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Effects of MHD on the Unsteady Rotating Flow of a Generalized Maxwell Fluid with Oscillating Gradient Between Coaxial Cylinders

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Abstract

The aim of this paper is studied the effect of magnetic field on the unsteady rotating flow of a generalized Maxwell fluid with fractional derivative between two infinite straight circular cylinder .The velocity field and the shear stress are obtained by means of discrete Laplace transform and finite Hankel transform. The exact solution for the velocity field and the shear stress that have been obtained by integral and series form in terms of the generalized G functions and Mitting –leffer function .the graphs are plotted to show the effects of the fractional parameter on the fluid dynamic characteristics with MHD on the velocity and shear stress.

1.Introduction

The modeling of the equations governing the non-Newtonian fluids give rise to be a nonlinear differential equation .Such nonlinear fluids are now consider to play a more important and appropriate role in technological application in comparison with Newtonian fluids . The application of non-Newtonian fluid in engineering problems such as magnetohydrodtnamical(MHD) and have now become the focus of extension study for example plastics ,polymer fluids ,exotic lubricant ,food stuffs and polymers are handled extensively by chemical industries where us biological and the rheological properties of many materials are described by there constitutive equation .

In the recent years, the fractional derivatives are found to be quite flexible in describing the behaviors viscotastic fluid and are studied by many mathematicians considering various motion of such fluids .In their studies ,the constrictive equation for generalized non-Newtonian fluids are modified from well known fluid models by replacing the time derivative of an integer order by precisely non-integer order integrals or derivatives. The fractional derivative models of the viscoelastic fluids are obtained by researchers especially in the problems of the motion of a fluid in rotating or translating cylinder is of interest to both theoretical and practical domains . The first exact solution for flows of non-Newtonian fluids in cylindrical problems are those of Ting[22],for second grade fluids, Srivastava [20] for Maxwell fluids and Waters and King [19] for Oldroyd –B fluids in a straight circular tube . Fetecau [4,5,7] studied some heical flows of Maxwell and Oldroyd -B fluids within an infinite cylinder .Exact solutions of generalized Maxwell fluid flow due to oscillatory and constantly accelerating plate obtained by Zheng[16]. The unsteady Couette flow problems have been considered in several works for the longitudinal time-dependent shear stress by Fetecau[3,17], Preziosi and Joseph [13]. and other various effects as in the book and paper by Joseph [9].Bernardin [8] and Preziosi[14]tan, Xu [24] and Pan [25] considered a plate surface suddenly set in motion in a viscoelastic fluid with fractional Maxwell equation or between two parallel plates . Liancun and zhang[15] studied the unsteady rotating flows of a viscoelastic generalized Maxwell fluid with oscillating pressure gradient between coaxial cylinders.

In this paper we studied the effect of magnetic field on the unsteady rotating flow of a generalized Maxwell fluid with fractional derivative between two infinite straight circular cylinder .The velocity field and the shear stress are obtained by means of discrete Laplace transform and finite Hankel transform the exact solution for the velocity field and the shear stress that have been obtained by integral and series form in terms of the generalized G functions and Mitting –leffer function .The graphs are plotted to show the effects of the fractional parameteron the fluid dynamic characteristics with MHD on the velocity and shear stress.

2.Basic governing equations

The constitutive equations of an incompressible fractional Maxwell fluid are given by [2,10]

$$\mathbf{T} = -p\boldsymbol{I} + \mathbf{S}\,, \mathbf{S} + \lambda\,\frac{\mathbf{DS}}{\mathbf{Dt}} = \mu\boldsymbol{A} \tag{1}$$

where - p**I** denoted the indeterminate spherical stress, **S** is the extra-stress tensor, **T** is the Cauchy stress tensor, $\mathbf{A} = \mathbf{L} + \mathbf{L}^{T}$ is the first Rivlin-Ericksen tensor with **L**=grad **V**, μ the dynamic viscosity of the fluid, λ the material constant ,α the fractional calculus parameter such that $0 \le \alpha \le 1$ and $\frac{\mathrm{DS}}{\mathrm{Dt}}$ is defined by

$$\frac{\mathrm{D}\mathbf{S}}{\mathrm{Dt}} = \mathrm{D}_t^{\alpha}\mathbf{S} + \mathbf{V}.\nabla\mathbf{S} - \mathbf{LS} - \mathbf{SL}^{\mathrm{T}} \tag{2}$$

In which ∇ is gradient operator, D_t^{α} is the fractional differential operator based on Riemann-Liouville's, defined as

$$D_t^{\alpha} f(\mathrm{t}) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}} d\tau, \quad 0 \le \alpha \le 1 \tag{3}$$

where $\Gamma(.)$ denotes the Gamma function. This model reduces to the ordinary Maxwell fluid model when $\alpha = 1$ and

$$D_t^{2\alpha} = D_t^{\alpha}(D_t^{\alpha}\mathbf{S}) \tag{4}$$

We assume that, the axial coquette flow velocity and shear stress in cylindrical coordinates (r, θ, z) is given by

$$\mathbf{V} = w(r, \mathrm{t})\, e_{\theta}\,, \quad \mathbf{S} = \mathbf{S}(r, \theta) \tag{5}$$

Where e_{θ} is the unit vector in the θ- axis and w is the velocity .Since **V** is dependent of r and t ,we also assume that **S** depends only on r and θ.

If the fluid is assumed to be at rest at the moment t=0, then

$$\mathbf{V}(r, 0) = 0\,, \quad \mathbf{S}(r, 0) = 0 \tag{6}$$

We can obtain

$$\tau(1 + \lambda\ D_t^{\alpha}) = \mu\left(\frac{\partial}{\partial r} - \frac{1}{r}\right) w \tag{7}$$

Where $S_{rr} = S_{zz} = S_{rz} = S_{\theta z} = S_{\theta\theta} = 0$, $\tau(r, \mathrm{t}) = S_{r\theta}\ (r, \mathrm{t})$ is the shear stress .

3. Momentum and continuity equations

we will write the formula of the momentum equation which governing the magnetohydrodynamic in θ -direction and consider the pressure gradient ,the balance of linear momentum leads to the relevant and meaningful equation

$$\rho \frac{\partial w}{\partial \mathrm{t}} = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{1}{r^2} \frac{\partial (r^2 \tau)}{\partial r} - \delta \beta_0^2\, w \tag{8}$$

Where ρ is the constant density of the fluid.

Eliminating $\tau(r,t)$ between Eqs, (7) and (8), yields

$$\frac{\partial w}{\partial \mathrm{t}}(1 + \lambda\ D_t^{\alpha}) = -\frac{1}{r\,\rho} \frac{\partial p}{\partial \theta}(1 + \lambda\ D_t^{\alpha}) + \frac{\mu}{\rho}\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial r^2}\right) w - \frac{\delta \beta_0^2}{\rho}\, w \tag{9}$$

Let $\upsilon = \frac{\mu}{\rho}$ is the kinematics viscosity and $\frac{\partial p}{\partial \theta} = -\rho\, p_0 \cos(\omega t)$ or $\frac{\partial p}{\partial \theta} = -\rho\, p_0 \sin(\omega t)$

Where p_0 is constant ,we get the governing equation

$$(1 + \lambda\ D_t^{\alpha}) \frac{\partial w}{\partial \mathrm{t}} = \frac{p_0}{r}(1 + \lambda\ D_t^{\alpha}) \cos(\omega t) + \upsilon\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right) w - \frac{\delta \beta_0^2}{\rho}\, w \tag{10}$$

4. Rotating flow between coaxial cylinders

this paper considers an incompressible generalized Maxwell fluid at rest in the annular region between two infinite circular cylinders of radius R_1 and R_2 ($R_2 > R_1$). At time t =0 , the inner cylinder is suddenly moved with a time-dependent pressure gradient in the θ axial direction. The associated initial and boundary conditions are

$$w(r, 0) = \frac{\partial w(r, 0)}{\partial \mathrm{t}} = 0\,, \quad r \,\epsilon [R_1\,, R_2] \tag{11}$$

$$w(R_1, t) = f e^{at}\,, \quad w(R_2, t) = 0, \quad \mathrm{t} > 0 \tag{12}$$

Where f is a constant.

5. Calculation of the velocity field

In this section ,the velocity field will be calculate for different case of pressure gradient.

