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A Common Unique Random Fixed Point Theorems in S – Metric Spaces

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Abstract: In this paper, we present some new definitions of S – metric spaces and prove some random fixed point theorem for two random functions in complete S – metric spaces. We get some improved versions of several fixed point theorems in S – metric spaces.

Key words: D^*- metric space, S – metric space, common fixed point theorem.

Introduction:

In 1922, the Polish mathematician, Banach, proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banach's fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways. In [8] Jungck introduced more generalized commuting mappings, called compatible mappings, which are more general than commuting and weakly commuting mappings. This concept has been useful for obtaining more comprehensive fixed point theorems. One such generalization is generalized metric space or D – metric space initiated by Dhage in 1992. He proved some results on fixed points for a self – map satisfying a contraction for complete and bounded D – metric spaces. Rhoades generalized Dhage's contractive condition by increasing the number of factors and proved the existence of unique fixed point of a self – map in D – metric space. Recently, motivated by the concept of compatibility for metric space. Singh and Sharma introduced the concept of D–compatibility of maps in D – metric space and proved some fixed point theorems using a contractive condition. Naidu observed that almost all fixed point theorems in D – metric spaces are not valid or of doubtful validity. Also, Sedghi and Shobe introduced D^*- metric space, by modifying the tetrahedral inequality in D – metric space and proved some basic result in it. In this paper, we introduce D^*- metric which is a probable modification of the definition of D – metric introduced by

Dhage and prove some basic propertied in D^*- metric space. We also prove a common fixed point theorem for six mappings under the condition of weakly compatible mappings in D^*- metric spaces.

In what follows (X, D^*) will denote a D^*- metric space, ℕ the set of all natural numbers, and R^+ the set of all positive real numbers.

The definition of D^*- metric as follows:

Definition 1: Let X be a nonempty set. A generalized metric (or D^*- metric) on X is a function D^*: $X^3 \rightarrow [0, \infty)$that satisfies the following conditions for each x, y, z, a ∈ X.

(1) D^* (x, y, z) ≥ 0,

(2) D^* (x, y, z) = 0 if and only if x = y = z,

(3) D^* (x, y, z) = D^* (p{x, y, z}), (symmetry) where p is a permutation function,

(4) D^* (x, y, z) ≤ D^* (x, y, a) + D^* (a, z, z).

The pair (X, D^*) is called a generalized metric (or D^*- metric) space.

In this paper we introduce new concept of a generalized metric space which is more generalized than D^*- metric space, that is S – metric space and prove some basic properties and some fixed point theorems in s – metric spaces.

Definition 2. Let X be a nonempty set. A generalized metric (or S – metric) on X is a function S: $X^3 \rightarrow [0, \infty)$that satisfies the following conditions for each x, y, z, a ∈ X,

(1)$S_{(x,y,z)} \geq 0$,

(2) $S_{(x,y,z)} = 0$ if and only if x = y = z,

(3) $S_{(x,y,z)} \leq S_{(a,y,z)} + S_{(a,x,x)}$.

The pair (X, S) is called a generalized metric (or S – metric) space.

Immediate examples of such a function are

(a) If X = R^nthen we define

$$S_{(x,y,z)} = \|y + x - 2z\| + \|y - z\|.$$

(b) $S_{(x,y,z)}$ = d(x, y) + d (x, z) here, d is the ordinary metric on X.

(c) If X = R^n then we define

$$S_{(x,y,z)} = \|x - z\| + \|y - z\|$$

(d) if X = ℝ then we define

$$S_{(x,y,z)} = |a^{y+z} - a^{2x}| + |y - z|,$$

For every x, y, z ∈ X, a > 0 and a ≠ 1. Here, d is an ordinary metric on X.

Remark 2. Let (X, S) be a S – metric space. If we define f: $X^2 \to [0, \infty)$ as $f_{(x,y)} = S_{(x,y,y)}$ for all x, y ∈ X then f is an ordinary metric on X.

