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SUMS OF HAZARD FUNCTIONS OF EXPONENTIAL MIXTURES AND ASSOCIATED CONVOLUTIONS OF MIXED POISSON DISTRIBUTIONS

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Abstract

A Sum of hazard functions of exponential mixtures characterizes a convolution of infinitely divisible mixed Poisson distributions which is also a convolution of compound Poisson distributions.
For each sum of two special cases of Hofmann hazard function, the following have been obtained:

- the probability generating function (pgf) of the convolution of the mixed Poisson distributions.
- the pgf of the independent and identically distributed (iid) random variables for the convolution of the compound Poisson distributions.
- the recursive form of the convolution of the compound Poisson distribution.

We also wish to find out whether Panjer's recursive model holds for all cases.

Key words: convolutions, exponential mixtures, mixed Poisson distribution, Hofmann hazard functions, characterization, compound Poisson distribution, Panjer's recursive model, Laplace transform

1 Introduction

The objective of this paper is to show that a sum of two hazard functions of exponential mixtures gives rise to a convolution of infinitely divisible mixed Poisson distributions and hence a convolution of compound Poisson distributions. Pairs of Hofmann hazard functions have been considered to identify the convolutions.

The rest of the paper is organised as follows: Section 2 briefly discusses the relationship between a hazard function of an exponential mixture and the corresponding infinitely divisible mixed Poisson distribution. Section 3 proves that a sum of two hazard functions of exponential mixtures gives rise to a convolution of two mixed Poisson distributions and a convolution of two corresponding compound Poisson distributions. Section 4 is an illustration of the results obtained using sums of various cases of Hofmann hazard function. Concluding remarks are given in section 5.

2 A Single Hazard Function of an Exponential Mixture

A mixed Poisson distribution can be expressed in terms of a Laplace transform as

$$p_n(t) = (-1)^n \frac{t^n}{n!} L_\Lambda^{(n)}(t) \qquad n = 0, 1, 2, ... \tag{2.1}$$

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where $L_\Lambda(t)$ is Laplace transform of the mixing distribution

and

$$L_\Lambda^{(n)}(t) = \frac{d^n}{d\,t^n}\, L_\Lambda(t) \tag{2.2}$$

When n = 0, we have

$$\begin{aligned} p_0(t) &= L_\Lambda(t) \\ &= e^{-In\,\frac{1}{L_\Lambda(t)}} \\ &= e^{-\theta(t)} \end{aligned} \tag{2.3}$$

where,

$$\theta(t) = In\frac{1}{L_\Lambda(t)} \tag{2.4}$$

$$\therefore \quad \theta'(t) = -\frac{L_\Lambda'(t)}{L_\Lambda(t)} \tag{2.5}$$
$$= h(t)$$

which is a hazard function of the exponential mixture.

Since $h(t) = \theta'(t)$ is completely monotone and $\theta(0) = 0$, then $p_0(t)$ is a Laplace transform of an infinitely divisible mixing distribution.

Hence the mixed Poisson distribution $p_n(t)$ is also infinitely divisible (Feller, Chapter XIII, Vol. 2, 1971).Furthermore, an infinitely divisible mixed Poisson distribution is a compound Poisson distribution (Feller, Chapter XII, Vol. I, 1968; Ospina and Gerbes, 1987) whose pgf is given by

$$H(s,t) = e^{-\theta(t)(1-G(s,t))} \tag{2.6}$$

where $G(s,t)$ is the pgf of the iid random variables.

Since the pgf of the mixed Poisson distribution is

$$H(s,t) = e^{-\theta(t-ts)} \tag{2.7}$$

by equating the two formulae for pgf, $H(s,t)$, we get

$$G(s,t) = 1 - \frac{\theta(t-ts)}{\theta(t)} \tag{2.8}$$

Therefore the probability mass functions (pmfs) of the iid random variables are

$$\begin{aligned} g_x(t) &= \frac{1}{x!}\frac{d^x}{ds^x}\, G(s,t)|_{s=0} \\ &= (-1)^{x-1}\,\frac{t^x}{x!}\,\frac{\theta^x(t)}{\theta(t)}, \qquad x = 1,2,3... \end{aligned} \tag{2.9}$$

and

$$g_0(t) = 0 \tag{2.10}$$

Let $x = i+1$, which implies that $x-1 = i$, and hence

$$g_{i+1}(t) = (-1)^i\,\frac{t^{i+1}}{(i+1)!}\,\frac{\theta^{i+1}(t)}{\theta(t)}, \qquad i = 0,1,2,3... \tag{2.11}$$

