

On Jordan Left Centralizer of Γ M-Modules

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Abstract

In this paper we present the definitions of left centralizer, Jordan left centralizer on Γ M-module and prove that any left centralizer of Γ - ring M into 2-torsion free prime right Γ M-module X then T is a left centralizer.

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1. Introduction

Let M and Γ be two additive abelian groups. M is called a Γ -ring if there exists a mapping $M \times \Gamma \times M \rightarrow M$ (sending (a,α,b) into $a\alpha b$) satisfy then following conditions:

$$i)(a+b)\alpha c = a\alpha c + b\alpha c$$

$$a(\alpha+\beta)b = a\alpha b + a\beta b$$

$$a\alpha(b+c) = a\alpha b + a\alpha c$$

$$ii) (a\alpha b)\beta c = a\alpha(b\beta c)$$

for all $a,b,c \in M$ and $\alpha, \beta \in \Gamma$. [1]

M is called prime Γ -ring if $a\Gamma M \Gamma b = (0)$ implies $a=0$ or $b=0$ for all $a,b \in M$; M is called semiprime Γ -ring if $a\Gamma M \Gamma a = (0)$ implies $a=0$ for all $a \in M$ and M is called 2-torsion free if $2a = 0$ implies $a=0$ for all $a \in M$. The commutator $[a,b]_\alpha$ its mean $a\alpha b - b\alpha a$ for all $a,b \in M$, $\alpha \in \Gamma$. [3]

Let M be a Γ -ring and X be an additive abelian group. X is a right Γ M-module if there exists a mapping $X \times \Gamma \times M \rightarrow X$ (sending (x,α,m) into $x\alpha m$) satisfy the following conditions:

$$i)(x_1 + x_2)\alpha m = x_1\alpha m + x_2\alpha m$$

$$iii) x\alpha(m_1+m_2) = x\alpha m_1 + x\alpha m_2$$

$$iv) (x_1\alpha x_2) \beta m = x_1\alpha(x_2 \beta m)$$

for all $x,x_1,x_2 \in X$, $m,m_1,m_2 \in M$ and $\alpha, \beta \in \Gamma$

X is prime if $x\Gamma M\Gamma m = (0)$ implies $x=0$ or $m=0$ for all $x \in X$ and $m \in M$, and X is called 2-torsion free if $2x=0$ implies $x=0$ for all $x \in X$.

B.Zalar in [4] defined left centralizer and Jordan left centralizer on a ring R , as follows. An additive mapping $T: R \rightarrow R$ is left centralizer if $T(ab) = T(a)b$ holds for all $a, b \in R$ and T is Jordan left centralizer on R if $T(a^2) = T(a)a$ holds for all $a \in R$ also proved that any left Jordan centralizer on a 2-torsion free semiprime ring is a left centralizer. M.Hoque and A.Paul [2] defined a centralizer and Jordan left centralizer on Γ -ring, as follows. An additive mapping $T: M \rightarrow M$ is a left centralizer if $T(a\alpha b) = T(a)\alpha b$ for all $a, b \in M, \alpha \in \Gamma$ and T is Jordan left centralizer on Γ -ring M if $T(a\alpha a) = T(a)\alpha a$ holds for all $a \in M, \alpha \in \Gamma$, also proved some properties of its.

In this paper we present the definitions of left centralizer, Jordan left centralizer and Jordan triple left centralizer of Γ -ring M into ΓM -module X also we prove that every left centralizer of Γ -ring M into 2-torsion free prime right ΓM -module X then T is a left centralizer.

2. Jordan Left Centralizer

In this section we present the definitions of centralizer, Jordan left centralizer and Jordan left centralizer of Γ -ring into ΓM -module also we study some properties of them.