5.1. *The case* $\frac{\partial p}{\partial \theta} = -\rho\, p_0\, Cos(\omega t)$

Applying the Laplace transform to Eqs. (10)-(12), using the Laplace transform of the sequential fractional derivatives [11], we find that

$$(q + \lambda q^{\alpha+1})\overline{w} = \frac{p_0}{r}\left[\left(1 + \lambda\, \omega^{\alpha} \cos\left(\frac{\pi}{2}\,\alpha\right)\right)\frac{q}{q^2 + \omega^2} - \lambda\, \omega^{\alpha} \sin\left(\frac{\pi}{2}\,\alpha\right)\frac{\omega}{q^2 + \omega^2}\right] + \upsilon\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\overline{w} - \frac{\delta \beta_0^2}{\rho}\,\overline{w}\,, r \,\epsilon [R_1\,, R_2] \tag{13}$$

Subject to boundary condition

$$\overline{w}(r, 0) = 0\,, \quad r \,\epsilon [R_1\,, R_2] \tag{14}$$

$$\bar{w}(R_1,q)=\frac{f}{(q-a)}\ ,\quad \bar{w}(R_2,q)=\ 0,\ \ t>0 \tag{15}$$

We use the finite Hankel transform [12] ,defined as follows

$$\bar{w}_H=\int_{R_1}^{R_2} r\,\bar{w}\,B_1(r\,r_n)dr\ ,n=1,2,3\ \dots\dots\dots \tag{16}$$

where r_n are the positive roots of equation $B_1(R_1\,r)=0$ and

$$B_1(r\,r_n)=J_1(r\,r_n)Y_1(R_2\,r_n)-J_1(R_2\,r_n)Y_1(r\,r_n) \tag{17}$$

where $J_n(.)$ and $Y_n(.)$ are Baseel functions of the first and second kind of older n, respectively.
Multiplying both sides of Eq. (13) by $r\,B_1(r\,r_n)$, integrating with respect to r from R_1 to R_2 and taking into account the conditions (14) and (15) and the identity

$$\int_{R_1}^{R_2} r\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right)\bar{w}\,B_1(r\,r_n)dr=-\frac{2}{\pi}\frac{f}{(q-a)}\frac{J_1(R_2\,r_n)}{J_1(R_1\,r_n)}-r_n^{\,2}\,\bar{w}_H \tag{18}$$

we find that

$$\begin{aligned}\bar{w}_H=&\ p_0\left(1+\lambda\,\omega^{\alpha}\cos\left(\frac{\pi}{2}\,\alpha\right)\right)\frac{\bar{B}_1(R_1\,r_n)-\bar{B}_1(R_2\,r_n)}{r_n}\ \frac{q}{(q^2+\omega^2)\left(q+\lambda q^{\alpha+1}+\upsilon r_n^2+\frac{\upsilon\delta\beta_0^2}{\rho}\right)}\\ &-\lambda\,\omega^{\alpha}\sin\left(\frac{\pi}{2}\,\alpha\right)\frac{\bar{B}_1(R_1\,r_n)-\bar{B}_1(R_2\,r_n)}{r_n}\times\frac{\omega}{(q^2+\omega^2)\left(q+\lambda q^{\alpha+1}+\upsilon r_n^2+\frac{\upsilon\delta\beta_0^2}{\rho}\right)}\\ &-\frac{2}{\pi\,r_n^2}\frac{f}{(q-a)}\frac{J_1(R_2\,r_n)}{J_1(R_1\,r_n)}+\frac{2}{\pi}\frac{f}{r_n^2}\frac{J_1(R_2\,r_n)}{J_1(R_1\,r_n)}\times\frac{\left(q+\lambda q^{\alpha+1}+\frac{\delta\beta_0^2}{\rho}\right)}{\left(q+\lambda q^{\alpha+1}+\upsilon r_n^2+\frac{\upsilon\delta\beta_0^2}{\rho}\right)}\end{aligned} \tag{19}$$

Where $\bar{B}_1(r\,r_n)=J_0(r\,r_n)Y_1(R_2\,r_n)-J_1(R_2\,r_n)Y_0(r\,r_n)$ [1].

Inverting the above results by means of the Hankel transform, one gets the expression for $\bar{w}$ in the form

$$\begin{aligned}\bar{w}=&\ \frac{R_1(R_2^2-r^2)f}{(R_2^2-R_1^2)r(q-a)}+\frac{\pi^2p_0\left(1+\lambda\,\omega^{\alpha}\cos\left(\frac{\pi}{2}\,\alpha\right)\right)}{2}\times\sum_{n=1}^{\infty}\frac{r_n\,J_1^2(R_1\,r_n)B_1(r\,r_n)(\bar{B}_1(R_1\,r_n)-\bar{B}_1(R_2\,r_n))}{J_1^2(R_1\,r_n)-\ J_1^2(R_2\,r_n)}\\ &\times\frac{q}{(q^2+\omega^2)\left(q+\lambda q^{\alpha+1}+\upsilon r_n^2+\frac{\upsilon\delta\beta_0^2}{\rho}\right)}-\frac{\pi^2p_0\,\lambda\,\omega^{\alpha}\sin\left(\frac{\pi}{2}\,\alpha\right)}{2}\\ &\times\sum_{n=1}^{\infty}\frac{r_n\,J_1^2(R_1\,r_n)B_1(r\,r_n)(\bar{B}_1(R_1\,r_n)-\bar{B}_1(R_2\,r_n))}{J_1^2(R_1\,r_n)-\ J_1^2(R_2\,r_n)}\times\frac{\omega}{(q^2+\omega^2)\left(q+\lambda q^{\alpha+1}+\upsilon r_n^2+\frac{\upsilon\delta\beta_0^2}{\rho}\right)}\\ &+\pi f\sum_{n=1}^{\infty}\frac{J_1(R_1\,r_n)J_1(R_2\,r_n)B_1(r\,r_n)}{J_1^2(R_1\,r_n)-\ J_1^2(R_2\,r_n)}\\ &\times\frac{\left(q+\lambda q^{\alpha+1}+\frac{\delta\beta_0^2}{\rho}\right)}{(q-\mathrm{a})\left(q+\lambda q^{\alpha+1}+\upsilon r_n^2+\frac{\upsilon\delta\beta_0^2}{\rho}\right)}\end{aligned} \tag{20}$$

where

$$\bar{w}=\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{r_n^2\,J_1^2(R_1\,r_n)B_1(r\,r_n)}{J_1^2(R_1\,r_n)-\ J_1^2(R_2\,r_n)}\ \bar{w}_H \tag{21}$$

now, applying discrete Laplace transform method, we use the expansion

$$\frac{q}{(q^2+\omega^2)\left(q+\lambda q^{\alpha+1}+\upsilon r_n^2+\frac{\upsilon\delta\beta_0^2}{\rho}\right)}=\frac{-q}{(q^2+\omega^2)}\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^{k+1}\frac{q^k}{(q^{\alpha+1}+\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1})^{k+1}}$$

$$\frac{\left(q+\lambda q^{\alpha+1}+\frac{\delta\beta_0^2}{\rho}\right)}{\left(q+\lambda q^{\alpha+1}+\upsilon r_n^2+\frac{\upsilon\delta\beta_0^2}{\rho}\right)}=\frac{(q+\lambda q^{\alpha+1})}{\left(q+\lambda q^{\alpha+1}+\upsilon r_n^2+\frac{\upsilon\delta\beta_0^2}{\rho}\right)}+\frac{\left(\frac{\delta\beta_0^2}{\rho}\right)}{\left(q+\lambda q^{\alpha+1}+\upsilon r_n^2+\frac{\upsilon\delta\beta_0^2}{\rho}\right)}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k(\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1})^k}{(\lambda^{-1}+q^{\alpha})^k}-\frac{\delta\beta_0^2}{\rho}\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^{k+1}\frac{q^k}{(q^{\alpha+1}+\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1})^{k+1}}$$

and using the following property of inverse Laplace transform

$$L^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\}=G_{a,b,c}(d,\tau) \tag{22}$$

$$L^{-1}\left\{\frac{n!\,s^{\alpha-\beta-1}}{(s^a\mp c)^{n+1}}\right\}=t^{\alpha n+\beta-1}E_{\alpha,\beta}^{(n)}\,(\mp c\,t^{\alpha}) \tag{23}$$

where [11]

$$G_{a,b,c}(d,\tau)=\sum_{j=0}^{\infty}\frac{(c)_j d^j t^{(j+c)a-b-1}}{j!\,\Gamma[(j+c)a-b]} \tag{24}$$

is the generalized *G* function and $(c)_j$ is pochhammer polynomial [6] and

$$E_{\alpha,\beta}^{n}(z)=\sum_{j=0}^{\infty}\frac{(c)_j d^j t^{(j+c)a-b-1}}{j!\,\Gamma[\alpha\, j+\alpha n+\beta]} \tag{25}$$

represents the generalized Mittag- Leffler function [18].
Finally ,We attain the following expressions for the velocity field.

