FIXED POINT THEOREM IN FUZZY METRIC SPACE WITH THE PROPERTY

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Abstract

The purpose of this paper is to prove a common fixed point theorem for semi-compatible and occasionally weakly compatible mappings in fuzzy metric space by using the property (E.A.) and implicit relation. Our result generalizes the result of [14].

Keywords: Fuzzy metric space, property (E.A.), semi-compatible and weakly compatible mappings.

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Introduction

The evolution of fuzzy mathematics solely rests on the notion of fuzzy sets which was introduced by Zadeh [18] in 1965 with a view to represent the vagueness in everyday life. In mathematical programming, the problems are often expressed as optimizing some goal functions equipped with specific constraints suggested by some concrete practical situations. There exist many real-life problems that consider multiple objectives, and generally, it is very difficult to get a feasible solution that brings us to the optimum of all the objective functions. Thus, a feasible method of resolving such problems is the use of fuzzy sets [16]. In fact, the richness of applications has engineered the all round development of fuzzy mathematics. Then, the study of fuzzy metric spaces has been carried out in several ways (e.g., [2, 7]). George and Veeramani [4] modified the concept of fuzzy metric space introduced by Kramosil and Michá lek [8] with a view to obtain a Hausdorff topology on fuzzy metric spaces, and this has recently found very fruitful applications in quantum particle physics, particularly in connection with both string and theory (see [3]). In recent years, many authors have proved fixed point and common fixed point theorems in fuzzy metric spaces. To mention a few, we cite [1, 5, 9, 10, 12, 13, 14, 16, 17]. As patterned in Jungck [6], a metrical common fixed point theorem generally involves conditions on commutatively, continuity, completeness together with a suitable condition on containment of ranges of involved mappings by an appropriate contraction condition. Thus, research in this domain is aimed at weakening one or more of these conditions. In this paper, we observe that the notion of common property (E.A.) relatively relaxes the required containment of the range of one mapping into the range of other and hence, we obtain a common fixed point theorem in fuzzy metric space using the concepts of semi-compatible, occasionally weakly compatible and (E.A.) property with implicit relation. Our result generalizes the result of Singh et. al. [14]. For the sake of completeness, we recall some definitions and known results in Fuzzy metric space.
Preliminaries

2.1. Definition: A binary operation \( *: [0, 1] \times [0, 1] \rightarrow [0, 1] \) is a continuous t-norm if \( * \) is satisfying the following conditions:

(a) \( * \) is commutative and associative;
(b) \( * \) is continuous;
(c) \( a \ast b = a \) for all \( a \in [0, 1] \);
(d) \( a\ast b \leq c \ast d \) whenever \( a \leq c \) and \( b \leq d \) and \( a, b, c, d \in [0, 1] \).

2.2. Definition: A 3-tuple \((X, M, \ast)\) is said to be a fuzzy metric space if \( X \) is an arbitrary set, \( \ast \) is a continuous t-norm and \( M \) is a fuzzy set on \( X^2 \times (0, \infty) \) satisfying the following conditions; for all \( x, y, z \in X \), \( s, t > 0 \).

(1) \( M(x, y, t) > 0 \);
(2) \( M(x, y, t) = 1 \) if and only if \( x = y \);
(3) \( M(x, y, t) = M(y, x, t) \);
(4) \( M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s) \);
(5) \( M(x, y, .): (0, \infty) \rightarrow [0, 1] \) is continuous.

Then \( M \) is called a fuzzy metric on \( X \). The function \( M(x, y, t) \) denote the degree of nearness between \( x \) and \( y \) with respect to \( t \).

2.3. Example: Let \((X, d)\) be a metric space. Denote \( a \ast b = a \cdot b \) for \( a, b \in [0, 1] \) and let \( M_d \) be a fuzzy set on \( X^2 \times (0, \infty) \) defined as follows: \( M_d (x, y, t) = \frac{t}{t + d(x, y)} \)

Then \((X, M_d, \ast)\) is a fuzzy metric space, we call this fuzzy metric induced by a metric \( d \) the standard intuitionistic fuzzy metric.

2.4. Definition: Let \((X, M, \ast)\) be a fuzzy metric space, then

(a) A sequence \( \{x_n\} \) in \( X \) is said to be convergent to \( x \) in \( X \) if for each \( \varepsilon > 0 \) and each \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( M(x_n, x, t) > 1 - \varepsilon \) for all \( n \geq n_0 \).
(b) A sequence \( \{x_n\} \) in \( X \) is said to be Cauchy if for each \( \varepsilon > 0 \) and each \( t > 0 \), there exist \( n_0 \in \mathbb{N} \) such that \( M(x_n, x_m, t) > 1 - \varepsilon \) for all \( n, m \geq n_0 \).
(c) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

2.5. Proposition: In a fuzzy metric space \((X, M, \ast)\), if \( a \ast a \geq a \) for \( a \in [0, 1] \) then \( a \ast b = \min \{a, b\} \) for all \( a, b \in [0, 1] \).
2.6. Lemma: Let \((X, M, T)\) be an \(L\)-fuzzy metric space. Then \(M(x, y, t)\) is nondecreasing with respect to \(t\), for all \(x, y \in X\).