Definition 2.1: Let M be a Γ -ring and X be a right ΓM -module. Let $T: M \rightarrow X$ be left centralizer if

$$T(a\alpha b) = T(a)\alpha b, \text{ for all } a, b \in M \text{ and } \alpha \in \Gamma.$$

T is called Jordan left centralizer if

$$T(a\alpha a) = T(a)\alpha a, \text{ for all } a \in M \text{ and } \alpha \in \Gamma.$$

T is called Jordan triple left centralizer if

$$T(a\alpha b\beta a) = T(a)\alpha b\beta a, \text{ for all } a, b \in M \text{ and } \alpha, \beta \in \Gamma.$$

Lemma 1: Let T be a Jordan left centralizer of Γ -ring M into right ΓM -module X then for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

$$\text{i) } T(a\alpha b + b\alpha a) = T(a)\alpha b + T(b)\alpha a$$

$$\text{ii) } T(a\alpha b\alpha c + c\alpha b\alpha a) = T(a)\alpha b\alpha c + T(c)\alpha b\alpha a$$

$$\text{iii) } T(a\alpha b\beta c + c\alpha b\beta a) = T(a)\alpha b\beta c + T(c)\alpha b\beta a$$

$$\begin{aligned} \text{Proof: i) } T((a+b)\alpha(a+b)) &= T(a+b)\alpha(a+b) \\ &= (T(a)+T(b))\alpha(a+b) \\ &= T(a)\alpha a + T(a)\alpha b + T(b)\alpha a + T(b)\alpha b \end{aligned} \quad \dots(1)$$

On the other hand

$$\begin{aligned} T((a+b)\alpha(a+b)) &= T(a\alpha a + a\alpha b + b\alpha a + b\alpha b) \\ &= T(a)\alpha a + T(b)\alpha b + T(a\alpha b + b\alpha a) \end{aligned} \quad \dots(2)$$

Comparing (1) and (2) we get

$$T(a\alpha b + b\alpha a) = T(a)\alpha b + T(b)\alpha a$$

ii) In Definition 2.1 replace $a+c$ for a and β for α we get

$$\begin{aligned} T((a+c)\alpha b\alpha(a+c)) &= T(a+c)\alpha b\alpha(a+c) \\ &= (T(a) + T(c))\alpha b\alpha(a+c) \\ &= T(a)\alpha b\alpha a + T(a)\alpha b\alpha c + T(c)\alpha b\alpha a + T(c)\alpha b\alpha c \end{aligned} \quad \dots(1)$$

On the other hand

$$\begin{aligned} T((a+c)\alpha b\alpha(a+c)) &= T(a\alpha b\alpha a + a\alpha b\alpha c + c\alpha b\alpha a + c\alpha b\alpha c) \\ &= T(a\alpha b\alpha a + c\alpha b\alpha c) + T(a\alpha b\alpha c + c\alpha b\alpha a) \\ &= T(a)\alpha b\alpha a + T(c)\alpha b\alpha c + T(a\alpha b\alpha c + c\alpha b\alpha a) \end{aligned} \quad \dots(2)$$

Comparing (1) and (2) we get

$$T(a\alpha b\alpha c + c\alpha b\alpha a) = T(a)\alpha b\alpha c + T(c)\alpha b\alpha a$$

iii) In Definition 2.1 replace $a+c$ for a and by the same way as in (ii) we get the require result.

Definition 2.2: Let T be a Jordan left centralizer of Γ - ring M into right ΓM -module X then we define

$$\Psi(a,b) = T(a\alpha b) - T(a)\alpha b \quad \text{for all } a,b \in M \text{ and } \alpha \in \Gamma.$$

In the following lemma we present the properties of $\Psi_\alpha(a,b)$.

Lemma 2: Let T be a Jordan left centralizer of Γ - ring M into right ΓM -module X then for all $a,b,c \in M$ and $\alpha, \beta \in \Gamma$.

- i) $\Psi_\alpha(a,b) = -\Psi_\alpha(b,a)$
- ii) $\Psi(a+c, b) = \Psi_\alpha(a,b) + \Psi_\alpha(c,b)$
- iii) $\Psi(a, b+c) = \Psi_\alpha(a,b) + \Psi_\alpha(a,c)$
- iv) $\Psi_{\alpha+\beta}(a,b) = \Psi_\alpha(a,b) + \Psi_\beta(a,b)$