$$w=\frac{R_1(R_2^2-r^2)f}{(R_2^2-R_1^2)r}e^{at}-\frac{\pi^2 p_0\left(1+\lambda\,\omega^{\alpha}\cos\left(\frac{\pi}{2}\,\alpha\right)\right)}{2}\times\sum_{n=1}^{\infty}\frac{r_n J_1^2(R_1 r_n)B_1(r\, r_n)\left(\bar{B}_1(R_1 r_n)-\bar{B}_1(R_2 r_n)\right)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}$$
$$\times\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^{k+1}\int_0^t\left\{Cos(\omega(t-\tau))G_{\alpha+1,k,k+1}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau$$
$$+\frac{\pi^2 p_0\,\lambda\,\omega^{\alpha}\sin\left(\frac{\pi}{2}\,\alpha\right)}{2}\times\sum_{n=1}^{\infty}\frac{r_n J_1^2(R_1 r_n)B_1(r\, r_n)\left(\bar{B}_1(R_1 r_n)-\bar{B}_1(R_2 r_n)\right)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}$$
$$\times\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^{k+1}\int_0^t\left\{Sin(\omega(t-\tau))G_{\alpha+1,k,k+1}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau$$
$$+\pi f\sum_{n=1}^{\infty}\frac{J_1(R_1 r_n)J_1(R_2 r_n)B_1(r\, r_n)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}\times\sum_{k=0}^{\infty}\left(\frac{-\upsilon(r_n^2+\frac{\delta\beta_0^2}{\rho})}{\lambda}\right)^{k+1}\int_0^t\left\{e^{\alpha(t-\tau)}G_{\alpha,-k,k}(-\lambda^{-1},\tau)\right\}d\tau$$
$$-\pi f\sum_{n=1}^{\infty}\frac{J_1(R_1 r_n)J_1(R_2 r_n)B_1(r\, r_n)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}\times\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^{k+1}\times\frac{\delta\beta_0^2}{\rho}$$
$$\times\int_0^t\left\{e^{\alpha(t-\tau)}G_{\alpha+1,k,k+1}\left(-\upsilon(r_n^2+\frac{\delta\beta_0^2}{\rho})\lambda^{-1},\tau\right)\right\}d\tau \tag{26}$$

5.2. *The case* $\frac{\partial p}{\partial\theta}=-\rho\, p_0\, Sin(\omega t)$

By similar method ,we can obtained the solution in the following form

$$w=\frac{R_1(R_2^2-r^2)f}{(R_2^2-R_1^2)r}e^{at}-\frac{\pi^2 p_0\left(1+\lambda\,\omega^\alpha\cos\left(\frac{\pi}{2}\,\alpha\right)\right)}{2}\times\sum_{n=1}^{\infty}\frac{r_n J_1^2(R_1 r_n)B_1(r\, r_n)\left(\bar{B}_1(R_1 r_n)-\bar{B}_1(R_2 r_n)\right)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}$$

$$\times\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^{k+1}\int_0^t\left\{Sin(\omega(t-\tau))G_{\alpha+1,k,k+1}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau$$

$$-\frac{\pi^2 p_0\,\lambda\,\omega^\alpha\sin\left(\frac{\pi}{2}\,\alpha\right)}{2}\times\sum_{n=1}^{\infty}\frac{r_n J_1^2(R_1 r_n)B_1(r\, r_n)\left(\bar{B}_1(R_1 r_n)-\bar{B}_1(R_2 r_n)\right)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}$$

$$\times\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^{k+1}\int_0^t\left\{Cos(\omega(t-\tau))G_{\alpha+1,k,k+1}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau$$

$$+\pi f\sum_{n=1}^{\infty}\frac{J_1(R_1 r_n)J_1(R_2 r_n)B_1(r\, r_n)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}$$

$$\times\sum_{k=0}^{\infty}\left(\frac{-\upsilon(r_n^2+\frac{\delta\beta_0^2}{\rho})}{\lambda}\right)^{k+1}\int_0^t\left\{e^{\alpha(t-\tau)}G_{\alpha,-k,k}(-\lambda^{-1},\tau)\right\}d\tau$$

$$-\pi f\sum_{n=1}^{\infty}\frac{J_1(R_1 r_n)J_1(R_2 r_n)B_1(r\, r_n)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}\times\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^{k+1}\times\frac{\delta\beta_0^2}{\rho}$$

$$\times\int_0^t\left\{e^{\alpha(t-\tau)}G_{\alpha+1,k,k+1}\left(-\upsilon(r_n^2+\frac{\delta\beta_0^2}{\rho})\lambda^{-1},\tau\right)\right\}d\tau \qquad (27)$$

6. calculation of the shear stress

In this section ,the shear stress is calculate for different type of pressure gradient.

6.1 *the case* $\frac{\partial p}{\partial\theta}=-\rho\, p_0\, cos(\omega t)$

The shear stress can be calculated from Eq(7) ,taking Laplace transform of Eq(7),we get

$$(1+\lambda q^\alpha)\bar{\tau}=\mu\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\bar{w} \qquad (28)$$

$$\bar{\tau}=\frac{1}{(1+\lambda q^\alpha)}\mu\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)\bar{w} \qquad (29)$$

$$\bar{\tau}=\frac{-2\mu R_1 R_2^2 f}{(R_2^2-R_1^2)r(q-a)(1+\lambda q^\alpha)}+\frac{\mu\,\pi^2 p_0\left(1+\lambda\,\omega^\alpha\,cos\left(\frac{\pi}{2}\,\alpha\right)\right)}{2}$$

$$\times\sum_{n=1}^{\infty}\frac{r_n J_1^2(R_1 r_n)(\bar{B}_1(R_1 r_n)-\bar{B}_1(R_2 r_n))}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}\times$$

$$\frac{q\,(r_n\bar{B}_1(R_1\,r_n)-\left(\frac{1}{r}\right)B_1(r\,r_n))}{(q^2+\omega^2)\left(q+\lambda q^{\alpha+1}+\upsilon r_n^2+\frac{\upsilon\delta\beta_0^2}{\rho}\right)(1+\lambda q^\alpha)}-\frac{\mu\pi^2 p_0\,\lambda\,\omega^\alpha\,sin\left(\frac{\pi}{2}\,\alpha\right)}{2}$$
$$\times\sum_{n=1}^{\infty}\frac{r_n\,J_1^2(R_1\,r_n)(\bar{B}_1(R_1\,r_n)-\bar{B}_1(R_2\,r_n))}{J_1^2(R_1\,r_n)-J_1^2(R_2\,r_n)}$$
$$\times\frac{\omega(r_n\bar{B}_1(R_1\,r_n)-\left(\frac{1}{r}\right)B_1(r\,r_n))}{(q^2+\omega^2)\left(q+\lambda q^{\alpha+1}+\upsilon r_n^2+\frac{\upsilon\delta\beta_0^2}{\rho}\right)(1+\lambda q^\alpha)}$$
$$+\pi f\sum_{n=1}^{\infty}\frac{J_1(R_1\,r_n)J_1(R_2\,r_n)(r_n\bar{B}_1(R_1\,r_n)-\left(\frac{1}{r}\right)B_1(r\,r_n))}{J_1^2(R_1\,r_n)-J_1^2(R_2\,r_n)}$$
$$\times\frac{\left(q+\lambda q^{\alpha+1}+\frac{\delta\beta_0^2}{\rho}\right)}{(q-a)\left(q+\lambda q^{\alpha+1}+\upsilon r_n^2+\frac{\upsilon\delta\beta_0^2}{\rho}\right)(1+\lambda q^\alpha)} \qquad (30)$$

we, applying discrete inverse Laplace transform for Eq(30) to obtain the shear stress in the following form

$$\tau=\frac{-2\mu R_1R_2^2}{(R_2^2-R_1^2)r}\left[e^{at}-\int_0^t\left\{e^{\alpha(t-\tau)}G_{\alpha,\alpha,1}(-\lambda^{-1},\tau)\right\}d\tau\right]+\frac{\mu\pi^2 p_0\left(1+\lambda\,\omega^\alpha\,cos\left(\frac{\pi}{2}\,\alpha\right)\right)}{2}$$
$$\times\sum_{n=1}^{\infty}\frac{r_n\,J_1^2(R_1\,r_n)(r_n\bar{B}_1(R_1\,r_n)-\left(\frac{1}{r}\right)B_1(r\,r_n))(\bar{B}_1(R_1\,r_n)-\bar{B}_1(R_2\,r_n))}{J_1^2(R_1\,r_n)-J_1^2(R_2\,r_n)}\times\left(\frac{1}{\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)}\right)$$
$$\times\left[\lambda^{-1}\int_0^t\left\{Cos(\omega(t-\tau))\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda^{-1}\,\tau)\right\}d\tau\right.$$
$$\left.+\sum_{k=1}^{\infty}\left(\frac{-1}{\lambda}\right)^k\int_0^t\left\{Cos(\omega(t-\tau))G_{\alpha+1,k,k}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau\right]$$
$$-\frac{\mu\pi^2 p_0\,\lambda\,\omega^\alpha\,Sin\left(\frac{\pi}{2}\,\alpha\right)}{2}$$
$$\times\sum_{n=1}^{\infty}\frac{r_n\,J_1^2(R_1\,r_n)(r_n\bar{B}_1(R_1\,r_n)-\left(\frac{1}{r}\right)B_1(r\,r_n))(\bar{B}_1(R_1\,r_n)-\bar{B}_1(R_2\,r_n))}{J_1^2(R_1\,r_n)-J_1^2(R_2\,r_n)}\times\left(\frac{1}{\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)}\right)$$
$$\times\left[\lambda^{-1}\int_0^t\left\{Sin(\omega(t-\tau))\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda^{-1}\tau)\right\}d\tau\right.$$
$$\left.+\sum_{k=1}^{\infty}\left(\frac{-1}{\lambda}\right)^k\int_0^t\left\{Sin(\omega(t-\tau))G_{\alpha+1,k,k}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau\right]$$
$$+\mu\pi f\sum_{n=1}^{\infty}\frac{J_1(R_1\,r_n)J_1(R_2\,r_n)\,\left(r_n\bar{B}_1(R_1\,r_n)-\left(\frac{1}{r}\right)B_1(r\,r_n)\right)}{J_1^2(R_1\,r_n)-J_1^2(R_2\,r_n)}\left\{\left[\sum_{k=1}^{\infty}\left(\frac{-1}{\lambda}\right)^k\right.\right.$$
$$\left.\times\int_0^t\left\{e^{\alpha(t-\tau)}G_{\alpha+1,k,k}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau\right]+\left(\frac{\frac{\delta\beta_0^2}{\rho}}{\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)}\right)$$
$$\left[\lambda^{-1}\int_0^t\left\{e^{\alpha(t-\tau)}\tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda^{-1}\tau)\right\}d\tau\right.$$
$$\left.\left.+\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^k\int_0^t\left\{e^{\alpha(t-\tau)}G_{\alpha+1,k,k}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau\right]\right\} \qquad (31)$$