Proof. Clearly $f_{(x,y)} \geq 0$ for all x, y ∈ X and $f_{(x,y)} = 0$ iff x = y.

$$f_{(x,y)} = S_{(x,y,y)} = S_{(y,x,x)} = f_{(y,x)} \text{ from Remark 1.}$$

From Definition 2 we have

$$\begin{aligned} f_{(x,y)} &= S_{(x,y,y)} \\ &\leq S_{(z,y,y)} + S_{(z,x,x)} \\ &= f_{(z,y)} + f_{(z,x)}. \end{aligned}$$

Thus f is a metric on X.

Let (X, S) be a S – metric space. For r > 0 define

$$B_s(x,r) = \{y \in X: S_{(x,y,y)} < r\}.$$

Definition3. Let (X, S) be a S- metric space and A ⊂ X.

(1) If for every x ∈ A there exists r > 0 such that $B_s(x,r) \subset A$, then subset A is called open subset of X.

(2) Subset A of X is said to be S – bounded if there exists r > 0 such that $S_{(x,y,y)} <$ r for all x , y Є A.

(3) A sequence $\{x_n\}$ in X converges to x if and only if

$$S_{(x_n,x,x)} = S_{(x,x_n,x_n)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

That is for each $\epsilon > 0$ there exists n_0 Є ℕ such that

$$\forall\, n \geq n_0 \rightarrow S_{(x,x_n,x_n)} < \epsilon.$$

(4) Sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exists n_0 Є ℕ such that $S_{(x_n,x_m,x_m)} < \epsilon$ for each n, m $\geq n_0$. The S – metric space (X, S) is said to be complete if every Cauchy sequence is convergent.

Let τ be the set of all A ⊂ X with x Є A if and only if there exists r > 0 such that $B_s(x,r) \subset$ A. Then τ is a topology on X (induced by the S – metric Space).

Lemma1. Let (X, S) be a S – metric space. If r > 0, then ball $B_s(x,r)$ with centre x Є X and radius r is open ball.

Proof: Let z Є $B_s(x,r)$, Hence $S_{(x,z,z)} <$ r. If set $S_{(x,z,z)} = \delta$ and $r' =$ r – δ then we prove that $B_s(z,r') \subseteq B_s(x,r)$. Let y Є $B_s(z,r')$, by triangular inequality we have

$$S_{(x,y,y)} = S_{(y,x,x)} \leq S_{(x,x,x)} + S_{(z,y,y)} < r' + \delta = r.$$

Hence $B_s(z,r') \subseteq B_s(x,r)$. That is ball $B_s(x,r)$ is open ball.

Lemma 2. Let (X, S) be a S – metric space. If there exists sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \rightarrow$ x and $y_n \rightarrow$ y, then

$$S_{(x_n,y_n,y_n)} \rightarrow S_{(x,y,y)}.$$

Proof: Since sequence $\{(x_n, y_n, y_n)\}$ in X^3 converges to a point (x, y, y) Є X^3 i. e.

$$\lim_{n\rightarrow\infty} x_n = x,\ \lim_{n\rightarrow\infty} y_n = y,$$

For each $\epsilon > 0$ there exists n_1 Є ℕ such that for every

$$n \geq n_1 \Longrightarrow S_{(x,x_n,x_n)} < \epsilon/2$$

and $n_2 \in \mathbb{N}$ such that for every

$$n \geq n_2 \Longrightarrow S_{(y,y_n,y_n)} < \epsilon/2.$$

If $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ by triangular inequality we have

$S_{(x_n,y_n,y_n)} \leq S_{(x,y_n,y_n)} + S_{(x,x_n,x_n)}$

$\leq S_{(y,y_n,y_n)} + S_{(y,x,x)} + S_{(x,x_n,x_n)}$

$< \epsilon/2 + \epsilon/2 + S_{(y,x,x)} = S_{(y,x,x)} + \epsilon.$

Hence we have $S_{(x_n,y_n,y_n)} - S_{(y,x,x)} < \epsilon.$

On the other hand

$S_{(y,x,x)} \leq S_{(x_n,x,x)} + S_{(x_n,y,y)}$

$\leq S_{(x_n,x,x)} + S_{(y_n,y,y)} + S_{(y_n,x_n,x_n)}$

$< \epsilon/2 + \epsilon/2 + S_{(x_n,y_n,y_n)}$

$= S_{(x_n,y_n,y_n)} + \epsilon.$

That is,

$$S_{(y,x,x)} - S_{(x_n,y_n,y_n)} < \epsilon .$$

Therefore we have $|S_{(x_n,y_n,y_n)} - S_{(x,y,y)}| < \epsilon$ i.e.

$$\lim_{n\to\infty} S_{(x_n,y_n,y_n)} = S_{(x,y,y)}$$

Lemma 3: Let (X, S) be a S – metric space. If sequence $\{x_n\}$ in X converges to x, then x is unique.

