On Jordan Left Centralizer of ΓM-Modules

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Abstract

In this paper we present the definitions of left centralizer, Jordan left centralizer on **Γ**M-module and prove that any left centralizer of **Γ** - ring M into 2-torsion free prime right **Γ**M-module X then T is a left centralizer.

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1. Introduction

Let M and **Γ** be two additive abelian groups. M is called a **Γ**-ring if there exists a mapping $M \times \boldsymbol{\Gamma} \times M \to M$ (sending (a,α,b) into aαb) satisfy then following conditions:

i)(a+b)αc = aαc + bαc

a(α+β)b = aαb + aβb

aα(b+c) = aαb + aαc

ii) (aαb)βc = aα(bβc)

for all a,b,c∈M and α, β∈**Γ**.[1]

M is called prime **Γ**-ring if a**Γ**M**Γ**b = (0) implies a=0 or b=0 for all a,b∈M ; M is called semiprime **Γ**-ring if a**Γ**M**Γ**a = (0) implies a=0 for all a ∈M and M is called 2-torsion free if 2a =0 implies a=0 for all a∈M . The commutater $[a,b]_{\alpha}$ its mean aαb-bαa for all a,b∈M, α∈**Γ**.[3]

Let M be a **Γ**-ring and X be an additive abelian group. X is a right **Γ**M-module if there exists a mapping $X\times \boldsymbol{\Gamma} \times M \to X$ (sending (x,α,m) into xαm) satisfy the following conditions:

i)$(x_1 + x_2)\alpha m = x_1\alpha\, m + x_2\alpha m$

iii) $x\alpha(m_1+m_2) = x\alpha m_1 + x\alpha m_2$

iv) $(x_1\alpha x_2)\,\beta m = x_1\alpha(x_2\,\beta m)$

for all x,x_1,x_2∈X, m,m_1,m_2∈M and α, β∈**Γ**

X is prime if $x\Gamma M\Gamma m=(0)$ implies x=0 or m=0 for all x∊X and m∊M, and X is called 2-torsion free if 2x=0 implies x=0 for all x∊X.

B.Zalar in [4] defined left centralizer and Jordan left centralizer on a ring R, as folldws. An additive maping T:R→R is left centralizer if T(ab) = T(a)b holed for all a,b∊R and T is Jordan left centralizer on R if $T(a^2) = T(a)a$ holed for all a∊R also proved that any left Jordan centralizer on a 2-torsion free semiprime ring is a left centralizer . M.Hoque and A.Paul [2] defined a centralizer and Jordan left centralizer on **Γ**-ring, as follows. An additive mapping T:M→M is a left centralizer if $T(a\alpha b) = T(a)\alpha b$ for all a,b∊M,α∊**Γ** and T is Jordan left centralizer on **Γ**-ring M if $T(a\alpha a) = T(a)\alpha a$ holed for all a∊M , α∊**Γ**, also proved some properties of its.

In this paper we present the definitions of left centralizer, Jordan left centralizer and Jordan triple left centralizer of **Γ**-ring M into **Γ**M-module X also we prove that every left centralizer of **Γ** - ring M into 2-torsion free prime right **Γ**M-module X then T is a left centralizer.

2. Jordan Left Centralizer

In this section we present the definitions of centralizer, Jordan left centralizer and Jordan left centralizer of **Γ**-ring into **Γ**M-module also we study some properties of them.

Definition 2.1: Let M be a **Γ**- ring and X be a right **Γ**M-module. Let T: M→X be left centralizer if

$T(a\alpha b) = T(a)\alpha b$, for all a,b∊M and α∊**Γ**.

T is called Jordan left centralizer if

$T(a\alpha a) = T(a)\alpha a$, for all a∊M and α∊**Γ**.

T is called Jordan triple left centralizer if

$T(a\alpha b\beta a) = T(a)\alpha b\beta a$, for all a,b∊M and α, β∊**Γ**.