6.2 *the case* $\frac{\partial p}{\partial \theta} = -\rho\, p_0\, Sin(\omega t)$

Proceeding in similar as before ,we can find the solution in the following form

$$\tau = \frac{-2\mu R_1 R_2^2}{(R_2^2 - R_1^2) r}\left[e^{at} - \int_0^t \{e^{\alpha(t-\tau)} G_{\alpha,\alpha,1}(-\lambda^{-1},\tau)\}\, d\tau\right] + \frac{\mu\pi^2 p_0 \left(1 + \lambda\, \omega^\alpha \cos\left(\frac{\pi}{2}\,\alpha\right)\right)}{2}$$
$$\times \sum_{n=1}^{\infty} \frac{r_n J_1^2(R_1 r_n)(r_n \bar{B}_1(r\, r_n) - \left(\frac{1}{r}\right) B_1(r\, r_n))\left(\bar{B}_1(R_1 r_n) - \bar{B}_1(R_2 r_n)\right)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} \times \left(\frac{1}{\upsilon\left(r_n^2 + \frac{\delta\beta_0^2}{\rho}\right)}\right)$$
$$\times \left[\lambda^{-1} \int_0^t \{Sin(\omega(t-\tau))\tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda^{-1}\,\tau)\} d\tau\right.$$
$$\left. + \sum_{k=1}^{\infty} \left(\frac{-1}{\lambda}\right)^k \int_0^t \left\{Sin(\omega(t-\tau)) G_{\alpha+1,k,k}\left(-\upsilon\left(r_n^2 + \frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1}, \tau\right)\right\} d\tau\right]$$
$$+ \frac{\mu\pi^2 p_0\, \lambda\, \omega^\alpha\, Sin\left(\frac{\pi}{2}\,\alpha\right)}{2} \times \sum_{n=1}^{\infty} \frac{r_n J_1^2(R_1 r_n)(r_n \bar{B}_1(r\, r_n) - \left(\frac{1}{r}\right) B_1(r\, r_n))\left(\bar{B}_1(R_1 r_n) - \bar{B}_1(R_2 r_n)\right)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)}$$
$$\times \left(\frac{1}{\upsilon\left(r_n^2 + \frac{\delta\beta_0^2}{\rho}\right)}\right) \times \left[\lambda^{-1} \int_0^t \{Cos(\omega(t-\tau))\tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda^{-1}\tau)\} d\tau\right.$$
$$\left. + \sum_{k=1}^{\infty} \left(\frac{-1}{\lambda}\right)^k \int_0^t \left\{Cos(\omega(t-\tau)) G_{\alpha+1,k,k}\left(-\upsilon\left(r_n^2 + \frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1}, \tau\right)\right\} d\tau\right]$$
$$+ \mu\pi f \sum_{n=1}^{\infty} \frac{J_1(R_1 r_n) J_1(R_2 r_n)\left(r_n \bar{B}_1(r\, r_n) - \left(\frac{1}{r}\right) B_1(r\, r_n)\right)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)}$$
$$\left\{\left[\sum_{k=1}^{\infty} \left(\frac{-1}{\lambda}\right)^k \times \int_0^t \left\{e^{\alpha(t-\tau)} G_{\alpha+1,k,k}\left(-\upsilon\left(r_n^2 + \frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1}, \tau\right)\right\} d\tau\right]\right.$$
$$+ \left(\frac{\frac{\delta\beta_0^2}{\rho}}{\upsilon\left(r_n^2 + \frac{\delta\beta_0^2}{\rho}\right)}\right)\left[\lambda^{-1} \int_0^t \{e^{\alpha(t-\tau)} \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda^{-1}\tau)\} d\tau\right.$$
$$\left.\left. + \sum_{k=0}^{\infty} \left(\frac{-1}{\lambda}\right)^k \int_0^t \left\{e^{\alpha(t-\tau)} G_{\alpha+1,k,k}\left(-\upsilon\left(r_n^2 + \frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1}, \tau\right)\right\} d\tau\right]\right\} \qquad (32)$$

7. Special limiting cases for

In this section ,we recover some limiting case that have been studies before .there solution are obtained from our solution.

7.1 *The case* $\frac{\partial p}{\partial \theta} = -\rho\, p_0\, cos(\omega t)$

In special case Making the limit of Eq.(26) and Eq.(31) when $\alpha \to 1$, we get the solution of velocity field and shear stress reduces to

$$w=\frac{R_1(R_2^2-r^2)f}{(R_2^2-R_1^2)r}e^{at}-\frac{\pi^2 p_0}{2}\times\sum_{n=1}^{\infty}\frac{r_n J_1^2(R_1 r_n)B_1(r r_n)\left(\bar{B}_1(R_1 r_n)-\bar{B}_1(R_2 r_n)\right)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}$$
$$\times\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^{k+1}\int_0^t\left\{Cos\left(\omega(t-\tau)\right)G_{2,k,k+1}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau+\frac{\pi^2 p_0\,\lambda\,\omega}{2}$$
$$\times\sum_{n=1}^{\infty}\frac{r_n J_1^2(R_1 r_n)B_1(r r_n)\left(\bar{B}_1(R_1 r_n)-\bar{B}_1(R_2 r_n)\right)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}$$
$$\times\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^{k+1}\int_0^t\left\{Sin\left(\omega(t-\tau)\right)G_{2,k,k+1}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau$$
$$+\pi f\sum_{n=1}^{\infty}\frac{J_1(R_1 r_n)J_1(R_2 r_n)B_1(r r_n)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}$$
$$\times\sum_{k=0}^{\infty}\left(\frac{-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)}{\lambda}\right)^{k+1}\int_0^t\left\{e^{\alpha(t-\tau)}G_{1,-k,k}\left(-\lambda^{-1},\tau\right)\right\}d\tau$$
$$-\pi f\sum_{n=1}^{\infty}\frac{J_1(R_1 r_n)J_1(R_2 r_n)B_1(r r_n)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}\times\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^{k+1}\times\frac{\delta\beta_0^2}{\rho}$$
$$\times\int_0^t\left\{e^{\alpha(t-\tau)}G_{2,k,k+1}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau \tag{33}$$

$$\tau=\frac{-2\mu R_1 R_2^2}{(R_2^2-R_1^2)r}\left[e^{at}-\int_0^t\left\{e^{\alpha(t-\tau)}G_{1,-k,k}\left(-\lambda^{-1},\tau\right)\right\}d\tau\right]+\frac{\mu\pi^2 p_0}{2}$$
$$\times\sum_{n=1}^{\infty}\frac{r_n J_1^2(R_1 r_n)(r_n\bar{B}_1(R_1 r_n)-\left(\frac{1}{r}\right)B_1(r r_n))\left(\bar{B}_1(R_1 r_n)-\bar{B}_1(R_2 r_n)\right)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}\times\left(\frac{1}{\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)}\right)$$
$$\times\left[\lambda^{-1}\int_0^t\left\{Cos\left(\omega(t-\tau)\right)E_{1,1}(-\lambda^{-1}\tau)\right\}d\tau\right.$$
$$+\sum_{k=1}^{\infty}\left(\frac{-1}{\lambda}\right)^{k}\int_0^t\left\{\mathrm{Cos}\left(\omega(t-\tau)\right)G_{2,k,k+1}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau$$
$$-\frac{\mu\pi^2 p_0\,\lambda\,\omega}{2}\times\sum_{n=1}^{\infty}\frac{r_n J_1^2(R_1 r_n)(r_n\bar{B}_1(r r_n)-\left(\frac{1}{r}\right)B_1(r r_n))\left(\bar{B}_1(R_1 r_n)-\bar{B}_1(R_2 r_n)\right)}{J_1^2(R_1 r_n)-J_1^2(R_2 r_n)}\times$$
$$\left(\frac{1}{\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)}\right)\times\left[\lambda^{-1}\int_0^t\left\{Sin\left(\omega(t-\tau)\right)E_{1,1}(-\lambda^{-1}\tau)\right\}d\tau\right.$$
$$\left.+\sum_{k=1}^{\infty}\left(\frac{-1}{\lambda}\right)^{k}\int_0^t\left\{Sin\left(\omega(t-\tau)\right)G_{2,k,k}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau\right]$$