Proof: i)

$$\begin{aligned} T(a\alpha b + b\alpha a) &= T(a)\alpha b + T(b)\alpha a \\ T(a\alpha b) - T(a)\alpha b &= -T(b\alpha a) - T(b)\alpha a \\ \Psi(a,b) &= -\Psi_\alpha(b,a) \end{aligned}$$

$$\begin{aligned} \text{ii) } \Psi(a+c, b) &= T((a+c)\alpha b) - T(a+c)\alpha b \\ &= T(a\alpha b + c\alpha b) - T(a)\alpha b - T(c)\alpha b \\ &= T(a\alpha b) - T(a)\alpha b + T(c\alpha b) - T(c)\alpha b \\ &= \Psi(a,b) + \Psi_\alpha(c,b) \end{aligned}$$

iii) and iv) as the same way as in (ii) we get the require result.

Remark 2.3: Note that T is left centralizer of Γ - ring M into right ΓM -module X if and only if $\Psi(a,b) = 0$.

Lemma 3: Let $T: M \rightarrow X$ be a Jordan left centralizer of Γ - ring M into right ΓM -module X then for all $a,b,c \in M$ and $\alpha, \beta \in \Gamma$.

$$\Psi(a,b) \beta m \beta [a,b]_\alpha = 0$$

Proof : Let $w = a\alpha b m \beta b\alpha a + b\alpha a \beta m \alpha a \beta b$

$$\begin{aligned} T(w) &= T(a\alpha(bm\beta b)\alpha a + b\alpha(a\beta m\alpha a)\beta b) \\ &= T(a)\alpha(bm\beta b)\alpha a + T(b)\alpha(a\beta m\alpha a)\beta b \end{aligned} \quad \dots(1)$$

On the other hand

$$\begin{aligned} T(w) &= T((a\alpha b)m\beta(b\alpha a)) + (b\alpha a)\beta m\alpha(a\beta b) \\ &= T(a\alpha b)m\beta(b\alpha a) + T(b\alpha a)\beta m\alpha(a\beta b) \end{aligned} \quad \dots(2)$$

Comparing (1) and (2) we get

$$0 = (T(a\alpha b) - T(a)\alpha b) \beta m \beta b\alpha a + (T(b\alpha a) - T(b)) \alpha a \beta m \alpha a \beta b$$

$$= \Psi(a,b) \beta m \beta b \alpha a + \Psi_{\alpha}(b,a) \alpha a \beta m \alpha a \beta b$$

$$= \Psi(a,b) \beta m \beta [a,b]_{\alpha}$$

3) The Main Results

In this section we introduce the main results.

Theorem 4: Let $T: M \rightarrow X$ be a Jordan left centralizer of Γ -ring M into prime right ΓM -module X then for all $a, b, c, d \in M$ and $\alpha, \beta \in \Gamma$.

$$\Psi(a,b) \beta m \beta [a,b]_{\alpha} = 0$$

Proof: In Lemma 3 replace $a+c$ for a we get

$$\Psi(a+c,b) \beta m \beta [a+c,b]_{\alpha} = 0$$

$$\Psi_{\alpha}(a,b) \beta m \beta [a,b]_{\alpha} + \Psi_{\alpha}(a,b) \beta m \beta [c,b]_{\alpha} + \Psi_{\alpha}(c,b) \beta m \beta [a,b]_{\alpha} + \Psi_{\alpha}(c,b) \beta m \beta [c,b]_{\alpha} = 0$$

By Lemma 3 we get

$$\Psi(a,b) \beta m \beta [c,b]_{\alpha} + \Psi_{\alpha}(c,b) \beta m \beta [a,b]_{\alpha} = 0$$

Since

$$\Psi(a,b) \beta m \beta [c,b]_{\alpha} \beta m \beta \Psi_{\alpha}(a,b) \beta m \beta [c,b]_{\alpha} = 0$$

$$- \Psi(a,b) \beta m \beta [c,b]_{\alpha} \beta m \beta \Psi_{\alpha}(c,b) \beta m \beta [a,b]_{\alpha} = 0$$