Lemma 1: Let T be a Jordan left centralizer of **Γ**- ring M into right **Γ**M-module X then for all a,b,c∊M and α, β∊**Γ**.
i) $T(a\alpha b+ b\alpha a)= T(a)\alpha b + T(b)\alpha a$
ii) $T(a\alpha b\alpha c+ c\alpha b\alpha a)= T(a)\alpha b\alpha c+ T(c)\alpha b\alpha a$
iii) $T(a\alpha b\beta c+ c\alpha b\beta a)= T(a)\alpha b\beta c+ T(c)\alpha b\beta a$

Proof: i) $T((a+b)\alpha(a+b))= T(a+b)\alpha(a+b)$
$= (T(a)+T(b))\alpha(a+b)$
$= T(a)\alpha a+ T(a)\alpha b + T(b)\alpha a +T(b)\alpha b$ …(1)
On the other hand
$T((a+b)\alpha(a+b))= T(a\alpha a + a\alpha b + b\alpha a + b\alpha b)$
$= T(a)\alpha a + T(b)\alpha b + T(a\alpha b + b\alpha a)$ …(2)
Comparing (1) and (2) we get
$T(a\alpha b + b\alpha a)= T(a)\alpha b + T(b)\alpha a$

ii) In Definition 2.1 replace a+c for a and β for α we get

$T((a+c)\alpha b\alpha(a+c))) = T(a+c)\alpha b\alpha(a+c)$
$= (T(a) +T(c))\alpha b\alpha(a+c)$
$= T(a)\alpha b\alpha a + T(a)\alpha b\alpha c +T(c)\alpha b\alpha a +T(c)\alpha b\alpha c$ …(1)

On the other hand

$T((a+c)\alpha b\alpha(a+c))) = T(a\alpha b\alpha a + a\alpha b\alpha c +c\alpha b\alpha a +c\alpha b\alpha c)$
$= T(a\alpha b\alpha a + c\alpha b\alpha c) + T(a\alpha b\alpha c +c\alpha b\alpha a)$
$= T(a)\alpha b\alpha a + T(c)\alpha b\alpha c + T(a\alpha b\alpha c +c\alpha b\alpha a)$ …(2)

Comparing (1) and (2) we get

$T(a\alpha b\alpha c +c\alpha b\alpha a) = T(a)\alpha b\alpha c +T(c)\alpha b\alpha a$

iii) In Definition 2.1 replace a+c for a and by the same way as in (ii) we get the require result.

Definition 2.2: Let T be a Jordan left centralizer of **Γ**- ring M into right **Γ**M-module X then we define

$\Psi(a,b) = T(a\alpha b) – T(a)\alpha b$ for all $a,b\in M$ and $\alpha\in\Gamma$.

In the following lemma we present the properties of $\Psi_\alpha(a,b)$.

Lemma 2: Let T be a Jordan left centralizer of **Γ**- ring M into right **Γ**M-module X then for all $a,b,c\in M$ and $\alpha, \beta\in\Gamma$.

i) $\Psi_\alpha(a,b) = - \Psi_\alpha(b,a)$
ii) $\Psi(a+c ,b) = \Psi_\alpha(a,b) + \Psi_\alpha(c,b)$
iii) $\Psi(a,b+c) = \Psi_\alpha(a,b) + \Psi_\alpha(a,c)$
iv) $\Psi_{\alpha+\beta} (a,b) = \Psi_\alpha(a,b) + \Psi_\beta(a,b)$

Proof: i)

$T(a\alpha b + b\alpha a)= T(a)\alpha b + T(b)\alpha a$
$T(a\alpha b) - T(a)\alpha b = - T(b\alpha a) – T(b)\alpha a$
$\Psi(a,b) = - \Psi_\alpha(b,a)$

ii) $\Psi(a+c ,b) = T((a+c)\alpha b) – T(a+c)\alpha b$
$= T(a\alpha b + c\alpha b) - T(a)\alpha b - T(c)\alpha b$
$= T(a\alpha b) - T(a)\alpha b + T(c\alpha b) - T(c)\alpha b$
$= \Psi(a,b) + \Psi_\alpha(c,b)$

iii) and iv) as the same way as in (ii) we get the require result.