$$+\mu\pi f\sum_{n=1}^{\infty}\frac{J_1(R_1\, r_n)J_1(R_2\, r_n)\left(r_n\bar{B}_1(r\, r_n)-\left(\frac{1}{r}\right)B_1(r\, r_n)\right)}{J_1^2(R_1\, r_n)-\,J_1^2(R_2\, r_n)}$$

$$\times\left\{\left[\sum_{k=1}^{\infty}\left(\frac{-1}{\lambda}\right)^k\int_0^t\left\{e^{\alpha(t-\tau)}G_{2,k,k}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau\right]\right.$$

$$+\left(\frac{\frac{\delta\beta_0^2}{\rho}}{\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)}\right)\times\left[\lambda^{-1}e^{\alpha(t-\tau)}E_{1,1}(-\lambda^{-1}\ \tau)\right.$$

$$\left.+\sum_{k=1}^{\infty}\left(\frac{-1}{\lambda}\right)^k\int_0^t\left\{e^{\alpha(t-\tau)}G_{2,k,k}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau\right] \qquad (34)$$

If we Making the limit of Eq.(26) and Eq.(31) when $\alpha\to 1$, $\delta\to 0$ and $\beta_0\to 0$,we get similar solution velocity field and shear stress for ordinary Maxwell fluids, as obtained by Liancun and zhang[15].

7.2 *The case* $\frac{\partial p}{\partial\theta}=-\rho\, p_0\, Sin(\omega t)$

In special case Making the limit of Eq.(27) and Eq.(32) when $\alpha\to 1$, we get the solution of velocity field and shear stress reduces to

$$w=\frac{R_1(R_2^2-r^2)f}{(R_2^2-R_1^2)r}e^{\alpha t}-\frac{\pi^2 p_0}{2}\times\sum_{n=1}^{\infty}\frac{r_n\, J_1^2(R_1\, r_n)B_1(r\, r_n)\left(\bar{B}_1(R_1\, r_n)-\bar{B}_1(R_2\, r_n)\right)}{J_1^2(R_1\, r_n)-\,J_1^2(R_2\, r_n)}$$

$$\times\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^{k+1}\int_0^t\left\{Sin\big(\omega(t-\tau)\big)G_{2,k,k+1}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau+\frac{\pi^2 p_0\,\lambda\,\omega}{2}$$

$$\times\sum_{n=1}^{\infty}\frac{r_n\, J_1^2(R_1\, r_n)B_1(r\, r_n)\left(\bar{B}_1(R_1\, r_n)-\bar{B}_1(R_2\, r_n)\right)}{J_1^2(R_1\, r_n)-\,J_1^2(R_2\, r_n)}\times$$

$$\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^{k+1}\int_0^t\left\{Cos\big(\omega(t-\tau)\big)G_{2,k,k+1}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau$$

$$+\,\pi f\sum_{n=1}^{\infty}\frac{J_1(R_1\, r_n)J_1(R_2\, r_n)B_1(r\, r_n)}{J_1^2(R_1\, r_n)-\,J_1^2(R_2\, r_n)}$$

$$\times\sum_{k=0}^{\infty}\left(\frac{-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)}{\lambda}\right)^{k+1}\int_0^t\left\{e^{\alpha(t-\tau)}G_{1,-k,k}(-\lambda^{-1},\tau)\right\}d\tau$$

$$-\,\pi f\sum_{n=1}^{\infty}\frac{J_1(R_1\, r_n)J_1(R_2\, r_n)B_1(r\, r_n)}{J_1^2(R_1\, r_n)-\,J_1^2(R_2\, r_n)}\times\sum_{k=0}^{\infty}\left(\frac{-1}{\lambda}\right)^{k+1}\times\frac{\delta\beta_0^2}{\rho}$$

$$\times\int_0^t\left\{e^{\alpha(t-\tau)}G_{2,k,k+1}\left(-\upsilon\left(r_n^2+\frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1},\tau\right)\right\}d\tau \qquad (35)$$

$$\tau = \frac{-2\mu R_1 R_2^2}{(R_2^2 - R_1^2) r}\left[e^{at} - \int_0^t \{e^{\alpha(t-\tau)} G_{1,-k,k}(-\lambda^{-1}, \tau)\} d\tau\right] + \frac{\mu \pi^2 p_0}{2}$$

$$\times \sum_{n=1}^{\infty} \frac{r_n J_1^2(R_1 r_n)(r_n \bar{B}_1(r r_n) - \left(\frac{1}{r}\right) B_1(r r_n))(\bar{B}_1(R_1 r_n) - \bar{B}_1(R_2 r_n))}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} \times \left(\frac{1}{\upsilon\left(r_n^2 + \frac{\delta\beta_0^2}{\rho}\right)}\right)$$

$$\times \left[\lambda^{-1} \int_0^t \{Sin(\omega(t-\tau)) E_{1,1}(-\lambda^{-1} \tau)\} d\tau\right.$$

$$+ \sum_{k=1}^{\infty} \left(\frac{-1}{\lambda}\right)^k \int_0^t \left\{Sin(\omega(t-\tau)) G_{2,k,k+1}\left(-\upsilon\left(r_n^2 + \frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1}, \tau\right)\right\} d\tau$$

$$+ \frac{\mu\pi^2 p_0 \lambda \omega}{2} \times \sum_{n=1}^{\infty} \frac{r_n J_1^2(R_1 r_n)(r_n \bar{B}_1(r r_n) - \left(\frac{1}{r}\right) B_1(r r_n))(\bar{B}_1(R_1 r_n) - \bar{B}_1(R_2 r_n))}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)} \times$$

$$\left(\frac{1}{\upsilon\left(r_n^2 + \frac{\delta\beta_0^2}{\rho}\right)}\right) \times \left[\lambda^{-1} \int_0^t \{Cos(\omega(t-\tau)) E_{1,1}(-\lambda^{-1} \tau)\} d\tau\right.$$

$$\left. + \sum_{k=1}^{\infty} \left(\frac{-1}{\lambda}\right)^k \int_0^t \left\{Cos(\omega(t-\tau)) G_{2,k,k}\left(-\upsilon\left(r_n^2 + \frac{\delta\beta_0^2}{\rho}\right)\lambda^{-1}, \tau\right)\right\} d\tau\right]$$

$$+ \mu\pi f \sum_{n=1}^{\infty} \frac{J_1(R_1 r_n) J_1(R_2 r_n)\left(r_n \bar{B}_1(r r_n) - \left(\frac{1}{r}\right) B_1(r r_n)\right)}{J_1^2(R_1 r_n) - J_1^2(R_2 r_n)}$$

$$\times \left\{\left[\sum_{k=1}^{\infty} \left(\frac{-1}{\lambda}\right)^k \int_0^t \left\{e^{\alpha(t-\tau)} G_{2,k,k}\left(-\upsilon\left(r_n^2 + \frac{\delta\beta_0^2}{\rho}\right) \lambda^{-1}, \tau\right)\right\} d\tau\right]\right.$$

$$+ \left(\frac{\frac{\delta\beta_0^2}{\rho}}{\upsilon\left(r_n^2 + \frac{\delta\beta_0^2}{\rho}\right)}\right) \times [\lambda^{-1} e^{\alpha(t-\tau)} E_{1,1}(-\lambda^{-1} \tau)$$

$$\left. + \sum_{k=1}^{\infty} \left(\frac{-1}{\lambda}\right)^k \int_0^t \left\{e^{\alpha(t-\tau)} G_{2,k,k}\left(-\upsilon\left(r_n^2 + \frac{\delta\beta_0^2}{\rho}\right) \lambda^{-1}, \tau\right)\right\} d\tau\right] \quad (36)$$

if we Making the limit of Eq.(27) and Eq.(32) when $\alpha \to 1$, $\delta \to 0$ and $\beta_0 \to 0$,we get similar solution velocity field and shear stress for ordinary Maxwell fluids, as obtained by Liancun and zhang[15].

8. Numerical results and conclusions

In this paper ,we established the effect of magnetic field on the unsteady rotating flow of a generalized Maxwell through two infinite straight circular cylinders , the exact solution for velocity field and shear stress are obtained by using Hankel and Laplace transforms .The solution that have been determined ,written under integral and series from in terms of generalized G-function and Mittag –leffler function .Some figures are plotted to show the behavior of various parameter involved in the expressions of velocity field w and shear stress $\boldsymbol{\tau}$.

In the case $\alpha \to 1$,we have obtained the solution for ordinary and generalized Maxwell fluids.