Proof. Let $x_n \to$ y and and y ≠ x. Since $\{x_n\}$ converges to x and y, for each $\epsilon > 0$ there exist $n_1 \in \mathbb{N}$ such that for every n ≥ $n_1 \Rightarrow S_{(x_n ,x,x)} < \epsilon /2$ and $n_2 \in \mathbb{N}$ such that for every n ≥ $n_2 \Rightarrow S_{(x_n,y,y)} < \epsilon /2$.

If $n_0 = \max\{n_1, n_2\}$, then for every n ≥ n_0 by triangular inequality we have

$S_{(x,y,y)} \leq S_{(x_n ,x,x)} + S_{(x_n,y,y)} < \epsilon/ 2+ \epsilon/ 2 = \epsilon$.

Hence $S_{(x,y,y)} = 0$ is a contradiction. So, x = y.

Lemma 4. Let (X, S) be a S- metric space. If sequence $\{x_n\}$in X converges to x, then sequence $\{x_n\}$ is a Cauchy sequence.

Proof. Since $x_n \to$ x for each $\epsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that

for every n ≥ $n_1 \Rightarrow S_{(x,x_n,x_n)} < \epsilon/ 2$

and $n_2 \in \mathbb{N}$ such that for every m ≥ $n_2 \Rightarrow S_{(x,x_m,x_m)} < \epsilon/ 2$.

If $n_0 = \max\{n_1, n_2\}$, then for every n ≥ n_0 by triangular inequality we have

$S_{(x_n,x_m,x_m)} \leq S_{(x,x_n,x_n)} + S_{(x,x_m,x_m)} < \epsilon/ 2+ \epsilon/ 2 = \epsilon$.

Hence sequence $\{x_n\}$ is a Cauchy sequence.

Main Results:

Definition4: Let F: $\mathbb{R} \times X \to X$ be a function, where X is a nonempty set. Then function g: $\mathbb{R} \to X$ is said to be a random fixed point of the function F if $F_{(t,g(t))} = g(t)$ for all t in $\mathbb{R}$.

We shall prove the following theorem.

Theorem 1: Let (X, S) be a complete S – metric space and let F, G: $\mathbb{R} \times X \to X$ be two functions satisfying the following condition:

$S(F_{(t,x)}, G_{(t,y)}, G_{(t,y)}) \leq k_1 S(x, F_{(t,x)}, F_{(t,x)}) + k_2(S(t, G_{(t,y)}, G_{(t,y)})$

$$+ k_3 S_{(x,y,y)}.$$

For every x, y ∈ X, t ∈ ℝ where $k_i \geq 0$ for i = 1, 2, 3 and $0 < k_1 + k_2 + k_3 < 1$. Then F and G have a unique common fixed point.

Proof: We define a sequence of functions $\{g_n\}$ as $g_n: \mathbb{R} \to X$ is arbitrary function for t ∈ ℝ, and n = 0, 1, 2, 3, . . .

$$g_{2n+1}(t) = F_{(t, g_{2n}(t)}, \; g_{2n+2}(t) = G_{(t, g_{2n+1}(t)},$$

If $g_{2n}(t) = g_{2n+1}(t) = g_{2n+2}(t)$ for t ∈ ℝ, for some n then we set that $g_{2n}(t)$ is a random fixed point of F and G. Therefore, we suppose that no two consecutive terms of sequence $\{g_n\}$ are equal. Now by using (i) in all t ∈ ℝ we have

$$S_{(g_{2n+1}(t), g_{2n+2}(t), g_{2n+2}(t)} = S_{(F_{(t,g_{2n}(t)}, G_{(t,g_{2n+1}(t)}, G_{(t,g_{2n+1}(t))}}$$

$$\leq k_1 S_{(g_{2n}(t), F_{(t,g_{2n}(t)}, F_{(t,g_{2n}(t))}}$$

$$+ k_2 S_{(g_{2n+1}(t), G_{(t,g_{2n+1}(t)}, G_{(t,g_{2n+1}(t)}}$$

$$+ k_3 S_{(g_{2n}(t), g_{2n+1}(t), g_{2n+1}(t)}$$

Therefore,

$$S_{(g_{2n+1}(t), g_{2n+2}(t), g_{2n+2}(t)} \leq \frac{k_1 + k_3}{1 - k_2} S_{(g_{2n}(t), g_{2n+1}(t), g_{2n+1}(t)}$$

