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ON STABILITY AND SOME STATISTICAL PROPERTIES TO THE DYNAMIC SOLUTIONS OF DAMPED WAVE EQUATION

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Abstract

in this paper we discuss and study stability of damped wave equation take the formula

$$u_{tt} = c^2u_{xx} - \gamma u_t \qquad (1)$$

using the concept of " Lypunove stability" , and we study some statistical properties to explain its dynamic behavior after solving it using separation of variable and turn it into a probability density function (pdf) in order to calculate the first and second moment and then the variance.

Key words asymptoticallystable, dynamic solution , statistical property , damped wave equation.

1.Introduction

Wave equation is one of the most important and popular partial differential equations (pde),it describe many phenomena in physics such as sound wave ,light wave,heat wave,water wave ….etc.

In [1] Talha,A. and Khaled,O. solve the non linear damped generalized regularized long-wave using Adomain decomposition method , in [2] Yener studied the damped wave equation with special class of boundary conditions and he solve it by Laplace transformation , in [3] Zhigang and Zhilin and Tia present and work on global solution to a class of non linear wave operator equations , in[4] Noor studied in her Msc. research " Some statistical properties of the solution of some stochastic Fredholm integral equation contains Gamma process " , in [5] Lu studied in his Phd. Research the application of micro local analyses in δ-evalution equations , in [6] Fibich and Helfand studied statistical properties of wave in random medium , in [7] Vajiac and Tolosa give us introduction in (pde) in the under graduate curriculum , in[8] page 160 Zauderer introduce stability conditions for (pde) depending on Lypunove stability concept in his book " Partial differential equation of applied mathematics" , in [9] we depend on two definitions to study the stability of equation (1) .

In this research we improve the study in [4]and [9] to cover the works over (pde) and introduce damped wave equation in (1) as an example to our study.

2. Stability of damped wave equation

In this section we will analyze stability of equa. (1) giving two conditions under which the analytic solution of (1) is asymptotically stable . A damped wave equation of length one in (1) can be solved in many methods, in our work we use separation method , let us rewrite our problem

$u_{tt}=c^2u_{xx}-\gamma u_t$ (1)

with initial condition

$u_0(x,t)=u(x,0)=x(1-x)$ (2)

and initial velocity

and the boundary conditions end points are fixed $u(0,t)=u(1,t)=0$

if we let $u(x,t)=X(x)T(t)$ then

$X(x)T''(t) = c^2 X''(x)T(t)-\gamma X(x)T'(t)$ (4)

Divide (4) by $X(x)T(t)$ we get

$\frac{X''(x)}{X(x)}=\lambda$, $X(0)=X(1)=0$ (5)

using the characteristic equation $m^2-\lambda=0$ (6)

where λ is an eigenvalue then $m_{1,2}=\mp\sqrt{\lambda}$ is the solution of (6)

Case 1 : If $\lambda<0$ then $X(x)=c_1\cos(\sqrt{-\lambda}\,x)+c_2\sin(\sqrt{-\lambda}\,x)$ (7)

$X(0)=0 \mapsto c_1=0$

$X(1)=0 \mapsto c_2\sin(\sqrt{-\lambda}\,x)=0$

So we get $\sqrt{-\lambda}=n\pi$ and $\lambda_n=-n^2\pi^2$ for $n=0,\mp1,\mp2,\ldots$.

Case 2 : If $\lambda=0$ then $X(x)=c_1+c_2x$ (8)

$X(0)=0 \mapsto c_1=0$

$X(1)=0 \mapsto c_2=0$

There is no eigenvalue .

Case 3 : If $\lambda>0$ then $X(x)=c_1e^{\sqrt{\lambda}x}+c_2e^{-\sqrt{\lambda}x}$ (9)

The boundary conditions $X(0)=X(1)=0 \mapsto c_1=c_2=0$

There is no eigenvalue .