2.7. Definition: Two self-mappings \(A\) and \(S\) of a fuzzy metric space \((X, M, *)\) are called compatible if \(\lim_{n \to \infty} M(AX_n, SX_n, t) = 1\) whenever \(\{x_n\}\) is a sequence in \(X\) such that \(\lim_{n \to \infty} AX_n = \lim_{n \to \infty} SX_n = x\) for some \(x\) in \(X\).

2.8. Definition: Two self-maps \(A\) and \(B\) of a fuzzy metric space \((X, M, *)\) are called weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if \(Ax = Bx\) for some \(x \in X\) then \(ABx = BAx\).

2.9. Remark: If self-maps \(A\) and \(B\) of a fuzzy metric space \((X, M, *)\) are compatible then they are weakly compatible.

Let \((X, M, *)\) be a fuzzy metric space with the following condition:

\[
\lim_{t \to \infty} M(x, y, t) = 1 \quad \text{for all} \quad x, y \in X.
\]

2.10. Lemma: Let \((X, M, *)\) be a fuzzy metric space. If there exists \(k \in [0, 1]\) such that \(M(x, y, kt) \geq M(x, y, t)\) then \(x = y\).

2.11. Lemma: Let \(\{x_n\}\) be a sequence in a fuzzy metric space \((X, M, *)\) with the condition (6). If there exists \(k \in [0, 1]\) such that \(M(y_{n+1}, y_n, t) \geq M(y_{n-1}, y_n, t)\) for all \(t > 0\) and \(n \in \mathbb{N}\). Then \(\{y_n\}\) is a Cauchy sequence in \(X\).

2.12. Lemma: Let \(X\) be a set, \(f, g\) owc self maps of \(X\). If \(f\) and \(g\) have unique point of coincidence, \(w = f(x) = g(x)\), then \(w\) is the unique common fixed point of \(f\) and \(g\).

2.13. Definition: A Class of Implicit Function

Let \(\Phi\) be the set of all real continuous functions. \(F: [0,1]^6 \to R\) non decreasing in the first argument satisfying the following conditions:

(a) For \(u, v \geq 0\), \(F(u, v, u, v, 1, 1) \geq 0\) implies that \(u \geq v\).

(b) \(F(u, 1, 1, u, 1, 1) \geq 0\) or \(F(u, u, 1, 1, u, 1) \geq 0\) or \(F(u, 1, u, 1, u, 1) \geq 0\) implies that \(u \geq 1\).

2.14. Definition E A property Let \(A\) and \(B\) be self maps on a fuzzy metric space \((X, M, *)\). They are said to satisfy (EA) property if there exists a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} AX_n = \lim_{n \to \infty} SX_n = x\) for some \(x \in X\).

Main Result

**Theorem 3.1:** Let \(A, B, S\) and \(T\) be self mappings of a fuzzy metric space \((X, M, *)\) assume that exists \(\emptyset, \varphi \in \Phi\) such that

\[
(3.1) \quad \emptyset \bigg\{ M(Ax, By, kt), M(Sx, Ty, t), M(Ax, Sx, t), \bigg\} \geq 0,
\]
for all $x, y \in X$, $k \in (0,1)$ and $t > 0$. Suppose that the pair $(A,S)$ and $(B,T)$ share the common property (E.A.) and $S(x)$ and $T(x)$ the closed subsets of $X$. then the pair $(A,S)$ as well as $(B,T)$ have a point of coincidence each. Further $A,B,S$ and $T$ have a unique common fixed point provided the pair $(A,S)$ is semi compatible and $(B,T)$ is weakly compatible.

**Proof:** Since the pair $(A,S)$ and $(B,T)$ share the common (E.A.), then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that.

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = z, \text{ for some } z \in X.$$

Since $S(X)$ is closed subset of $X$, then $\lim_{n \to \infty} Ty_n = z \in S(X)$.

Therefore, there exists a point $u \in X$ such that $Su = z$.

Then by (3.1), we have

$$\emptyset \left\{ M(Au, By_n, kt), M(Su, Ty_n, t), M(Au, Su, t), M(By_n, Ty_n, kt), M(Au, Ty_n, t), M(Su, By_n, kt) \right\} \geq 0,$$

which on making $n \to \infty$ reduces to

$$\emptyset \left\{ M(Au, z, kt), M(Su, z, t), M(Au, Su, t), M(z, z, kt), M(Au, z, kt), M(Su, z, t) \right\} \geq 0,$$

or equivalently,

$$\emptyset (M(Au, z, kt), 1, M(Au, z, t), 1, M(Au, z, kt), 1) \geq 0,$$

$$\varphi (M(Au, z, kt), 1, M(Au, z, t), 1, M(Au, z, kt), 1) \geq 0,$$

which gives $M(Au, z, t) = 1$, for all $t > 0$, that is $Au = z$. Hence, $A = Su$. Therefore, $u$ is a coincidence point of the pair $(A,S)$. Since $T(X)$ is closed subset of $X$, then

$$\lim_{n \to \infty} Ty_n = z \in T(X).$$

Therefore, there exists a point $w \in X$ such that $Tw = z$.