The velocity and shear stresses are plotted about the case $\frac{\partial p}{\partial \theta} = -\rho\, p_0 \cos(\omega t)$ by using Mathematica package .Fig (1) and (2) are sketched to show the velocity field of Maxwell model with fractional derivative at different value of time and magnetic field . From these figures, it is obvious that velocity decreasing as the time decreases .When r approaches to R_1 , the velocity is decrease with increase the time . Fig (3) and (4) are depicted to show the changes of the velocity with fractional parameter α and magnetic field .The velocity decreases with increasing of the parameter α . Fig (5) displays the behavior of parameters λ with magnetic field . It is observed that the velocity is increase with increase of the parameter λ. Fig (6) are prepared to show the effect of the kinematic viscosity on the velocity field with magnetic field .The velocity is decrease with increase υ . Fig (7) is established to show the behavior of different value of time with λ=8 , $\alpha \to 1$ and magnetic field .The velocity field is decreases with increasing of time. Fig (8) provides the graphical illustration for the effect of the different values of r with magnetic field. Fig(1-7) ,we noted that for fixed instant at time the velocity first increases and start to decrease ,it has maximum in the radius range of (0.75-0.85) in this profile ,the pressure gradient leads to fluctuations of the velocity .It is notes that the fluctuations is big as r various from 0.7 to 0.8 and become small as r various from 0.8 to 0.95 . This is the case for both ordinary and generalized Maxwell models . Fig (9-12)shows the changes of shear stress for different values of (time t , fractional parameter α , kinematic viscosity υ) with magnetic field . the results indicated that the shear stress increase rapidly with increase of different parameters. Fig (13)and (16) provide the graphically illustration for shear stress with parameter λ and for different value of r with magnetic field respectively . Fig (14) is established to shows the effect of different value of kinematic viscosity υ with magnetic field on the shear stress .

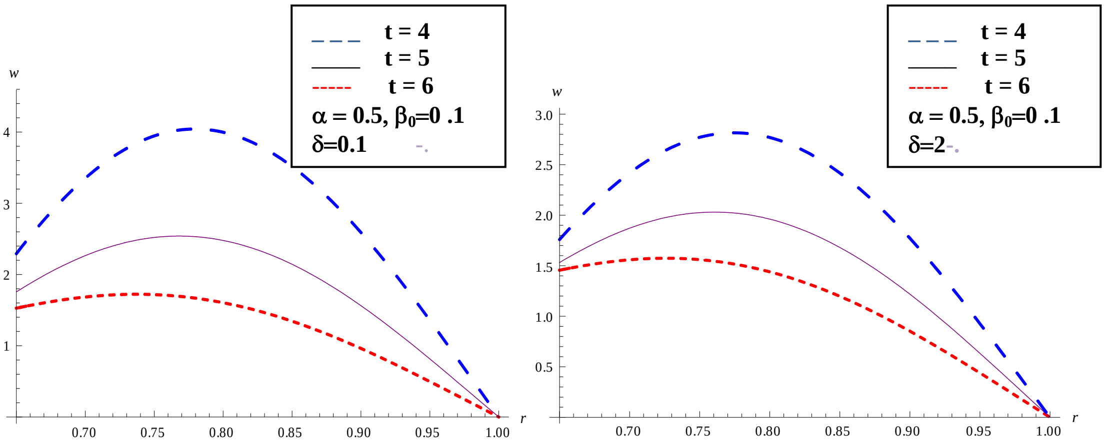

Fig .1. the velocity for different value of time t : λ=10 , υ=0.2, R_1=0.5 ,R_2=1 , a=-0.2 , α=0.5 , ω=0.1 , po=2 , ρ=0.02 , m_1=7.8999 , f=4

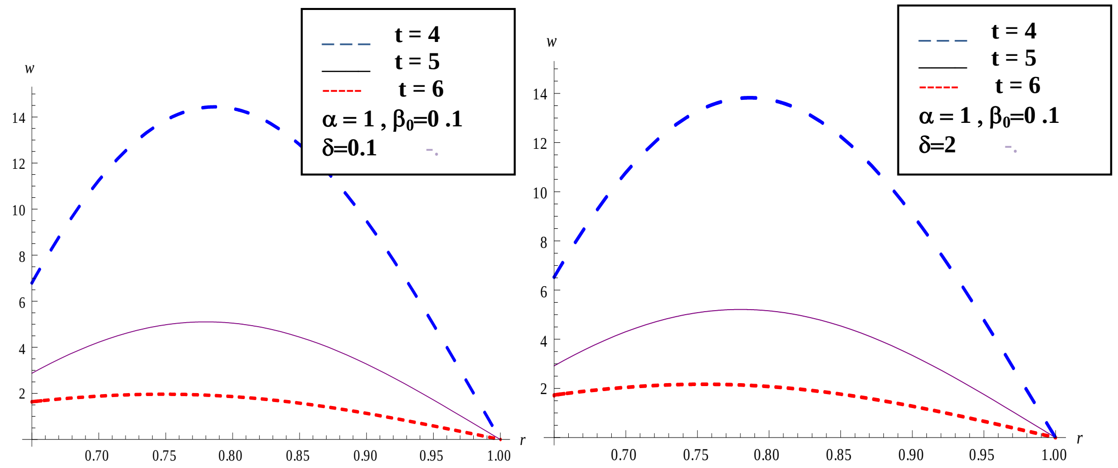

Fig .2. the velocity for different value of time t : λ=10 , R_1=0.5 ,R_2=1, υ=0.2 , a=-0.2 , α=0.5 , ω=0.1 , po=2 , ρ=0.02 , m_1=7.8999 , f=4

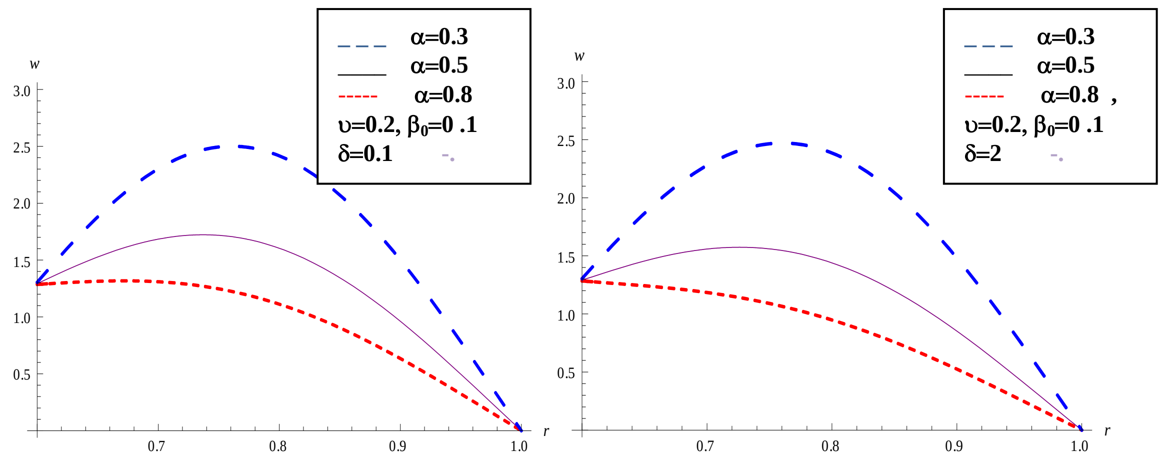

Fig .3. the velocity for different value of fractional parameter α : λ=10 , R_1=0.5 ,R_2=1 , a=-0.2 , ω=0.1 , po=2 , ρ=0.02 , m_1=7.8999 , f=4

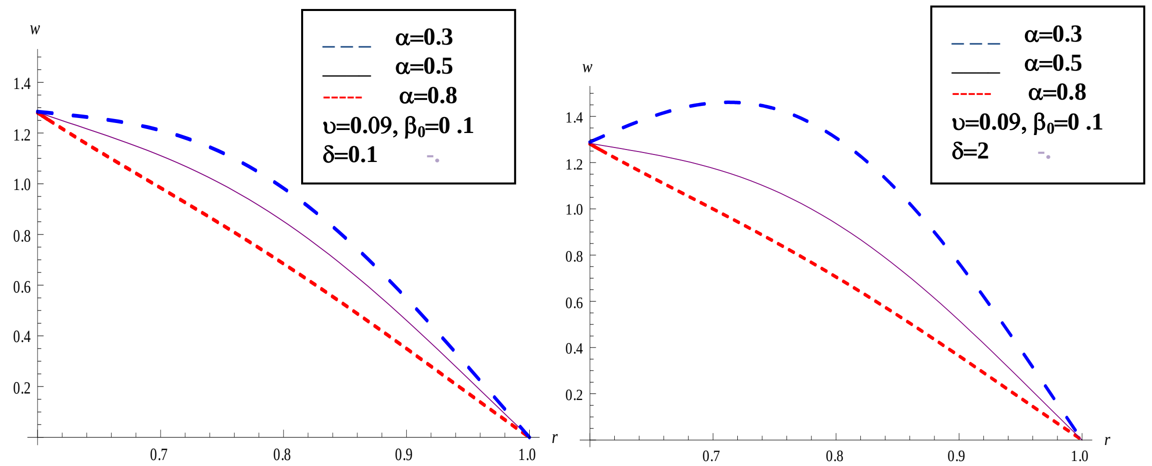

Fig .4. the velocity for different value of fractional parameter α : λ=10 , R_1=0.5 ,R_2=1 , a=-0.2 , ω=0.1 , po=2 , ρ=0.02 , m_1=7.8999 , f=4