The recursive form for the compound Poisson distribution is

$$n\,p_n(t) = \theta(t)\sum_{x=0}^{n} x\,g_x(t)p_{n-x}(t) \qquad n = 1,2,3...$$

or

$$\begin{aligned} (n+1)p_{n+1}(t) &= \theta(t)\sum_{i=0}^{n}(i+1)g_{i+1}(t)p_{n-i}(t) \\ &= \sum_{i=0}^{n}(-1)^i\,\frac{t^{i+1}}{i!}\theta^{i+1}(t)p_{n-i}(t) \qquad n = 0,1,2,3... \end{aligned} \tag{2.12}$$

Using the recursive relation, $p_n(t)$ can be obtained iteratively.

3 A sum of two hazard functions of exponential mixtures

3.1 *Derivations of key results for convolutions*

Let

$$h_1(t) = \theta_1'(t) \quad \text{and} \quad h_2(t) = \theta_2'(t) \tag{3.1}$$

be two hazard functions of exponential mixtures.

Further, let

$$\theta(t) = \theta_1(t) + \theta_2(t) \tag{3.2}$$

Applying equation (2.7), the pgf of the mixed Poisson distribution is

$$\begin{aligned} H(s,t) &= e^{-\theta(t-ts)} \\ &= e^{-\{\theta_1(t-ts)+\theta_2(t-ts)\}} \\ &= e^{-\theta_1(t-ts)}e^{-\theta_2(t-ts)} \end{aligned} \tag{3.3}$$

which is a product of two pgfs of mixed Poisson distributions.

Hence a sum of two hazard functions of exponential mixtures gives rise to a convolution of two random variables from mixed Poisson distributions.

Since infinitely divisible mixed Poisson distributions are also compound Poisson distributions, then the pgf can be expressed as

$$\begin{aligned} H(s,t) &= e^{-\{\theta_1(t)+\theta_2(t)\}\{1-G(s,t)\}} \\ &= e^{-\theta_1(t)\{1-G(s,t)\}}e^{-\theta_2(t)\{1-G(s,t)\}} \end{aligned} \tag{3.4}$$

implying a convolution of two compound Poisson random variables.

Equating the two formulae for $H(s,t)$, we get the pgf of the iid random variables for the convolution of the compound Poisson random variables

$$G(s,t) = 1 - \frac{\theta_1(t-ts)+\theta_2(t-ts)}{\theta_1(t)+\theta_2(t)} \tag{3.5}$$

with corresponding pmf of the iid random variables being

$$g_0(t) = 0 \tag{3.6}$$

and

$$g_x(t) = (-1)^{x-1}\,\frac{t^x}{x!}\,\frac{\theta_1^x(t)+\theta_2^x(t)}{\theta(t)}, \qquad x = 1,2,3... \tag{3.7}$$

or

$$g_{i+1}(t) = (-1)^i\,\frac{t^{i+1}}{(i+1)!}\,\frac{\theta_1^{i+1}(t)+\theta_2^{i+1}(t)}{\theta(t)}, \qquad i = 0,1,2,... \tag{3.8}$$

The recursive form for the compound Poisson distribution is either given by

$$n\,p_n(t) = \sum_{x=1}^{n} (-1)^{x-1}\,\frac{t^x}{(x-1)!}\,\left(\theta_1^x(t)+\theta_2^x(t)\right)p_{n-x}(t), \qquad n = 1,2,3,... \tag{3.9a}$$

or

$$(n+1)p_{n+1}(t) = \sum_{i=0}^{n} (-1)^i\,\frac{t^{i+1}}{i!}\,\left(\theta_1^{i+1}(t)+\theta_2^{i+1}(t)\right)p_{n-i}(t), \qquad n = 0,1,2,3... \tag{3.9b}$$

Using this recursive relation, $p_n(t)$ can be obtained iteratively.