Now, we assert that $Bw = z$

Indeed, again using (3.1) we have
\[ \varphi \left\{ \left( M(Ax_n, Bw, kt), M(Sx_n, Tw, t), M(Ax_n, Sx_n, t) \right), \left( M(Bw, Tw, kt), M(Ax_n, Tw, kt), M(Sx_n, Bw, t) \right) \right\} \geq 0, \]
\[ \varphi \left\{ \left( M(Ax_n, Bw, kt), M(Sx_n, Tw, t), M(Ax_n, Sx_n, kt) \right), \left( M(Bw, Tw, t), M(Ax_n, Tw, t), M(Sx_n, Bw, kt) \right) \right\} \geq 0, \]

which on making \( n \to \infty \) reduces to
\[ \varphi \left\{ \left( M(z, Bw, kt), M(z, z, t), M(z, z, t) \right), \left( M(Bw, z, kt), M(z, z, kt), M(z, Bw, t) \right) \right\} \geq 0, \]
\[ \varphi \left\{ \left( M(z, Bw, kt), M(z, z, t), M(z, z, kt) \right), \left( M(Bw, z, t), M(z, z, t), M(z, Bw, kt) \right) \right\} \geq 0, \]
or equivalently,
\[ \varphi(M(z, Bw, kt), 1, 1, M(Bw, z, kt), 1, M(z, Bw, t)) \geq 0, \]
\[ \varphi(M(z, Bw, kt), 1, 1, M(Bw, z, t), 1, M(z, Bw, kt)) \geq 0, \]

which gives \( M(z, Bw, t) = 1 \) for all \( t > 0 \), that is \( Bw = z \). Hence \( Tw = Bw = z \), which shows that \( w \) is a coincidence point of the pair \((B, T)\). Since \((A, S)\) is semi-compatible, so \( \lim_{n \to \infty} ASx_n = Sz \). Also \( \lim_{n \to \infty} ASx_n = Az \).

Since the limit in fuzzy metric space is unique, so \( Sz = Az \).

Now we assert that \( z \) is a common fixed point of the pair \((A, S)\). Using (3.1), we have
\[ \varphi \left\{ \left( M(Az, Bw, kt), M(Sz, Tw, t), M(Az, Sz, t) \right), \left( M(Bw, Tw, kt), M(Az, Tw, kt), M(Sz, Bw, t) \right) \right\} \geq 0, \]
\[ \varphi \left\{ \left( M(Az, Bw, kt), M(Sz, Tw, t), M(Az, Sz, kt) \right), \left( M(Bw, Tw, t), M(Az, Tw, t), M(Sz, Bw, kt) \right) \right\} \geq 0, \]
or
\[ \varphi \left\{ \left( M(Az, z, kt), M(Az, z, t), M(Az, Az, t) \right), \left( M(z, z, kt), M(Az, z, kt), M(Az, z, t) \right) \right\} \geq 0, \]
\[ \varphi \left\{ \left( M(Az, z, kt), M(Az, z, t), M(Az, Az, kt) \right), \left( M(z, z, t), M(Az, z, t), M(Az, z, kt) \right) \right\} \geq 0, \]
or
\[ \varphi(M(Az, z, kt), M(Az, z, t), 1, 1, M(Az, z, kt), M(Az, z, t)) \geq 0, \]
\[ \varphi(M(Az, z, kt), M(Az, z, t), 1, 1, M(Az, z, t), M(Az, z, kt)) \geq 0, \]

which gives \( M(Az, z, t) = 1 \), for all \( t > 0 \).

Hence, since \( w \) is a coincidence point of \( B \) and \( T \) and the pair \((B, T)\) is weakly compatible, we have \( BTw = TBw \) implies \( Bz = Tz = z \).

Hence \( z \) is a common fixed point of both the pairs \((A, S) \text{ and } (B, T)\).
For uniqueness, let \( v(v \neq z) \) be another common fixed point of \( A, B, S \) and \( T \), taking \( x = z \) and \( y = v \) in (3.1), we have

\[
\emptyset \left\{ \begin{array}{l}
\phi\left( M(Az, Bv, kt), M(Sz, Tv, t), M(Az, Sz, k) \right), \\
\phi\left( M(Bv, Tv, t), M(Az, Bv, kt), M(Sz, Bv, t) \right)
\end{array} \right\} \geq 0,
\]

which gives \( z = v \).

Hence \( z \) is unique common fixed point of the mappings \( A, B, S \) and \( T \).

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