From the above cases only case 1 give us $-n^2\pi^2$ periodic eigenvalue , we solve (1) using separable value and Fourier series to get the solution

$$u(x,t) = \sum_{n=1}^{\infty} \sin \sqrt{\lambda}x\, e^{-pt}\left(an \cos\left(\sqrt{p^2+\lambda c^2}\right) t + bn \sin\left(\sqrt{p^2+\lambda c^2}\right)t \right) \qquad (10)$$

where $p=\frac{\gamma}{2}$

$an = \frac{2}{\pi}\int_0^{\pi} x(1-x)\sin nx\, dx$

$bn = \frac{p\, an}{\sqrt{p^2+\lambda c^2}}$

In [8] Zauderer refers that as following :

Definition 1 (pde) of wave equation is stable depending on eignvalues and classified into three kinds

Strictly stable if $\lambda<0$, *unstable* if $\lambda>0$, *neutrally stable* if $\lambda=0$.

Corollary 1 *the damped wave equation (1) is strictly stable .*

Proof :

From case 1 in the above solution and definition 1 *strictly stable* appear i.e. equa. (1) is *strictly stable*.

In the other hand from [7] our work take two definitions and apply it on (1) so we have

Definition 2 solution of equation (1) is a function u(x,t) satisfies the conditions :

i- u(x,t) is continuous in $\mathcal{B}=[0,1]\times[0,\infty]$

ii- the partial derivativs u_t , u_x , u_{xx} , u_{tt} exist and are continuous in $\mathcal{B}$ with the possible exception of the points (x,n) (n=0,1,2,…) where one – sided derivatives exist.

iii- u(x,t) satisfies (1) , t>0 , in $\mathcal{B}$ with boundary conditions u(o,t)=u(1,t)=0 and initial velocity $u_t(x,0)=0=g(x)$, and initial conditions $u_0(x,t)=u(x,0)=x(1-x)=f(x)$.

Definition 3 if any solution u(x,t) of (1) satisfies $\lim_{t\to\infty} u(x,t) = 0$, $x \in [0,1]$ then te zero solution of (1) is called *asymptotically stable* .

Corollary 2 *the damped wave equation (1) is asymptotically stable.*

Proof :

from

(10) $\lim_{t\to\infty} u(x,t) =$

$$\lim_{t\to\infty} \sum_{n=1}^{\infty} \sin\sqrt{\lambda}x\, e^{-pt}\left(an\cos\left(\sqrt{p^2+\lambda c^2}\right)t + bn\sin(\sqrt{p^2+\lambda c^2})t\right. = 0$$

From def. 3 the equa. (1) is *asymptotically stable* for all $x \in [0,1]$.

3.Stochastic solution of damped wave equation

In general the subject of stochastic equation is one of the most useful tools in booth pure and applied mathematics , it has enormous in many physical and engineering and technology problems[4],many initial and boundary value problems associated with stochastic partial differential equations can be transformed into problems of solving stochastic partial differential equation .

In this paper we are interesting to define the stochastic solution of the damped wave equation in (1) .

In order to find stochastic solution of u(x,t) in (10) over the intervals 0<x<1 and 0<t<1 , in [4] Noor use method we apply it in this research on equ. (1) , so the stochastic solution is a (pdf) of damped wave equation in order to get it we multiply this stochastic solution by **B** which makes the solution as a (pdf)of wave equation , i.e. , so we write :

$\int_0^1 B\, u(x,t)dx = 1$ and let $\emptyset(x,t) = B\, u(x,t)$ now $\emptyset(x,t)$ is the (pdf) of the wave equation as the following steps :

$$u(x,t) = \sum_{n=1}^{\infty} \sin\sqrt{\lambda}x\, e^{-pt}\left(an\cos\left(\sqrt{p^2+\lambda c^2}\right)t + bn\sin(\sqrt{p^2+\lambda c^2})t\right)$$

where

$$a_n = \frac{2}{\pi}\int_0^{\pi} x(1-x)\sin nx\, dx$$

$$b_n = \frac{p\, an}{\sqrt{p^2+\lambda c^2}}$$

by computing the above integration we get

$$a_n = \left(\frac{2}{n} - \frac{2\pi}{n} + \frac{8}{n^2\pi}\right)(-1)^n$$

$$b_n = \frac{p\, an}{\sqrt{p^2+\lambda c^2}} \qquad for \qquad -\lambda = n^2\pi^2$$

we will *calculate the stochastic solution for many value of* ***n*** *and depend on the results appears to generalize it for any* ***n,*** *depending on the* *<u>concept of uniformly continuous property</u>* .[9]