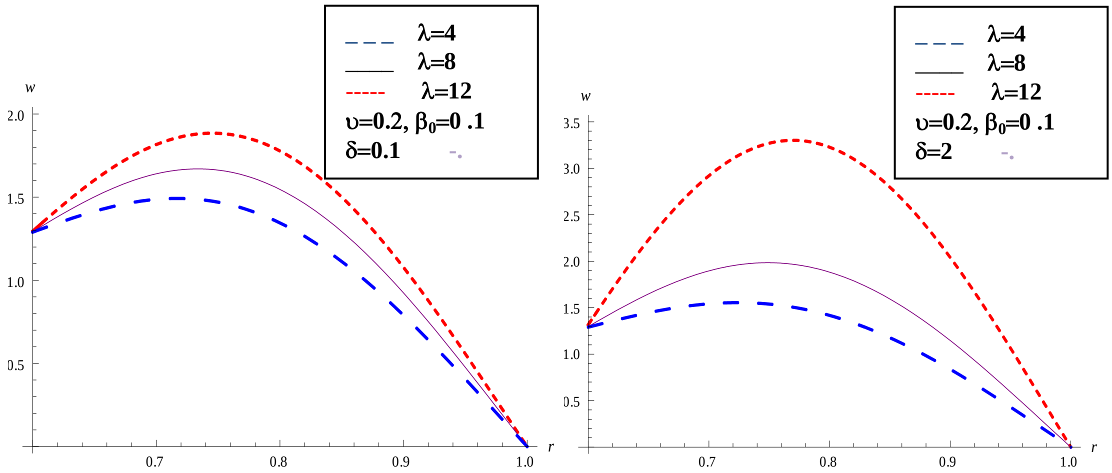

Fig .5. the velocity for different value of λ : $\upsilon \to 0.2$, $R_1=0.5$, $R_2=1$, a=-0.2 , t=4 , ω=0.1 , po=2 , ρ=0.02 , $\alpha \to 0.5$, $m_1=7.8999$, f=4 .

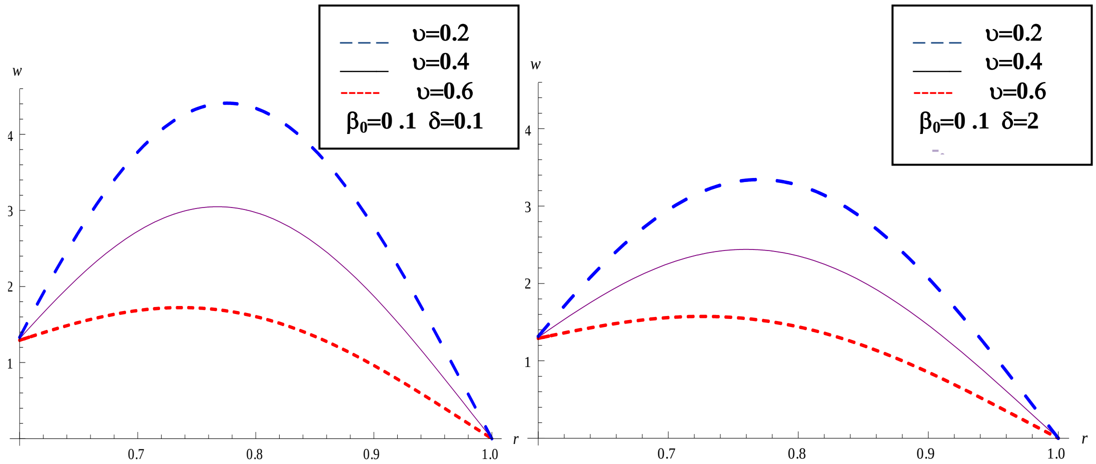

Fig .6. the velocity for different value of kinematic viscosity υ : λ =10 , $R_1=0.5$, $R_2=1$, a=-0.2 , t=4 , ω=0.1 , po=2 , ρ=0.02 , $\alpha \to 0.5$, $m_1=7.8999$, f=4 .

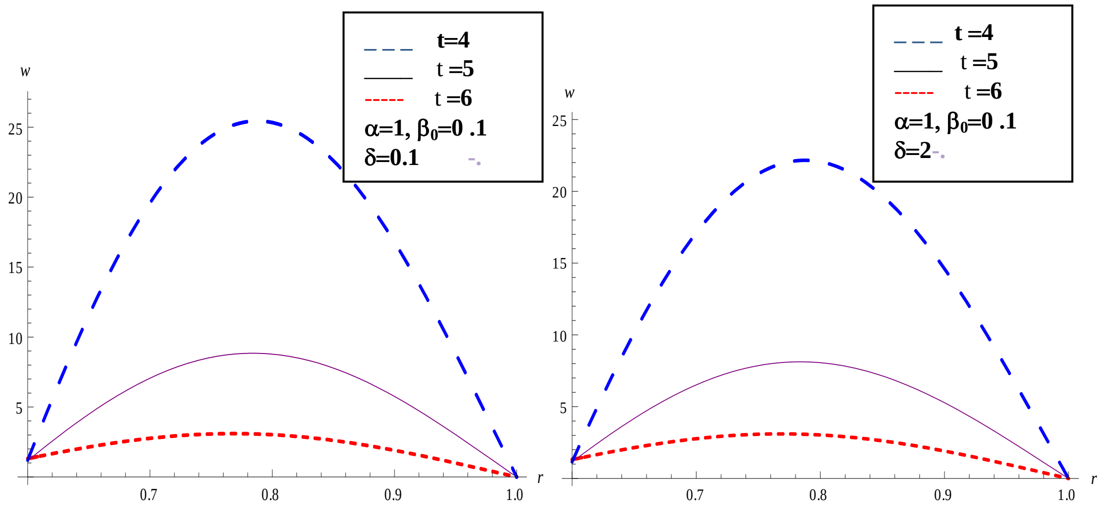

Fig .7. the velocity for different value of time t : λ =8 , $R_1=0.5$, $R_2=1$, a=-0.2 , t=4 , ω=0.1 , po=2 , ρ=0.02 , υ=0.2 , $m_1=7.8999$, f=4 .

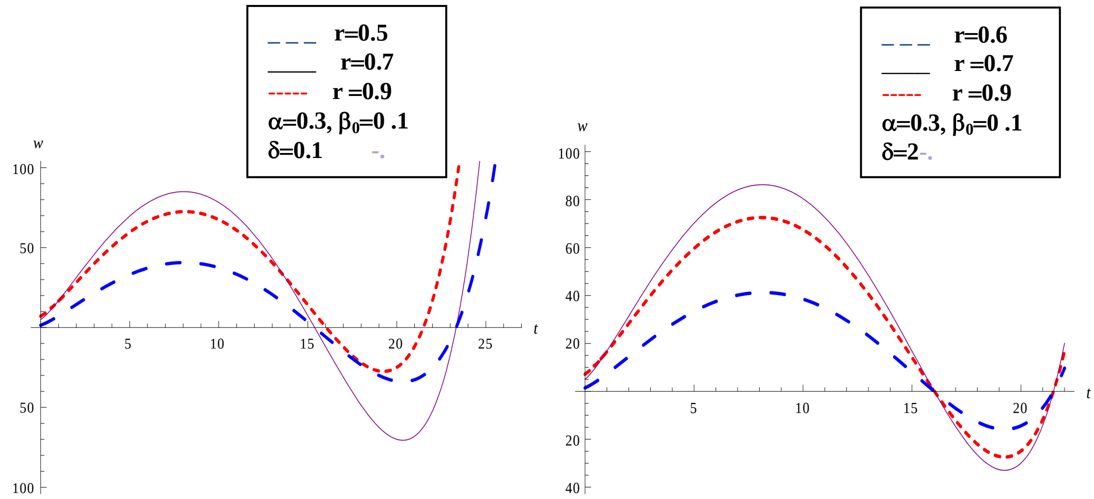

Fig .8. the velocity for different value of r : λ =15 , R_1=0.5 , R_2=1 , a=-0.2, ω=0.1 , po=10 , ρ=0.02 , υ=0.03 , m_1=5.79 , f=10 .