3.2 *A Special Case*

When the first hazard function is a constant, we have:

$$h_1(t) = \theta_1'(t) = \delta \tag{3.10}$$

and hence

$$\theta_1(t) = \delta t \tag{3.11}$$

Therefore,

$$\theta_1'(t) + \theta_2'(t) = \delta + \theta_2'(t) \tag{3.12a}$$
$$\theta_1(t) + \theta_2(t) = \delta t + \theta_2(t) \tag{3.12b}$$

and

$$H(s,t) = e^{-\delta t(1-s)} e^{-\theta_2(t-ts)} \tag{3.13}$$

which is a product of the pgf of a Poisson distribution with parameter δt and a pgf of a mixed Poisson distribution.

The pgf of the iid random variables of the convolution of the compound Poisson distributions

$$\begin{aligned} G(s,t) &= 1 - \frac{\theta_1(t-ts) + \theta_2(t-ts)}{\theta_1(t) + \theta_2(t)} \\ &= 1 - \frac{\delta * (t-ts) + \theta_2(t-ts)}{\delta t + \theta_2(t)} \end{aligned} \tag{3.14}$$

By differentiating $G(s,t)$

$$\begin{aligned} \frac{\partial\, G(s,t)}{\partial s} &= -\frac{-\delta t - t\theta_2'(t-ts)}{\delta t + \theta_2(t)} \\ &= \frac{\delta t + t\theta_2'(t-ts)}{\delta t + \theta_2(t)} \end{aligned} \tag{3.15}$$

$$\frac{\partial^2\, G(s,t)}{\partial s^2} = \frac{-t^2\theta_2''(t-ts)}{\delta t + \theta_2(t)} \tag{3.16}$$

we obtain

$$G^x(s,t) = \frac{(-1)^{x-1} t^x \theta_2^x(t-ts)}{\delta t + \theta_2(t)} \qquad x = 2,3,... \tag{3.17}$$

and the pmfs of the iid random variables

$$\therefore \quad g_0(t) = G(0,t) = 0 \tag{3.18a}$$

$$g_1(t) = \frac{\partial\, G(s,t)}{\partial s}|_{s=0} = \frac{\delta t + t\theta_2'(t)}{\delta t + \theta_2(t)} \tag{3.18b}$$

$$\begin{aligned} g_x(t) &= (-1)^{x-1} \frac{t^x}{x!} \frac{\partial^x\, G(s,t)}{\partial s^x}|_{s=0} \\ &= (-1)^{x-1} \frac{t^x}{x!} \frac{\theta_2^x(t)}{\delta t + \theta_2(t)} \qquad x = 2,3,... \end{aligned} \tag{3.18c}$$

The recursive form of the compound Poisson distribution is

$$
\begin{aligned}
n\ p_n(t) &= \theta(t) \sum_{x=1}^{n} x\ g_x(t)\ \ p_{n-x}(t) \\
&= \theta(t) g_1(t) p_{n-1}(t) + \theta(t) \sum_{x=2}^{n} x\ g_x(t)\ \ p_{n-x}(t) \\
&= (\delta t + t\theta_2'(t))\, p_{n-1}(t) + \sum_{x=2}^{n} x\ \frac{(-1)^{x-1}\ t^x\ \theta_2^x(t)}{x!}\ p_{n-x}(t) \\
&= (\delta t + t\theta_2'(t))\, p_{n-1}(t) + \sum_{x=2}^{n} \frac{(-1)^{x-1}\ t^x\ \theta_2^x(t)}{(x-1)!}\ p_{n-x}(t) \qquad (3.19)
\end{aligned}
$$

as given by Walhin and Paris (2002)

Replacing n by $n+1$ in (3.19), we have

$$
(n+1)\ p_{n+1}(t) = (\delta t + t\theta_2'(t))\, p_n(t) + \sum_{x=2}^{n+1} \frac{(-1)^{x-1}\ t^x\ \theta_2^x(t)}{(x-1)!}\ p_{n-(x-1)}(t) \qquad (3.20)
$$

Let $x = i+1$, which implies that $x-1 = i$, and therefore

$$
(n+1)\ p_{n+1}(t) = (\delta t + t\theta_2'(t))\, p_n(t) + \sum_{i=1}^{n} \frac{(-1)^{i}\ t^{i+1}\ \theta_2^{i+1}(t)}{i!}\ p_{n-i}(t) \qquad n = 0, 1, 2 ... \qquad (3.21)
$$