Hence by primness we get

$$\Psi(a,b) \beta m \beta [c,b]_{\alpha} = 0 \quad \dots(1)$$

Replace $b+d$ for b in Lemma 3 we get

$$\Psi(a,b+d) \beta m \beta [a,b+d]_{\alpha} = 0$$

$$\Psi_{\alpha}(a,b) \beta m \beta [a,b]_{\alpha} + \Psi_{\alpha}(a,b) \beta m \beta [a,d]_{\alpha} + \Psi_{\alpha}(a,d) \beta m \beta [a,b]_{\alpha} + \Psi_{\alpha}(a,d) \beta m \beta [a,d]_{\alpha} = 0$$

By Lemma 3 we get

$$\Psi(a,b) \beta m \beta [a,d]_{\alpha} + \Psi_{\alpha}(a,d) \beta m \beta [a,b]_{\alpha} = 0$$

Since

$$\Psi(a,b) \beta m \beta [a,d]_{\alpha} \beta m \beta \Psi_{\alpha}(a,b) \beta m \beta [a,d]_{\alpha} = 0$$

$$- \Psi(a,b) \beta m \beta [a,d]_{\alpha} \beta m \beta \Psi_{\alpha}(a,d) \beta m \beta [a,b]_{\alpha} = 0$$

Hence by primness we get

$$\Psi(a,b) \beta m \beta [a,d]_{\alpha} = 0 \quad \dots(2)$$

Thus

$$\Psi(a,b) \beta m \beta [a+c,b+d]_{\alpha}=0$$

$$\Psi_{\alpha}(a,b) \beta m \beta [a,b]_{\alpha} + \Psi_{\alpha}(a,b) \beta m \beta [a,d]_{\alpha} + \Psi_{\alpha}(a,b) \beta m \beta [c,b]_{\alpha} + \Psi_{\alpha}(a,b) \beta m \beta [c,d]_{\alpha}=0$$

By Lemma 3 and (1), (2) we get

$$\Psi(a,b) \beta m \beta [c,d]_{\alpha}=0$$

Theorem 5: Let $T: M \rightarrow X$ be a Jordan left centralizer of Γ -ring M into 2-torsion free prime right ΓM -module X then T is a left centralizer.

Proof: Since X is prime right ΓM -module and by Theorem 4 we get

Either $\Psi_{\alpha}(a,b) = 0$ or $[c,d]_{\alpha} = 0$, for all $a, b, c, d \in M$ and $\alpha \in \Gamma$.

If $[c,d]_{\alpha} \neq 0$ then $\Psi_{\alpha}(a,b) = 0$, for all $a, b, c, d \in M$ and $\alpha \in \Gamma$, by Remark 2.3 we get the require result.

If $[c,d]_{\alpha} = 0$, for all $c, d \in M$ and $\alpha \in \Gamma$, then M is commutative Γ -ring and by Lemma 2.1 we get

$$T(2a\alpha b) = 2T(a)\alpha b$$

Since X is 2-torsion free right ΓM -module then T is a left centralizer.

Theorem 6: Let T be a Jordan left centralizer of Γ -ring M which satisfy $a\alpha b\beta a = a\beta b\alpha a$ into 2-torsion free right ΓM -module X then T is a left centralizer.

Proof: Replace b by $a\beta b + b\beta a$ in Definition 2.1 we get

$$\begin{aligned} T(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) &= T(a)\alpha(a\beta b + b\beta a) + T(a\beta b + b\beta a)\alpha a \\ &= T(a)\alpha a\beta b + T(a)b\beta a + T(a)\beta b\alpha a + T(b)\beta a\alpha a \quad \dots(1) \end{aligned}$$

On the other hand

$$\begin{aligned} T(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) &= T(a\alpha a\beta b + a\alpha b\beta a + a\beta b\alpha a + b\beta a\alpha a) \\ &= T(a)\alpha a\beta b + T(b)\beta a\alpha a + T(a\alpha b\beta a + a\beta b\alpha a) \quad \dots(2) \end{aligned}$$

Comparing (1) and (2) since $a\alpha b\beta a = a\beta b\alpha a$ we get

$$2T(a\alpha b\beta a) = 2T(a)\alpha b\beta a$$

Since X is 2-torsion free we get

$$T(a\alpha b\beta a) = T(a)\alpha b\beta a$$

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