FIXED POINT THEOREM IN FUZZY METRIC SPACE WITH THE PROPERTY

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Abstract

The purpose of this paper is to prove a common fixed point theorem for semi-compatible and occasionally weakly compatible mappings in fuzzy metric space by using the property (E.A.) and implicit relation. Our result generalizes the result of [14].

Keywords: Fuzzy metric space, property (E.A.), semi-compatible and weakly compatible mappings.

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Introduction

The evolution of fuzzy mathematics solely rests on the notion of fuzzy sets which was introduced by Zadeh [18] in 1965 with a view to represent the vagueness in everyday life. In mathematical programming, the problems are often expressed as optimizing some goal functions equipped with specific constraints suggested by some concrete practical situations. There exist many real-life problems that consider multiple objectives, and generally, it is very difficult to get a feasible solution that brings us to the optimum of all the objective functions. Thus, a feasible method of resolving such problems is the use of fuzzy sets [16]. In fact, the richness of applications has engineered the all round development of fuzzy mathematics. Then, the study of fuzzy metric spaces has been carried out in several ways (e.g., [2, 7]). George and Veeramani [4] modified the concept of fuzzy metric space introduced by Kramosil and Michálek [8] with a view to obtain a Hausdorff topology on fuzzy metric spaces, and this has recently found very fruitful applications in quantum particle physics, particularly in connection with both string and theory (see [3]). In recent years, many authors have proved fixed point and common fixed point theorems in fuzzy metric spaces. To mention a few, we cite [1, 5, 9, 10, 12, 13, 14, 16, 17]. As patterned in Jungck [6], a metrical common fixed point theorem generally involves conditions on commutatively, continuity, completeness together with a suitable condition on containment of ranges of involved mappings by an appropriate contraction condition. Thus, research in this domain is aimed at weakening one or more of these conditions. In this paper, we observe that the notion of common property (E.A.) relatively relaxes the required containment of the range of one mapping into the range of other and hence, we obtain a common fixed point theorem in fuzzy metric space using the concepts of semi-compatible, occasionally weakly compatible and (E.A.) property with implicit relation. Our result generalizes the result of Singh et. al. [14]. For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.

Preliminaries

2.1. Definition: A binary operation *: [0, 1] × [0, 1] → [0, 1] is a continuous t-norm if * is satisfying the following conditions:

(a) * is commutative and associative;

(b) * is continuous;

(c) a * b = a for all a ∈ [0, 1];

(d) a* b ≤ c * d whenever a ≤ c and b ≤ d and a, b, c, d ∈ [0, 1].

2.2. Definition: A 3-tuple (X, M, *) is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions; for all x, y, z ∈ X, s, t > 0.

(1) M (x, y, t) > 0;

(2) M(x, y, t) =1 if and only if x = y;

(3) M(x, y, t) = M(y, x, t);

(4) M(x, y, t)* M(y, z, s) ≤ M(x, z, t + s);

(5) M(x, y, .): (0, ∞)→[0, 1] is continuous.

Then M is called a fuzzy metric on X. The function M(x, y, t) denote the degree of nearness between x and y with respect to t.

2.3. Example: Let (X, d) be a metric space. Denote $a * b = a\, b$ for a, b ∈ [0, 1] and let M_d be a fuzzy set on $X^2 \times (0, \infty)$ defined as follows: M_d (x, y, t) $= \frac{t}{t+d(x,\ y)}$

Then (X, M_d, *) is a fuzzy metric space, we call this fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric.

2.4. Definition: Let (X, M,*) be a fuzzy metric space, then

(a) A sequence $\{x_n\}$ in X is said to be convergent to x in X if for each ε > 0 and each t > 0, there exists n_0 ∈ N such that M (x_n, x, t) > 1 − ε for all n ≥ n_0.

(b) A sequence $\{x_n\}$ in X is said to be Cauchy if for each ε > 0 and each t > 0, there exist n_0 ∈ N such that M (x_n, x_m, t) > 1 − ε for all n, m ≥ n_0 .

(c) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

2.5. Proposition: In a fuzzy metric space (X, M, *), if a * a ≥ a for a ∈ [0, 1] then a * b = min {a, b} for all a, b ∈ [0, 1].

2.6. Lemma: Let (X, M, T) be an L-fuzzy metric space. Then $M(x, y, t)$ is nondecreasing with respect to t, for all $x, y \; in \; X$.