4 Sums of Hofmann hazard functions

Walhin and Paris (1999) defined Hofmann distribution as:

$$
p_0(t) = e^{-\theta(t)}
$$

and

$$
p_n(t) = (-1)^n \frac{t^n}{n!} p_0^n(t) \qquad n = 1, 2, 3, ...
$$

where

$$
\theta'(t) = \frac{p}{(1+ct)^a} \qquad p > 0, \quad c > 0, \quad a \geq 0
$$

and

$$
\theta(0) = 0
$$

Wakoli and Ottieno (2015) determined that $\theta'(t)$ is in fact a hazard function of an exponential mixture and referred to it as Hofmann hazard function. Let the sum of two hazard functions of exponential mixture be in the form of Hofmann hazard functions; i.e,

$$
h(t) = \frac{p_1}{(1+c_1t)^{a_1}} + \frac{p_2}{(1+c_2t)^{a_2}}
$$

We wish to obtain the following:

- the pgf of mixed Poisson distribution.
- the pgf of the iid random variables.
- the recursive form of the compound Poisson distribution for cases of a_i, where $i = 1, 2$

We also wish to find out whether Panjer's recursive model still holds for all cases.

4.1 *When the first hazard function is a constant*

4.1.1 *The case of* $a_1 = 0$ *and* $a_2 = \frac{1}{2}$

$$h(t) = \theta'(t) = p_1 + \frac{p_2}{(1+c_2t)^{\frac{1}{2}}} \qquad p_1 > 0, \quad p_2 > 0, \quad c_2 > 0$$

where the second hazard function is that of an exponential-inverse Gaussian distribution.

Therefore

$$\theta_1(t) = p_1 t \qquad ; \qquad \theta_2(t) = \frac{2p_2}{c_2}\left((1+c_2t)^{\frac{1}{2}} - 1\right) \tag{4.1a}$$

$$\theta_1(t-ts) = p_1 * (t-ts) \quad ; \qquad \theta_2(t-ts) = \frac{2p_2}{c_2}\left((1+c_2t-c_2ts)^{\frac{1}{2}} - 1\right) \tag{4.1b}$$

The pgf of the convolution is

$$\begin{aligned} H(s,t) &= e^{-\theta_1(t-ts)}e^{-\theta_2(t-ts)} \\ &= e^{-p_1t(1-s)}\, e^{\frac{2p_2}{c_2}\left((1+c_2t-c_2ts)^{\frac{1}{2}}-1\right)} \end{aligned} \tag{4.2}$$