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$$\leq \frac{k_1 + k_3}{1 - k_2}^{2n+1} S_{(g_0(t), g_1(t), g_1(t))}$$

Similarly we have

$$S_{(g_{2n}(t),g_{2n+1}(t),g_{2n+1}(t))} \leq (\frac{k_1+k_3}{1-k_2})^{2n} S_{(g_0(t),g_1(t),g_1(t))}$$

Thus for every n ∈ ℕ we get,

$$S_{(g_n(t),g_{n+1}(t),g_{n+1}(t)} \leq (\frac{k_1+k_3}{1-k_2})^{n} S_{(g_0(t),g_1(t),g_1(t))}$$

Now we show that $\{g_n(t)\}$ is a Cauchy sequence.

$$S_{(g_n(t),g_m(t),g_m(t)} \leq S_{(g_{n+1}(t),g_m(t),g_m(t)} + S_{(g_{n+1}(t),g_n(t),g_n(t)})$$

$$\leq S_{(g_{n+2}(t),g_m(t),g_m(t)}$$

$$+ S_{(g_{n+2}(t),g_{n+1}(t),g_{n+1}(t)}$$

$$+ S_{(g_{n+1}(t),g_n(t),g_n(t)} + \ldots$$

$$\leq S_{(g_{n-1}(t),g_m(t),g_m(t)}$$

$$+ S_{(g_{n+2}(t),g_{n+1}(t),g_{n+1}(t))}$$

$$+ S_{(g_{n+1}(t),g_n(t),g_n(t))}$$

$$= S_{(g_{m-1}(t),g_m(t),g_m(t))}$$

$$+ \ldots + S_{(g_{n+1}(t),g_{n+2}(t),g_{n+2}(t))}$$

$$+ S_{(g_n(t),g_{n+1}(t),g_{n+1}(t))}.$$

If $q = \frac{k_1+k_3}{1-k_2}$ then

$$S_{(g_n(t),g_m(t),g_m(t))} \leq q^{m-1} S_{(g_0(t),g_1(t),g_1(t))}$$

$$+ q^{m-2} S_{(g_0(t),g_1(t),g_1(t))}$$

$$+ \ldots + q^n S_{(g_0(t),g_1(t),g_1(t))}$$

$$= \frac{q^n - q^m}{1-q} \quad S_{(g_0(t), g_1(t), g_1(t))} \to 0.$$

Thus $\{g_n(t)\}$ is a Cauchy and by the completeness of X, $\{g_n(t)\}$ converges to g(t) in X. Now we prove that $F_{(\mathrm{t,g(t)})} = \mathrm{g(t)}$.

Replace x = g(t) and y = $g_{2n+1}(t)$ in inequality (i) we have

$$S_{(F_{(\mathrm{t,g(t)})}, G_{(\mathrm{t}, g_{2n}(t))}, G_{(\mathrm{t}, g_{2n}(t))})} \leq k_1\, S_{(\mathrm{g(t)}, F_{(\mathrm{t,g(t)})}, F_{(\mathrm{t,g(t)})})}$$

$$+ k_2\, S_{(g_{2n}(t), G_{(\mathrm{t}, g_{2n}(t))}, G_{(\mathrm{t}, g_{2n}(t))})}$$

$$+ k_3\, S_{(\,\mathrm{g(t)}, g_{2n}(t), g_{2n}(t))}.$$

On making n → ∞ in the above inequality we get

$$S_{(F_{(\mathrm{t,g(t)})}, \mathrm{g(t)}, \mathrm{g(t)})} \leq k_1 S_{(\mathrm{g(t)}, F_{(\mathrm{t,g(t)})}, F_{(\mathrm{t,g(t)})})} + k_2 S_{(\mathrm{g(t)}, \mathrm{g(t)}, \mathrm{g(t)})}$$