2.7. Definition: Two self-mappings A and S of a fuzzy metric space (X, M, *) are called compatible if $\lim_{n\to\infty}$ M (ASx_n, SAx_n, t) = 1 whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$ = x for some x in X.

2.8. Definition: Two self-maps A and B of a fuzzy metric space (X, M, *) are called weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Ax = Bx$ for some $x \in X$ then $ABx = BAx$.

2.9. Remark: If self-maps A and B of a fuzzy metric space (X, M, *) are compatible then they are weakly compatible. Let (X, M,*) be a fuzzy metric space with the following condition: (6) $\lim_{t\to\infty}$ M (x, y, t) =1 for all x, y ∈ X.

2.10. Lemma: Let (X, M, *) be a fuzzy metric space. If there exists k ∈[0, 1] such that M(x, y, kt) ≥ M(x, y, t) then x = y.

2.11. Lemma: Let $\{x_n\}$ be a sequence in a fuzzy metric space (X, M, *) with the condition (6). If there exists k ∈[0, 1] such that M (y_n,y_{n+1}, kt) ≥ M (y_{n-1}, y_n, t) for all t > 0 and n ∈ N. Then $\{y_n\}$ is a Cauchy sequence in X.

2.12. Lemma: Let X be a set, f, g owc self maps of X. If f and g have unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g.

2.13. Definition: A Class of Implicit Function

Let Φbe the set of all real continuous functions. $F: [0,1]^6 \to R$ non decreasing in the first argument satisfying the following conditions:

(a) For $u, v \geq 0, F(u, v, u, v. 1,1) \geq 0 \; implies \; that \; u \geq v.$

(b) $F\{u, 1,1, u, u, 1\} \geq 0 \; or \; F\{u, u, 1,1, u, 1\} \geq 0 \; or \; F\{u, 1, u, 1, u, 1\} \geq 0 implies \; that \; u \geq 1.$

2.14. Definition E A property Let A and B be self maps on a fuzzy metric space (X, M, *). They are said to satisfy (EA) property if there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = x$ for some x ∈ X.

Main Result

Theorem 3.1: Let $A, B, S \; and \; T$ be self mappings of a fuzzy metric space $(X, M,*)$ assume that exists $\emptyset, \varphi \in \Phi$ such that

$$(3.1) \quad \emptyset \begin{Bmatrix} M(Ax, By, kt), M(Sx, Ty, t), M(Ax, Sx, t), \\ M(By, Ty, kt), M(Ax, Ty, t), M(Sx, By, kt) \end{Bmatrix} \geq 0,$$

$$\varphi\begin{Bmatrix} M(Ax,By,kt), M(Sx,Ty,t), M(Ax,Sx,t), \\ M(By,Ty,kt), M(Ax,Ty,t), M(Sx,By,kt)\end{Bmatrix} \geq 0,$$

for all $x, y \in X$, $k \in (0,1)$ and $t > 0$. Suppose that the pair $(A,S) and$ (B,T) share the common property (E.A.) and $S(x)$ and $T(x)$ the closed subsets of X. then the pair (A,S) as $well$ as (B,T) have a point of coincidence each. Further A,B,S and T have a unique common fixed point provided the pair (A,S) is $semi$ $compatible$ and (B,T) is weakly compatible.

Proof: Since the pair $(A,S) and$ (B,T) share the common (E.A.), then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that.

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$, for some $z \in X$.

Since $S(X)$ is closed subset of X, then $\lim_{n\to\infty} Ty_n = z \in S(X)$.

Therefore, there exists a point $u \in X$ such that $Su = z$.

Then by (3.1), we have

$$\emptyset\begin{Bmatrix} M(Au,By_n,kt), M(Su,Ty_n,t), M(Au,Su,t), \\ M(By_n,Ty_n,kt), M(Au,Ty_n,t), M(Su,By_n,kt)\end{Bmatrix} \geq 0,$$

$$\varphi\begin{Bmatrix} M(Au,By_n,kt), M(Su,Ty_n,t), M(Au,Su,t), \\ M(By_n,Ty_n,kt), M(Au,Ty_n,t), M(Su,By_n,kt)\end{Bmatrix} \geq 0,$$

which on making $n \to \infty$ reduces to

$$\emptyset\begin{Bmatrix} (M(Au,z,kt), M(Su,z,t), M(Au,Su,t), \\ M(z,z,kt), M(Au,z,kt), M(Su,z,t)\end{Bmatrix} \geq 0,$$