The the sum of hazard functions of exponential distribution and that of the exponential-inverse Gaussian distribution, therefore, gives rise to the convolution of the Poisson distribution and the Poisson-inverse Gaussian (Sichel) distribution. Using (3.5) the pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$\begin{aligned}
G(s,t) &= 1 - \frac{1}{\theta(t)}\left\{p_1t - p_1ts + \frac{2p_2}{c_2}\left\{(1+c_2t-c_2ts)^{\frac{1}{2}} - 1\right\}\right. \\
G'(s,t) &= \frac{1}{\theta(t)}\left\{p_1t + \frac{p_2}{c_2}\,(c_2t)^1(1+c_2t-c_2ts)^{-\frac{1}{2}}\right\} \\
G''(s,t) &= \frac{1}{\theta(t)}\,(\frac{1}{2})\,\frac{p_2}{c_2}\,(c_2t)^2(1+c_2t-c_2ts)^{-\frac{3}{2}} \\
G'''(s,t) &= \frac{1}{\theta(t)}\,(\frac{1}{2})(\frac{3}{2})\,\frac{p_2}{c_2}\,(c_2t)^3(1+c_2t-c_2ts)^{-\frac{5}{2}} \\
G^{iv}(s,t) &= \frac{1}{\theta(t)}\,(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})\,\frac{p_2}{c_2}\,(c_2t)^4(1+c_2t-c_2ts)^{-\frac{7}{2}} \\
G^{v}(s,t) &= \frac{1}{\theta(t)}\,\frac{2.3-1}{2}\,\frac{2.2-1}{2}\,\frac{2.1-1}{2}\,\frac{p_2}{c_2}\,(c_2t)^4(1+c_2t-c_2ts)^{-\frac{(2.4-1)}{2}} \\
G^{x}(s,t) &= \frac{1}{\theta(t)}\left(\frac{2(x-1)-1}{2}\right)\left(\frac{2(x-2)-1}{2}\right)...\left(\frac{2.2-1}{2}\right)\left(\frac{2.1-1}{2}\right) \\
&\quad \frac{p_2}{c_2}\,(c_2t)^x\{(1+c_2t-c_2ts)^{-\frac{(2x-1)}{2}} \\
&= \frac{1}{\theta(t)}\left(x-1-\frac{1}{2}\right)\left(x-2-\frac{1}{2}\right)...\left(2-\frac{1}{2}\right)\left(1-\frac{1}{2}\right) \\
&\quad \frac{p_2}{c_2}\,(c_2t)^x\{(1+c_2t-c_2ts)^{-x+\frac{1}{2}} \\
&= \frac{1}{\theta(t)}\left(-\frac{1}{2}+x-1\right)\left(-\frac{1}{2}+x-2\right)...\left(-\frac{1}{2}+x-x+2\right)\left(-\frac{1}{2}+x-x+1\right) \\
&\quad \frac{p_2}{c_2}\,(c_2t)^x\{(1+c_2t-c_2ts)^{-x+\frac{1}{2}} \\
&= \frac{(x-1)!}{\theta(t)}\binom{-\frac{1}{2}+x-1}{x-1}\,\frac{p_2}{c_2}\,(c_2t)^x\{(1+c_2t-c_2ts)^{\frac{1}{2}-x} \\
&= \frac{(x-1)!}{\theta(t)}\,p_2t\binom{\frac{1}{2}+x-1-1}{x-1}\,(\frac{c_2t}{(1+c_2t-c_2ts})^{x-1}\,(\frac{1}{(1+c_2t-c_2ts})^{\frac{1}{2}} \\
&\quad for\ x = 2, 3, ...
\end{aligned} \tag{4.3}$$

Therefore the pmfs of the iid random variables are

$$g_0(t) = 0 \tag{4.4a}$$

$$g_1(t) = \frac{1}{\theta(t)} \left\{ p_1 t + \frac{p_2 t}{(1+c_2 t)^{\frac{1}{2}}} \right\} \tag{4.4b}$$

$$g_x(t) = \frac{1}{\theta(t)} \frac{1}{x} p_2 t \binom{-\frac{1}{2}}{x-1} (-\frac{c_2 t}{(1+c_2 t})^{x-1} (1+c_2 t)^{-\frac{1}{2}} \quad for\ x = 2, 3, ...$$

where

$$\theta(t) = \theta_1(t) + \theta_2(t)$$
$$= p_1 t + \frac{2p_2}{c_2} \left((1+c_2 t)^{\frac{1}{2}} - 1 \right) \tag{4.4c}$$

And so

$$\frac{g_x(t)}{g_{x-1}(t)} = \frac{\frac{1}{x} \binom{-\frac{1}{2}}{x-1} (-\frac{c_2 t}{(1+c_2 t})^{x-1}}{\frac{1}{(x-1)} \binom{-\frac{1}{2}}{x-2} (-\frac{c_2 t}{(1+c_2 t})^{x-2}}$$
$$= \frac{x-1}{x} \frac{\binom{-\frac{1}{2}}{x-1}}{\binom{-\frac{1}{2}}{x-2}} (\frac{-c_2 t}{(1+c_2 t})$$
$$= (\frac{x-1}{x}) (\frac{\frac{3}{2}-x}{x-1}) (\frac{-c_2 t}{1+c_2 t})$$
$$= \frac{c_2 t}{(1+c_2 t)} - \frac{3\ c_2 t}{2\ (1+c_2 t)} \frac{1}{x}$$

which is in Panjer's recursive form with

$$a = \frac{c_2 t}{(1+c_2 t)} \qquad and \qquad b = -\frac{3}{2} \frac{c_2 t}{(1+c_2 t)}$$