If n=1 :

$a_1 = \frac{-2\pi+2\pi^2+8}{\pi}$, $b_1 = \frac{a1\,p}{\sqrt{p^2-c^2\pi^2}}$

$$u_1(x,t) = \sin\pi x\, e^{-pt}\left(a_1\cos(\sqrt{p^2+\pi^2c^2}\,t\right) + b_1\sin(\sqrt{p^2+\pi^2c^2}\,t))$$

$u_1(x,t) = k_1\sin\pi x + k_2\sin\pi x = (k_1+k_2)\sin\pi x$,

where $k_1 = e^{-pt}(a_1\cos(\sqrt{p^2+\pi^2c^2}\,t))$, $k_{2=}e^{-pt}(b_1\sin(\sqrt{p^2+\pi^2c^2}\,t))$

$\int_0^1 B\,u_1(x,t)dx = 1 \Rightarrow$ B $=\frac{1}{\int_0^1 u_{1(x,t)dx}} = \frac{\pi}{2k_1+2k_2}$

The (pdf) for n =1 is $\emptyset_1(x,t) = Bu_1(x,t) = \frac{\pi}{2}\sin\pi x$

The first moment (mean) is $E(\emptyset_1(x,t),x) = \mathcal{M}_1 = \frac{\pi}{2}\int_0^1 x\sin\pi x\;dx = \frac{1}{2}$

The second moment is $E(\emptyset_1(x,t),x^2) = \frac{\pi}{2}\int_0^1 x^2\sin\pi x\,dx = \frac{1}{2} - \frac{2}{\pi^2} = \mathcal{M}_{11}$

If n=2

$a_2 = \frac{\pi-\pi^2-2}{\pi}$, $b_2 = \frac{a_2p}{\sqrt{p^2-4\pi^2c^2}}$

$u_2(x,t)$= $u_1(x,t)$ $+sin2\pi x(e^{-pt}a_2\cos(\sqrt{p^2-4\pi^2c^2}$ t$)+e^{-pt}b_2\sin(\sqrt{p^2-4\pi^2c^2}$ t))

$u_2(x,t) = \sin\pi x\,(k_1+k_2) + sin2\pi x(e^{-pt}a_2\cos(\sqrt{p^2-4\pi^2c^2}$
t$)+e^{-pt}b_2\sin(\sqrt{p^2-4\pi^2c^2}$ t))

Define K_3 and K_4 as :

$K_3 = e^{-pt}a_2\cos(\sqrt{p^2-4\pi^2c^2}$ t

$K_4 = e^{-pt}b_2\sin(\sqrt{p^2-4\pi^2c^2}$ t

So $u_2(x,t) = \sin\pi x(k_1+k_2) + sin2\pi x$ (K_3+K_4)

B$= \frac{1}{\int_0^1 u_{2(x,t)dx}} = \frac{\pi}{2(K_1+K_{2)}}$

$\emptyset_2(x,t) = Bu_2(x,t)$ now we will calculate first and second moment for n = 2

First moment (mean) $\mathcal{M}_2 = \int_0^1 \emptyset_2(x,t)\,xdx = \mathcal{M}_1 + B\int_0^1(K_3+K_4)x\,sin\;2x\;dx$

$\mathcal{M}_2 = \frac{1}{2} - \frac{(K_3+K_4)}{4(K_1+K_2)}$

Second moment $\mathcal{M}_{22} = \int_0^1 \emptyset_2(x,t)x^2dx = \mathcal{M}_{11} + B\int_0^1(K_3+K_4)x^2sin2xdx$