$$\varphi\begin{Bmatrix} (M(Au,z,kt), M(Su,z,t), M(Au,Su,t), \\ M(z,z,kt), M(Au,z,kt), M(Su,z,t)\end{Bmatrix} \geq 0,$$

or equivalently,

$$\emptyset(M(Au,z,kt), 1, M(Au,z,t), 1, M(Au,z,kt), 1) \geq 0,$$

$$\varphi(M(Au,z,kt), 1, M(Au,z,kt), 1, M(Au,z,t), 1) \geq 0,$$

which gives $M(Au,z,t) = 1$, for all $t > 0$, that is $Au = z. Hence, A = Su. Therefore, u$ is a coincidence point of the par (A,S). Since $T(X)$ is closed subset of X, then

$\lim_{n\to\infty} Ty_n = z \in T(X)$.

Therefore, there exists a point $w \in X$such that $Tw = z$.

Now, we assert that $Bw = z$

Indeed, again using (3.1) we have

$$\emptyset\left\{\begin{matrix}(M(Ax_n,Bw,kt),M(Sx_n,Tw,t),M(Ax_n,Sx_n,t),\\ M(Bw,Tw,kt),M(Ax_n,Tw,kt),M(Sx_n,Bw,t))\end{matrix}\right\}\geq 0,$$

$$\varphi\left\{\begin{matrix}(M(Ax_n,Bw,kt),M(Sx_n,Tw,t),M(Ax_n,Sx_n,kt),\\ M(Bw,Tw,t),M(Ax_n,Tw,t),M(Sx_n,Bw,kt))\end{matrix}\right\}\geq 0,$$

which on making $n\to\infty$ redues to

$$\emptyset\left\{\begin{matrix}(M(z,Bw,kt),M(z,z,t),M(z,z,t),\\ M(Bw,z,kt),M(z,z,kt),M(z,Bw,t))\end{matrix}\right\}\geq 0,$$

$$\varphi\left\{\begin{matrix}(M(z,Bw,kt),M(z,z,t),M(z,z,kt),\\ M(Bw,z,t),M(z,z,t),M(z,Bw,kt))\end{matrix}\right\}\geq 0,$$

or equivalently,

$$\emptyset(M(z,Bw,kt),1,1,M(Bw,z,kt),1,M(z,Bw,t))\geq 0,$$

$$\varphi(M(z,Bw,kt),1,1,M(Bw,z,t),1,M(z,Bw,kt))\geq 0,$$

which gives $M(z,Bw,t)=1$ for all $t>0$, that is $Bw=z$. Hence $Tw=Bw=z$, which shows that w is a coincidence poit of the pair (B,T). Since (A,S) is semi-compatible, so $\lim_{n\to\infty} ASx_n=Sz$. Also $\lim_{n\to\infty} ASx_n=Az$.

Since the limit in fuzzy metric space is unique, so $Sz=Az$.

Now we assert that z is a common fixed point of the pair (A,S). Using (3.1), we have

$$\emptyset\left\{\begin{matrix}(M(Az,Bw,kt),M(Sz,Tw,t),M(Az,Sz,t),\\ M(Bw,Tw,kt),M(Az,Tw,kt),M(Sz,Bw,t))\end{matrix}\right\}\geq 0,$$

$$\varphi\left\{\begin{matrix}(M(Az,Bw,kt),M(Sz,Tw,t),M(Az,Sz,kt),\\ M(Bw,Tw,t),M(Az,Tw,t),M(Sz,Bw,kt))\end{matrix}\right\}\geq 0,$$

or $$\emptyset\left\{\begin{matrix}(M(Az,z,kt),M(Az,z,t),M(Az,Az,t),\\ M(z,z,kt),M(Az,z,kt),M(Az,z,t))\end{matrix}\right\}\geq 0,$$

$$\varphi\left\{\begin{matrix}(M(Az,z,kt),M(Az,z,t),M(Az,Az,kt),\\ M(z,z,t),M(Az,z,t),M(Az,z,kt))\end{matrix}\right\}\geq 0,$$

or $$\emptyset(M(Az,z,kt),M(Az,z,t),1,1,M(Az,z,kt),M(Az,z,t))\geq 0,$$

$$\varphi(M(Az,z,kt),M(Az,z,t),1,1,M(Az,z,t),M(Az,z,kt))\geq 0,$$

which gives $M(Az,z,t)=1$, for all $t>0$.