$$+ k_3 S_{(\mathrm{g(t)}, \mathrm{g(t)}, \mathrm{g(t)})}$$

$$= k_1 S_{(\mathrm{g(t)}, F_{(\mathrm{t,g(t)})},, F_{(\mathrm{t,g(t)})},)}$$

Therefore $S_{(\mathrm{g(t)}, F_{(\mathrm{t,g(t)})},, F_{(\mathrm{t,g(t)})},)} = 0$ that is $F_{(\mathrm{t,g(t)})} = \mathrm{g(t)}$.

Replace x = g(t) and y = g(t) in inequality (i) we have

$$S_{(F_{(\mathrm{t,g(t)})}, G_{(\mathrm{t,g(t)})},, G_{(\mathrm{t,g(t)})}\,)} \leq k_1\, S_{(\mathrm{g(t)}, F_{(\mathrm{t,g(t)})}, F_{(\mathrm{t,g(t)})})}$$

$$+ k_2\, S_{(\mathrm{g(t)}, G_{(\mathrm{t,g(t)})}, G_{(\mathrm{t,g(t)})})}$$

$$+ k_3\, S_{(\mathrm{g(t)}, \mathrm{g(t)}, \mathrm{g(t)})}$$

$$= k_2\, S_{(\mathrm{g(t)}, G_{(\mathrm{t,g(t)})}, G_{(\mathrm{t,g(t)})})}$$

Therefore $S_{(F_{(\mathrm{t,g(t)}}, G_{(\mathrm{t,g(t)})},, G_{(\mathrm{t,g(t)})})} = 0$ that is $F_{(\mathrm{t,g(t)}} = G_{(\mathrm{t,g(t)}} = \mathrm{g(t)}$. Thus g(t) is a common random fixed point of F and G.

Now to prove uniqueness let if possible h(t) ≠ g(t) be another common random fixed point of F and G. Then by inequality (i) we have

$$S_{(\mathrm{g(t),h(t),h(t)})} = \mathrm{S}(F_{(\mathrm{t,g(t)})}, G_{(\mathrm{t,h(t)})}, G_{(\mathrm{t,h(t)})})$$

$$\leq k_1\, S_{(\mathrm{g(t)}, F_{(\mathrm{t,g(t)})}, F_{(\mathrm{t,g(t)})})}$$

$$+ k_2 S_{(\mathrm{h(t)}, G_{(\mathrm{t,h(t)})}, G_{(\mathrm{t,h(t)})})}$$

$$+ k_3 S_{(\mathrm{g(t),h(t),h(t)})})$$

$$= k_3 S_{(\mathrm{g(t),h(t),h(t)})}$$

Therefore $S_{(\mathrm{g(t),h(t),h(t)})}) = 0$ that is g(t) = h(t) . Thus g(t) is a unique common random fixed point of F and G.

Corollary 2: Let (X, S) be a complete S – metric space and let F: ℝ × X → X be function satisfying the following condition:

$$S_{(F_{(\mathrm{t,x})},\, F_{(t,y)} F_{(t,y)},)} \leq k_1\, S_{(\mathrm{x}, F_{(t,x)}, F_{(t,x)}\,)} + k_2 S_{(\mathrm{y}, F_{(t,y)}, F_{(t,y)})}$$

$$+ k_3 S_{(\mathrm{x,y,y})},$$

For every x, y Є X, t Є ℝ where $k_i \geq 0$ for i = 1, 2, 3 and $0 < k_1 + k_2 + k_3 < 1$. Then F have a unique common fixed point.

Proof: By Theorem 1, it is enough set $F_{(t,y)} = G_{(t,y)}$.

Corollary 3: Let (X, S) be a complete S – metric space and let F: ℝ × X → X be function satisfying the following condition:

$$S_{(F_{(\mathrm{t,x})},\, F_{(t,y)} F_{(t,y)},)} \leq \mathrm{k}\, S_{(\mathrm{x,y,y})},$$

for every x, y Є X, t Є ℝ where 0 < k<1. Then F have a unique common fixed point.

Proof: By Corollary 2 , it is enough set $k_1 = k_2 = 0$.

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