Remark 2.3: Note that T is left centralizer of **Γ**- ring M into right **Γ**M-module X if and only if $\Psi(a,b) =0$.

Lemma 3: Let $T: M\rightarrow X$ be a Jordan left centralizer of **Γ**- ring M into right **Γ**M-module X then for all $a,b,c\in M$ and $\alpha, \beta\in\Gamma$.

$\Psi(a,b)\, \beta m\, \beta\, [a,b]_\alpha =0$

Proof : Let $w= a\alpha bm\, \beta b\alpha a + b\alpha a\beta m\alpha a\beta b$

$T(w) = T(a\alpha(bm\, \beta b)\alpha a + b\alpha(a\beta m\alpha a)\beta b)$
$= T(a)\alpha(bm\, \beta b)\alpha a + T(b)\alpha(a\beta m\alpha a)\beta b$ …(1)

On the other hand

$T(w) = T((a\alpha b)m\beta(b\alpha a)) + (b\alpha a)\, \beta m\alpha(a\beta b))$
$= T(a\alpha b)m\beta(b\alpha a)) + T(b\alpha a)\beta m\alpha(a\beta b)$ …(2)

Comparing (1) and (2) we get

$0= (T(a\alpha b)- T(a)\alpha b)\, \beta m\beta b\alpha a +(T(b\alpha a) - T(b))\alpha a\beta m\alpha a\beta b$

$= \Psi(a,b)\, \beta m \beta b \alpha a + \Psi_{\alpha}(b,a)\, \alpha a \beta m \alpha a \beta b$
$= \Psi(a,b)\, \beta m \beta\, [a,b]_{\alpha}$

3) The Main Results

In this section we introduce the main results.

Theorem 4: Let T: M→X be a Jordan left centralizer of **Γ** - ring M into prime right **Γ**M-module X then for all a,b,c,d∈M and α, β∈**Γ**.

$\Psi(a,b)\, \beta m \beta\, [a,b]_{\alpha} = 0$

Proof: In Lemma 3 replace a+c for a we get

$\Psi(a+c,b)\, \beta m \beta\, [a+c,b]_{\alpha} = 0$

$\Psi_{\alpha}(a,b)\, \beta m \beta\, [a,b]_{\alpha} + \Psi_{\alpha}(a,b)\, \beta m \beta\, [c,b]_{\alpha} + \Psi_{\alpha}(c,b)\, \beta m \beta\, [a,b]_{\alpha} + \Psi_{\alpha}(c,b)\, \beta m \beta\, [c,b]_{\alpha} = 0$

By Lemma 3 we get

$\Psi(a,b)\, \beta m \beta\, [c,b]_{\alpha} + \Psi_{\alpha}(c,b)\, \beta m \beta\, [a,b]_{\alpha} = 0$

Since

$\Psi(a,b)\, \beta m \beta\, [c,b]_{\alpha}\, \beta m \beta\, \Psi_{\alpha}(a,b)\, \beta m \beta\, [c,b]_{\alpha} = 0$

$- \Psi(a,b)\, \beta m \beta\, [c,b]_{\alpha}\, \beta m \beta\, \Psi_{\alpha}(c,b)\, \beta m \beta\, [a,b]_{\alpha} = 0$

Hence by primness we get

$\Psi(a,b)\, \beta m \beta\, [c,b]_{\alpha} = 0$ …(1)

Replace b+d for b in Lemma 3 we get

$\Psi(a,b+d)\, \beta m \beta\, [a,b+d]_{\alpha} = 0$

$\Psi_{\alpha}(a,b)\, \beta m \beta\, [a,b]_{\alpha} + \Psi_{\alpha}(a,b)\, \beta m \beta\, [a,d]_{\alpha} + \Psi_{\alpha}(a,d)\, \beta m \beta\, [a,b]_{\alpha} + \Psi_{\alpha}(a,d)\, \beta m \beta\, [a,d]_{\alpha} = 0$