The recursive form for the convolution of compound Poisson distributions is

$$n\ p_n(t) = \theta(t) \sum_{x=1}^{n} x\ g_x(t)\ \ p_{n-x}(t)$$
$$= \theta(t)\ g_1(t)\ p_{n-1}(t) + \theta(t) \sum_{x=2}^{n} x\ g_x(t)\ \ p_{n-x}(t)$$
$$= \left(p_1 t + \frac{p_2 t}{(1+c_2 t)^{\frac{1}{2}}} \right) p_{n-1}(t) +$$
$$(1+c_2 t)^{-\frac{1}{2}}\ p_2 t \sum_{x=2}^{n} \binom{-\frac{1}{2}}{x-1} (-\frac{c_2 t}{(1+c_2 t})^{x-1}\ p_{n-x}(t) \qquad n = 1, 2, ... \tag{4.5}$$

4.1.2 *The case of* $a_1 = 0$ *and* $a_2 = 1$

$$h(t) = p_1 + \frac{p_2}{(1+c_2 t)} \qquad p_1 > 0, \quad p_2 > 0, \quad c_2 > 0 \tag{4.6}$$

where the second hazard function is that of Pareto.

This sum of hazard functions can be obtained by considering an exponential mixture whose mixing distribution is the shifted-gamma distribution. The mixture is constructed below:

The pdf of the shifted-gamma distribution is

$$g(\lambda) = \frac{\beta^{\alpha}}{\Gamma\alpha}\, e^{-\beta(\lambda-\mu)}\, (\lambda-\mu)^{\alpha-1} \qquad \lambda > \mu, \quad \alpha > 0, \quad \beta > 0 \tag{4.7}$$

The survival function of the exponential mixture is

$$\begin{aligned} S(t) &= \int_{\lambda=0}^{\infty} S(t|\lambda)\, g(\lambda)\, d\lambda \\ &= \int_{\lambda=\mu}^{\infty} e^{-\lambda\, t}\, \frac{\beta^{\alpha}}{\Gamma\alpha}\, e^{-\beta(\lambda-\mu)}\, (\lambda-\mu)^{\alpha-1}\, d\lambda \end{aligned}$$

Let $y = \lambda - \mu$ hence $\lambda = y + \mu$ and $d\lambda = dy$

$$\begin{aligned} \therefore \quad S(t) &= \frac{\beta^{\alpha}}{\Gamma\alpha} \int_{y=0}^{\infty} e^{-(y+\mu)\, t}\, e^{-\beta\, y}\, y^{\alpha-1}\, dy \\ &= \frac{\beta^{\alpha}}{\Gamma\alpha}\, e^{-\mu\, t} \int_{y=0}^{\infty} e^{-y\, (\beta+t)}\, y^{\alpha-1}\, dy \\ &= \frac{\beta^{\alpha}}{\Gamma\alpha}\, e^{-\mu\, t}\, \frac{\Gamma\alpha}{(\beta+t)^{\alpha}} \\ &= \beta^{\alpha}\, e^{-\mu\, t}\, (\beta+t)^{-\alpha} \end{aligned} \tag{4.8}$$

$$\begin{aligned} -S'(t) &= \beta^{\alpha}\, (\mu)\, e^{-\mu\, t}\, (\beta+t)^{-\alpha} + \beta^{\alpha}\, e^{-\mu\, t}\, (\alpha)\, (\beta+t)^{-\alpha-1} \\ h(t) &= \frac{-S'(t)}{S(t)} \\ &= \frac{\beta^{\alpha}\, (\mu)\, e^{-\mu\, t}\, (\beta+t)^{-\alpha} + \beta^{\alpha}\, e^{-\mu\, t}\, (\alpha)\, (\beta+t)^{-\alpha-1}}{\beta^{\alpha}\, e^{-\mu\, t}\, (\beta+t)^{-\alpha}} \\ &= \mu + \alpha\, (\beta+t)^{-1} \\ &= \mu + \frac{\alpha}{\beta+t} \\ &= \mu + \frac{\frac{\alpha}{\beta}}{1+\frac{1}{\beta}\, t} \end{aligned} \tag{4.9}$$

which is the hazard function of the exponential-shifted gamma distribution.