Hence, since w is a coincidence point of B and T and the pair (B,T) is weakly compatible, we have $BTw=TBw$ implies $Bz=Tz=z$.

Hence z is a common fixed point of both the pairs $(A,S) and\ (B,T)$.

For uniqueness, let $v(v \neq z)$ be another common fixed point of $A, B, S \; and \; T$, taking $x = z$ and $y = v$ in (3.1), we have

$$\emptyset \left\{ \begin{matrix} (M(Az, Bv, kt), M(Sz, Tv, t), M(Az, Sz, t), \\ M(Bv, Tv, kt), M(Az, Tv, kt), M(Sz, Bv, t)) \end{matrix} \right\} \geq 0,$$

$$\varphi \left\{ \begin{matrix} (M(Az, Bv, kt), M(Sz, Tv, t), M(Az, Sz, kt), \\ M(Bv, Tv, t), M(Az, Tv, t), M(Sz, Bv, kt)) \end{matrix} \right\} \geq 0,$$

which gives $z = v$.

Hence z is unique common fixed point of the mappings $A, B, S \; and \; T$.

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REFERENCES

[1]. Abbas, M., Altun, I. and Gopal, D., Common fixed point theorems for non compatible mappings in fuzzy metric spaces. Bulletin of Mathematical Analysis and Applications, 1(2), (2009), 47–56.

[2]. Deng, Z., Fuzzy pseudometric spaces, Journal of Mathematical Analysis and Applications. 86(1), (1982), 74– 95.

[3]. El Naschie, M.S., On a fuzzy Khaler-like manifold which is consistent with two slit experiment, International Journal of Nonlinear Sciences and Numerical Simulation, 6 (2005), 95–98.

[4]. George, A. and Veeramani, P., On some results of analysis for fuzzy metric spaces, Fuzzy Sets and Systems, 90(3), (1997), 365–368.

[5]. Imdad, M. and Ali, J., Some common fixed point theorems in fuzzy metric spaces, Mathematical Communications, 11(2), (2006), 153–163.

[6]. Jungck, G., Commuting mappings and fixed points, The American Mathematical Monthly, 83(4), (1976), 261–263.

[7]. Kaleva, O. and Seikkala, S., On fuzzy metric spaces. Fuzzy Sets and Systems. 12(3), (1984), 215–229. [8]. Kramosil, I. and Michálek, J., Fuzzy metrics and statistical metric spaces. Kybernetika. 11(5), (1975), 336– 344.

[9]. Mihet, D., A generalization of a contraction principle in probabilistic metric spaces (II), International Journal of Mathematics and Mathematical Sciences, Vol. 2005 (2005), 729–736.

[10]. Mihet, D., Fixed point theorems in fuzzy metric spaces using property E.A. Nonlinear Analysis. 73 (2010), 2184–2188.

[11]. Mishra, S.N., Mishra, N. and Singh, S.L., Common fixed point of maps in fuzzy metric space, Int. J. Math. Math. Sci. 17(1994), 253-258.

[12]. Pant, V., Contractive conditions and common fixed points in fuzzy metric space, Journal of Fuzzy Mathematics, 14(2), (2006), 267–272.

[13]. Singh, B. and Chouhan, M.S., Common fixed points of compatible maps in Fuzzy metric spaces, Fuzzy sets and systems, 115 (2000), 471-475.

[14]. Singh, B. and Jain, S., Semi-compatibility and fixed point theorems in fuzzy metric space using implicit relation, International Journal of Mathematics and Mathematical Sciences, Vol. 16 (2005), 2617–2629.

[15]. Singh, B. and Jain, S., Weak-compatibility and fixed point theorems in fuzzy metric space, Ganita, 56(2), (2005), 167–176.

[16]. Turkoglu, D. and Rhoades, B.E., A fixed fuzzy point for fuzzy mapping in complete metric spaces, Mathematical Communications. 10(2), (2005), 115–121.

[17]. Vetro, C. and Vetro, P., Common fixed points for discontinuous mappings in fuzzy metric spaces, Rendiconti del Circolo Matematico di Palermo, 57(2), (2008), 295–303.

[18]. Zadeh, L.,A., Fuzzy sets, Information and Computation. 8 (1965), 338–353.