$\mathcal{M}_{22} = \mathcal{M}_{11} + \left(\frac{(K_3+K_4)}{(K_1+K_2)}\right)\left(\frac{1-\pi^2}{8\pi^2}\right)$

If n=3

$u3(x,t) = u_2(x,t) + \sin 3\pi x\,(K_5+K_6)$

$K_5 = e^{-pt} a_3 \cos(\sqrt{p^2-25\pi^2c^2}\,t)$

$K6 = e^{-pt}\, b_3 \sin(\sqrt{p^2-25\pi^2c^2}\,t)$

$B = \frac{3\pi}{6(K_1+K_2)-2(K_5+K_6)}$, $\emptyset_3(x,t) = B\,u_3(x,t)$

First moment (mean) for n =3 is $\mathcal{M}_3 = \int_0^1 \emptyset_3(x,t)x\,dx = \mathcal{M}_2 + B\int_0^1 (K_5+K_6)x\sin 3\pi x\,dx$

$\mathcal{M}_3 = \frac{6(K_1+K_2)-3(K_3+K_4)+2(K_5+K_6)}{12(K_1+K_2)-4(K_5+K_6)}$

$\mathcal{M}_{33} = \mathcal{M}_{22} + B\int_0^1 (K_5+K_6)x^2\sin 3\pi x\,dx$

$\mathcal{M}_{33} = \frac{3\pi}{6(K_1+K_2)-2(K_5+K_6)}\left((K_1+K_2)\left(\frac{\pi^2-4}{\pi^3}\right) + (K_3+K_4)\left(\frac{-4\pi^2+2}{8\pi^3}\right) + \right.$

$\left.(K_5+K_6)\left(\frac{9\pi^2-2}{27\pi^3}\right)\right)$

If n=4 in the same way of above we find

The first moment (mean) $\mathcal{M}_4 = \left(\frac{3\pi}{6(K_1+K_2)-2(K_5+K_6)}\right)\left(\frac{K_1+K_2}{\pi} - \frac{K_3+K_4}{2\pi} + \frac{K_5+K_6}{3\pi} - \frac{K_7+K_8}{4\pi}\right)$

Second moment $\mathcal{M}_{44} = \left(\frac{3\pi}{6(K_1+K_2)-2(K_5+K_6)}\right)\left((K_1+K_2)\left(\frac{\pi^2-4}{\pi^3}\right) + (K_3+K_4)\left(\frac{-4\pi^2+2}{8\pi^3}\right) + (K_5+K_6)\left(\frac{9\pi^2-2}{27\pi^3}\right) + (K_7+K_8)\left(\frac{-8\pi^2+1}{32\pi^3}\right)\right)$

And so on for any value of n .

4.Stochastic Variance

In probability theory and statistics, **variance** measures how far a set of numbers is spread out. A variance of zero indicates that all the values are identical. Variance is always non-negative: a small variance indicates that the data points tend to be very close to the mean (expected value) and hence to each other, while a high variance indicates that the data points are very spread out around the mean and from each other. [10]

According to section 3 we will calculate variance as $Var(x)=\sigma^2 = E(x^2) - (E(x))^2$ = second moment – (first moment)^2 for n= 1,2,3,4,… presented in Table 1

Results

1. In fig. 1for solve the three-dimensional x, y, t note that the development of natural water wave limit **t** = 0.8 and that the tremor got at **t** = 0.95 Surely a sharp fall of the wave will get as shown, followed by a large wave height at **t** = 0.975 and then landing and this is normal behavior for surface prompt influenced by the movement of the wind.
2. In table 1 we neglecting the negative values of the variance and took less positive value at every interview column for the time which got him When n = 1, we find that less variation when t = 0, and when n = 2 neglected, and when n = 3 less variation appear when t = 4 and finally when n = 4 less than the value of the variance appear when t = 0, and so if we continue snapped again for the values of n .....
3. The dynamic of the solution results controlled and differ according to the parameters ***c*** and ***p*** in equation (1) , which we call it free parameters see fig. 2
   Whene $0<x<2$ and $0<t<2$, $p=0.15, c=0.2$.