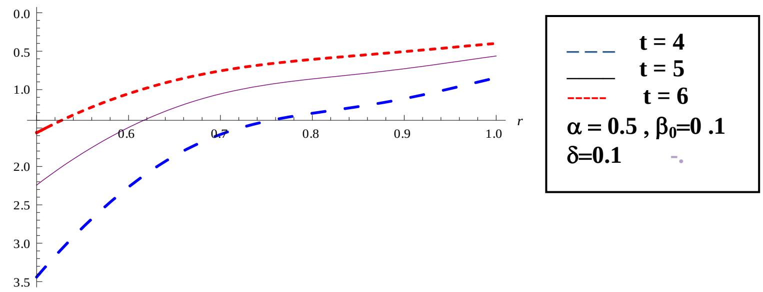

Fig.9. the shear stress for different value of time t : λ=10 , υ=0.2, R_1=0.5 ,R_2=1 , a=-0.2 , ω=0.1 , po=2 , ρ=0.02 , m_1=7.8999 , f=4

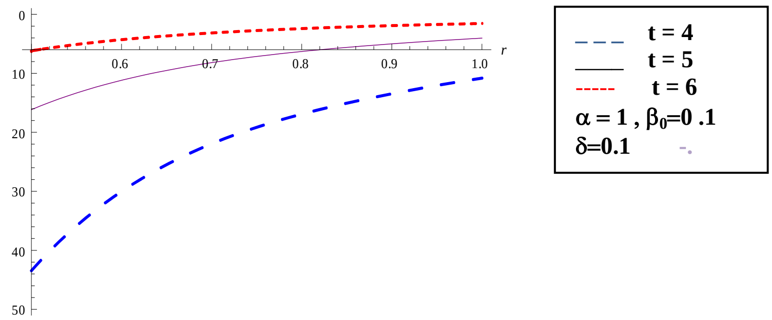

Fig .10. the shear stress for different value of time t : λ=10 , R_1=0.5 ,R_2=1, υ=0.2 , a=-0.2 , α=0.5 , ω=0.1 , po=2 , ρ=0.02 , m_1=7.8999 , f=4

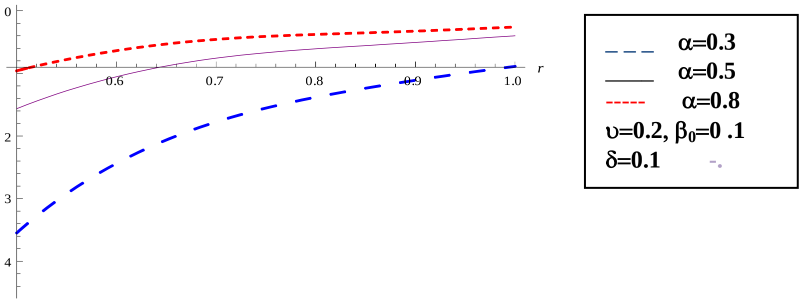

Fig 11 the shear stress for different value of fractional parameter α : λ=10 , R_1=0.5 ,R_2=1 , a=-0.2 , ω=0.1 , po=2 , ρ=0.02 , m_1=7.8999 , f=4

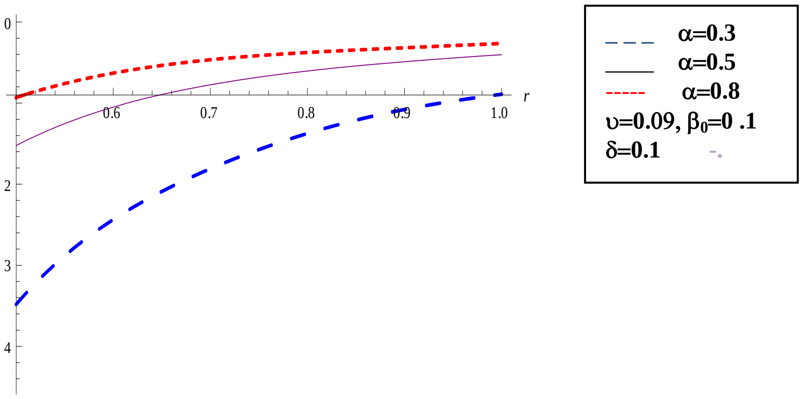

Fig .12 .the shear stress for different value of fractional parameter α : λ=10 , R_1=0.5 ,R_2=1 , a=-0.2 , ω=0.1 , po=2 , ρ=0.02 , m_1=7.8999 , f=4

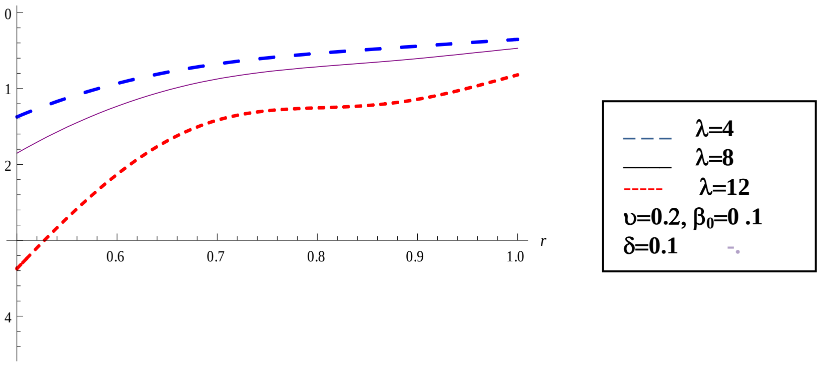

Fig .13. the shear stress for different value of λ : υ→0.2 , R_1=0.5 , R_2=1 , a=-0.2 , t=4 , ω=0.1 , po=2 , ρ=0.02 , α→0.5 , m_1=7.8999 , f=4 .

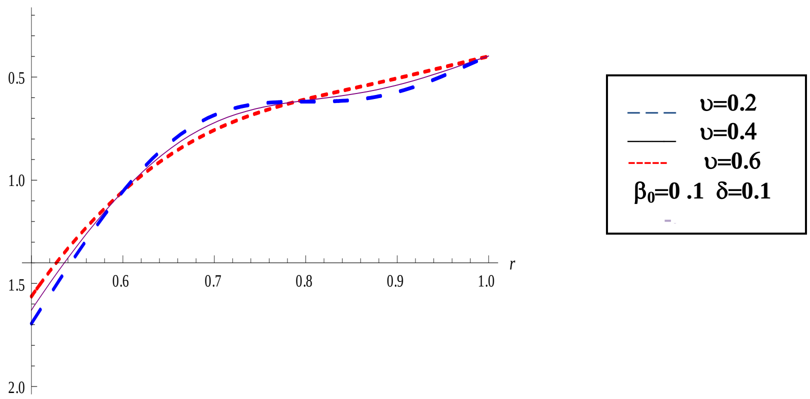

Fig .14. the shear stress for different value of kinematic viscosity υ : λ =10 , R_1=0.5 , R_2=1 , a=-0.2 , t=4 , ω=0.1 , po=2 , ρ=0.02 , α→0.5 , m_1=7.8999 , f=4 .

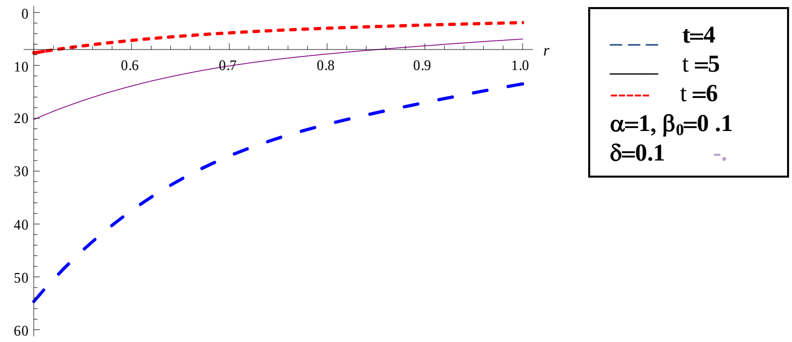

Fig .15. the shear stress for different value of time t : λ =8 , R_1=0.5 , R_2=1 , a=-0.2 , t=4 , ω=0.1 , po=2 , ρ=0.02 , υ=0.2 , m_1=7.8999 , f=4 .

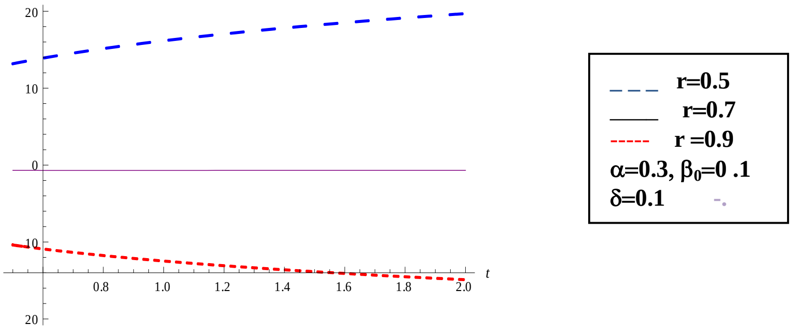

Fig .16. the velocity for different value of r : λ =15 , R_1=0.5 , R_2=1 , a=-0.2, ω=0.1 , po=10 , ρ=0.02 , υ=0.03 , m_1=5.79 , f=10 .

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