The hazard function of the exponential-shifted gamma distribution is therefore the sum of Hofmann hazard functions given by

$$h(t) = p_1 + \frac{p_2}{1+c_2 t}$$

where $p_1 = \mu,$ $p_2 = \frac{\alpha}{\beta}$ *and* $c_2 = \frac{1}{\beta}$

Therefore

$$\theta_1(t) = p_1 t \quad ; \quad \theta_2(t) = \frac{p_2}{c_2}\, In(1+c_2 t)$$

$$\theta_1(t-ts) = p_1 * (t-ts) \quad ; \quad \theta_2(t-ts) = \frac{p_2}{c_2}\, In(1+c_2 t - c_2 ts)$$

The pgf of the convolution is

$$\begin{aligned} H(s,t) &= e^{p_1 t(s-1)}\, e^{\frac{p_2}{c_2}\, In\left(\frac{1}{1+c_2 t - c_2 ts}\right)} \\ &= e^{p_1 t(s-1)}\, \left(\frac{1}{1+c_2 t - c_2 ts}\right)^{\frac{p_2}{c_2}} \\ &= e^{p_1 t(s-1)} \left(\frac{\frac{1}{1+c_2 t}}{1-\frac{ct}{1+c_2 t} s}\right)^{\frac{p_2}{c_2}} \\ &= e^{p_1 t(s-1)} \left(\frac{\frac{1}{1+c_2 t}}{1-\frac{ct}{1+c_2 t} s}\right)^{\frac{p_2}{c_2}} \end{aligned} \tag{4.10}$$

Therefore the sum of hazard functions of exponential distribution and that of the exponential-gamma (Pareto)distribution gives rise to the convolution of the Poisson distribution and the Poisson-gamma (negative binomial) distribution.

By differentiating the pgf of the iid random variables of the convolution of the compound Poisson distribution:

$$G(s,t) = 1 - \frac{1}{\theta(t)}\ \{p_1 t - p_1 ts + \frac{p_2}{c_2}\ In(1 + c_2 t - c_2 ts)\}$$

$$G^{'}(s,t) = \frac{\partial\ G(s,t)}{\partial s} = -\frac{1}{\theta(t)}\ \{-p_1 t + \frac{p_2}{c_2}\ (-c_2 t)(1 + c_2 t - c_2 ts)^{-1}\}$$

$$G^{''}(s,t) = -\frac{1}{\theta(t)}\ \{(-1)\frac{p_2}{c_2}\ (-c_2 t)^2 (1 + c_2 t - c_2 ts)^{-2})\}$$

$$G^{'''}(s,t) = -\frac{1}{\theta(t)}\ \{(-1)(-2)\frac{p_2}{c_2}\ (-c_2 t)^3 (1 + c_2 t - c_2 ts)^{-3})\}$$

$$G^{x}(s,t) = \frac{1}{\theta(t)}\ (x-1)!\ \frac{p_2}{c_2}\ (\frac{c_2 t}{1 + c_2 t - c_2 ts})^x \tag{4.11}$$

we obtain the pmfs of the iid random variables

$$g_0(t) = G(0,t) = 0 \tag{4.12a}$$

$$\begin{aligned} g_1(t) &= G'(0,t) \\ &= \frac{1}{\theta(t)}\ \left(p_1 t + \frac{p_2 t}{1 + c_2 t}\right) \end{aligned} \tag{4.12b}$$

$$\begin{aligned} g_x(t) &= \frac{1}{x!}\ G^{(x)}(s,t)|_{s=0} \\ &= \frac{1}{x!}\ (x-1)!\frac{p_2 t}{\theta(t)}\ (1 + c_2 t)^{-x}\ (c_2 t)^{x-1} \\ &= \frac{1}{x}\ \frac{1}{\theta(t)}\ \frac{p_2}{c_2}\ \left(\frac{c_2 t}{1 + c_2 t}\right)^x \\ &= \frac{1}{x}\ \left(\frac{1}{p_1 t + \frac{p_2}{c_2}\ In(1 + c_2 t)}\right)\ \frac{p_2}{c_2}\ \left(\frac{c_2 t}{1 + c_2 t}\right)^x \qquad x = 2, 3, ... \end{aligned} \tag{4.12c}$$

$$\begin{aligned} \therefore \qquad \frac{g_x(t)}{g_{x-1}(t)} &= \frac{x-1}{x}\ \frac{c_2 t}{1 + c_2 t} \\ &= \frac{ct}{1 + c_2 t} + \frac{-\frac{c_2 t}{1 + c_2 t}}{x} \qquad x = 2, 3, ... \end{aligned}$$

which is Panjer's recursive model with

$$a = \frac{c_2 t}{1 + c_2 t} \qquad and \qquad b = -\frac{c_2 t}{1 + c_2 t}$$

Remark 1 *This Panjer's model is the same as that of a logarithmic series distribution with parameter* $\frac{c_2 t}{1 + c_2 t}$