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t	$\mathcal{M}_1$	$\mathcal{M}_{11}$	$\sigma^2{}_1$	$\mathcal{M}_2$	$\mathcal{M}_{22}$	$\sigma^2{}_2$	$\mathcal{M}_3$	$\mathcal{M}_{33}$	$\sigma^2{}_3$	$\mathcal{M}_4$	$\mathcal{M}_{44}$	$\sigma^2{}_4$
0	.5	2.0309	1.7809	5.2436	15.1018	-12.3954	-0.3557	9.8664	9.7398	0.0580	0.4645	0.461
.1	.5	2.0556	1.8056	5.5382	15.1265	-15.5452	-0.3557	9.6706	9.5441	0.0587	0.4758	0.472
.2	.5	2.1297	1.8797	6.4737	15.2007	-26.7081	-0.3557	8.9335	8.807	0.0589	0.5100	0.51
.3	.5	2.2541	2.0041	8.2262	15.3251	-52.3453	-0.3557	7.1375	7.011	0.0522	0.5723	0.57
.4	.5	2.4308	2.1808	11.1345	15.5018	-108.4753	-0.3557	3.1796	3.0531	0.0225	0.6751	0.674
.5	.5	2.6630	2.413	15.7339	15.7339	-231.8217	-0.3557	-5.085	-5.212	-0.0680	0.8483	0.844
.6	.5	2.9553	2.7053	23.0461	16.0262	-515.0965	-0.3557	-21.71	-21.845	-0.3028	1.1766	1.085
.7	.5	3.3135	3.0635	34.4210	16.3844	-1168.421	-0.3557	-54.32	-54.452	-0.8561	2.0212	1.288
.8	.5	3.7451	3.4951	52.1829	16.8160	-2706.239	-0.3557	-117.07	-117.19	-2.0620	9.9626	5.711
.9	.5	4.2590	4.009	79.9292	17.3300	-6371.347	-0.3557	-236.27	-236.36	-4.4649	-2.6916	-22.63
1	.5	4.8662	4.6162	123.3067	17.9372	-15186.61	-0.3557	-460.42	-460.54	-8.6052	-0.9764	-75.02

Table 1 statistical mean and variance for stochastic solution of damped wave equation (1) .

