bifurcation of nonlinear problems. *Chaos, Solitons and Fractals.* 26(3) (2005), 827- 833.

[13] --, Comparison of homotopy perturbation method and homotopy analysis method. *Applied Mathematics and Computation.* 156 (2004), 527-539.

[14] --, Variational iteration method for autonomous ordinary differential systems. *Appl. Math. Comput.* 114 (2000) 115-123.

[15] --, Approximate analytical solution for seepage ow with fractional derivatives in porous media. *Comput. Meth. Appl. Mech. Eng.* 167 (1-2) (1998), 57-68.

[16] M. A. Al-Nimr and T. K. Aldoss, The Effect of the Macroscopic Local Inertial Term on the Non-Newtonian Fluid Flow in Channels Filled with Porous Medium. *International Journal of Heat and Mass Transfer,* 47 (2004) 125-133. doi:10.1016/S0017-9310(03)00382-X

[17] B. A. Abu-Hijleh and M. A. Al-Nimr, The Effect of the Local Inertial Term on the Fluid Flow in Channels Partially Filled with Porous Material. *International Journal of Heat Mass Transfer,* 44:8 (2001) 565-572. doi:10.1016/S0017-9310(00)00207-6

[18] A. F. Khadrawi and M. A. Al-Nimr, The Effect of the Local Inertial Term on the Free Convection Fluid Flow in Vertical Channels Partially Filled with Porous Media. *Journal of Porous Media,* 6:1 (2003) 1-12. doi:10.1615/JPorMedia.v6.i1.40

[19] M. L. Kaurangini and B. K. Jha, E_ect of Inertial on Generalized Couette Flow in Composite Parallel Plates with Uniform Porous Medium in the Presence of Suction and Injection. *International Journal of Modelling, Simulation and Control (AMSE)*, 2011, Article in Press.

[20] K. J. Basant and L. K. Muhammad, Approximate Analytical Solutions for the Nonlinear Brinkman-Forchheimer-Extended Darcy Flow Model, *Applied Mathematics*, 2 (2011) 1432-1436.

[21] S. Liao, Notes on the homotopy analysis method: some defnitions and theorems. *Communications in Nonlinear Science and Numerical Simulation*, 14:4 (2009) 983-997.

[22] P. D. Ariel, T. Hayat and S. Asghar, Homotopy perturbation method and axisymmetric flow over a stretching sheet, *Internat. J. Nonlinear Sci.* 7 (2006), 399-406.

[23] A. Belendez, A. Hernandez, T. Belendez et al., Application of He’s homotopy perturbation method to the Duffing- harmonic oscillator, *Internat. J. Nonlinear Sci.* 8 (2007), 79-88.

[24] J. Biazar, M. Eslami and H. Ghazvini, Homotopy perturbation method for systems of partial differential equations, *Internat. J. Nonlinear Sci.* 8 (2007), 413-418.

[25] D. D. Ganji and A. Sadighi, Application of Hes homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations, *Internat. J. Nonlinear Sci.* 7 (2006).

[26] Q. Ghori, M. Ahmed and A. M. Siddiqui, Application of homotopy perturbation method to squeezing flow of a Newtonian fluid, *Internat. J. Nonlinear Sci.* 8 (2007), 179-184.

[27] S. Momani and Z. Odibat, Application of variational iteration method to nonlinear differential equations of fractional order, *Internat. J. Nonlinear Sci.* 7 (2006), 27-34.

[28] H. Tari, D. D. Ganji and M. Rostamian, Approximate solutions of K(2, 2), KdV and modifed KdV equations by variational iteration method, homotopy perturbation method and homotopy analysis method *Internat. J. Nonlinear Sci.* 8 (2007), 203-210.