By Lemma 3 we get

$\Psi(a,b)\, \beta m \beta\, [a,d]_{\alpha} + \Psi_{\alpha}(a,d)\, \beta m \beta\, [a,b]_{\alpha} = 0$

Since

$\Psi(a,b)\, \beta m \beta\, [a,d]_{\alpha}\, \beta m \beta\, \Psi_{\alpha}(a,b)\, \beta m \beta\, [a,d]_{\alpha} = 0$

$-\Psi(a,b)\, \beta m \beta\, [a,d]_{\alpha}\, \beta m \beta\, \Psi_{\alpha}(a,d)\, \beta m \beta\, [a,b]_{\alpha} = 0$

Hence by primness we get

$\Psi(a,b)\, \beta m \beta\, [a,d]_{\alpha} = 0$ …(2)

Thus

$\Psi(a,b)\,\beta m\beta\,[a+c,b+d]_\alpha=0$

$\Psi_\alpha(a,b)\,\beta m\beta\,[a,b]_\alpha+\Psi_\alpha(a,b)\,\beta m\beta\,[a,d]_\alpha+\Psi_\alpha(a,b)\,\beta m\beta\,[c,b]_\alpha+\Psi_\alpha(a,b)\,\beta m\beta\,[c,d]_\alpha=0$

By Lemma 3 and (1) ,(2)we get

$\Psi(a,b)\,\beta m\beta\,[c,d]_\alpha=0$

Theorem 5: : Let T: M→X be a Jordan left centralizer of **Γ** - ring M into 2-torsion free prime right **Γ**M-module X then T is a left centralizer.

Proof: Since X is prime right **Γ**M-module and by Theorem 4 we get

Either $\Psi_\alpha(a,b)=0$ or $[c,d]_\alpha=0$,for all a,b,c,d∈M and $\alpha\in\Gamma$.

If $[c,d]_\alpha\neq 0$ then $\Psi_\alpha(a,b)=0$,for all a,b,c,d∈M and $\alpha\in\Gamma$, by Remark 2.3 we get the require result .

If $[c,d]_\alpha=0$,for all c,d∈M and $\alpha\in\Gamma$, then M is commutative **Γ**-ring and by Lemma 2.1 we get

$T(2a\alpha b) = 2\,T(a)\alpha b$

Since X is 2-torsion free right **Γ**M-module then T is a left centralizer.

Theorem 6: Let T be a Jordan left centralizer of **Γ** - ring M which satisfy $a\alpha b\beta a=a\beta b\alpha a$ into 2-torsion free right **Γ**M-module X then T is a left centralizer.

Proof: Replace b by aβb+bβa in Definition 2.1we get

$T(a\alpha(a\beta b+b\beta a) + (a\beta b+b\beta a)\alpha a)= T(a)\alpha(a\beta b+b\beta a) + T(a\beta b+b\beta a)\alpha a$

$= T(a)\alpha a\beta b + T(a)b\beta a + T(a)\beta b\alpha a +T(b)\beta a\alpha a \quad \ldots(1)$

On the other hand

$T(a\alpha(a\beta b+b\beta a) + (a\beta b+b\beta a)\alpha a)= T(a\alpha a\beta b +a\alpha b\beta a + a\beta b\alpha a + b\beta a\alpha a)$

$= T(a)\alpha a\beta b +T(b)\beta a\alpha a +T(a\alpha b\beta a + a\beta b\alpha a) \quad \ldots(2)$

Comparing (1) and (2) since $a\alpha b\beta a=a\beta b\alpha a$ we get

$2T(a\alpha b\beta a) = 2T(a)\alpha b\beta a$

Since X is 2-torsion free we get

$T(a\alpha b\beta a) = T(a)\alpha b\beta a$

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