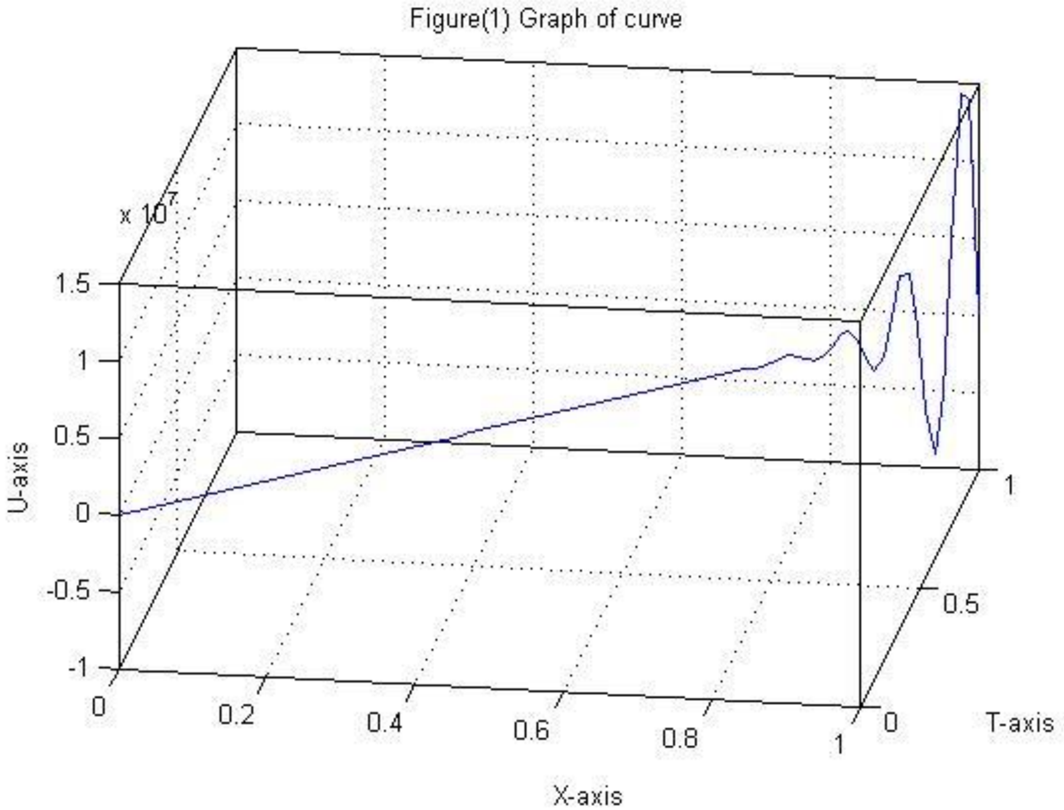

Fig. 1 dynamic solution of wave equation (1) with c= 0.1 and p= 0.5

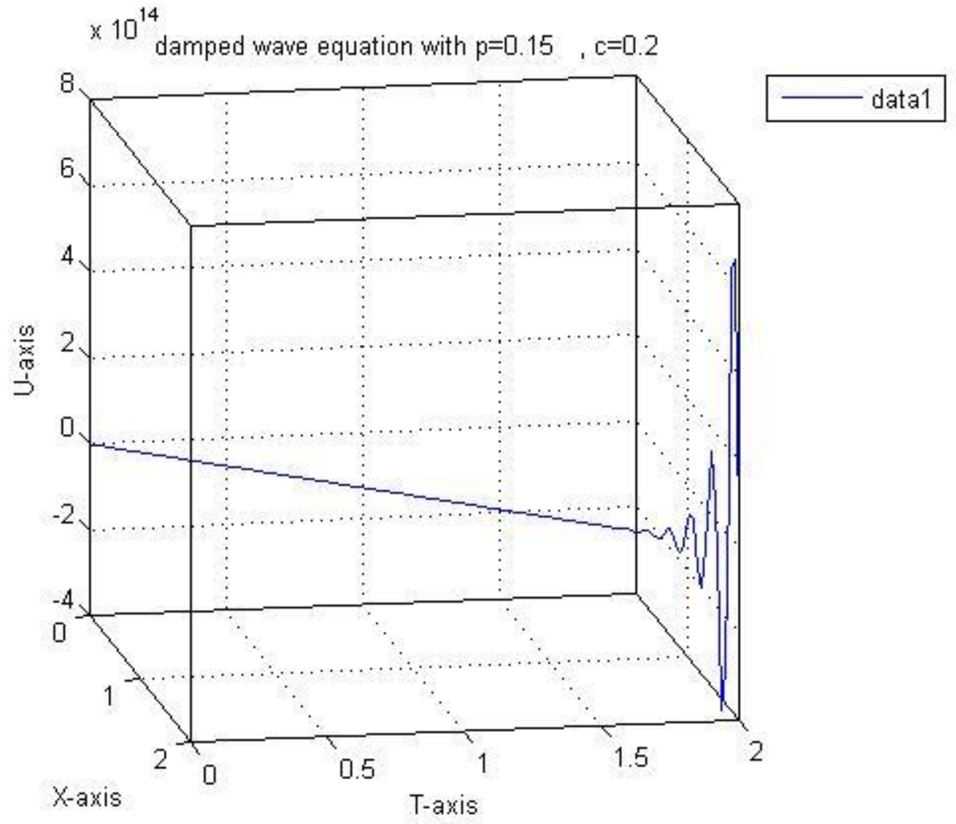

Fig. 2 dynamic solution of wave equation with p=0.15 and c=0.2 ,0<x<2 , 0<t<2 .