The recursive form for the convolution of compound Poisson distributions is:

$$\begin{aligned} n\ p_n(t) &= \theta(t)\ \sum_{x=1}^{n}\ x\ g_x(t)\ \ p_{n-x}(t) \\ &= \theta(t)\ g_1(t)\ p_{n-1}(t) +\ \theta(t)\ \sum_{x=2}^{n}\ x\ g_x(t)\ p_{n-x}(t) \\ &= \theta(t)\ \frac{1}{\theta(t)}\ \left(p_1 t + \frac{p_2 t}{1 + c_2 t}\right) p_{n-1}(t) + \theta(t)\ \sum_{x=2}^{n}\ x\ \frac{1}{x}\ \frac{1}{\theta(t)}\ \frac{p_2}{c_2}\ \left(\frac{c_2 t}{1 + c_2 t}\right)^x\ p_{n-x}(t) \\ &= \theta(t)\ g_1(t)\ p_{n-1}(t) + \sum_{x=2}^{n}\ \frac{p_2}{c_2}\ \left(\frac{c_2 t}{1 + c_2 t}\right)^x\ p_{n-x}(t) \qquad for \qquad n = 1, 2, 3, ... \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} p_0(t) &= e^{-\theta(t)} \\ &= e^{-p_1 t - \frac{p_2}{c_2}\, In(1+c_2 t)} \\ &= e^{-p_1 t} \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} \end{aligned} \tag{4.14}$$

Replace n by $n+1$ in (4.13)

$$(n+1)p_{n+1}(t) = \theta(t)\ g_1(t)\ p_n(t) + \sum_{x=2}^{n+1} \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^x\ p_{n-(x-1)}(t)$$

and let $x = i+1$, so that $x-1 = i$

$$(n+1)p_{n+1} = \theta(t)\ g_1(t)\ p_n(t) + \sum_{i=1}^{n} \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^{i+1}\ p_{n-i}(t) \tag{4.15}$$

To obtain $p_n(t)$ explicitly, we use

Method 1: The iteration technique

For $n = 1$

$$\begin{aligned} p_1(t) &= \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right)\ p_0(t) \\ &= \left(p_1 t + \frac{p_2}{c_2}\ \frac{c_2 t}{1+c_2 t}\right)\ p_0(t) \\ &= (p_1 t\ p_0(t)) + \frac{p_2}{c_2}\left(\frac{c_2 t}{1+c_2 t}\right)\ p_0(t) \end{aligned} \tag{4.16}$$

Substituting (4.14) into (4.16), we obtain

$$\begin{aligned} p_1(t) &= e^{-p_1 t}\ \frac{(p_1 t)^1}{1!}\left(\frac{c_2 t}{1+c_2 t}\right)^0 \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} + e^{-p_1 t}\ \frac{p_1 t)^0}{0!}\ \frac{p_2}{c_2}\left(\frac{c_2 t}{1+c_2 t}\right)\left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} \\ &= \sum_{k=0}^{1} \frac{e^{-p_1 t}\ (p_1 t)^{1-k}}{(1-k)!}\binom{\frac{p_2}{c_2}+k-1}{k}\left(\frac{c_2 t}{1+c_2 t}\right)^k \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} \end{aligned}$$

For $n = 2$

$$2\ p_2(t) = \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right)\ p_1(t) + \frac{p_2}{c_2}\left(\frac{c_2 t}{1+c_2 t}\right)^2\ p_0(t)$$

Using (4.12)

$$\begin{aligned} p_2(t) &= \left\{\left(p_1 t + \frac{p_2 t}{1+c_2 t}\right)^2 + \frac{p_2}{c_2}\left(\frac{c_2 t}{1+c_2 t}\right)^2\right\}\frac{p_0(t)}{2} \\ &= \left\{(p_1 t)^2 + 2\,p_1 t\ \frac{p_2 t}{1+c_2 t} + \left(\frac{p_2 t}{1+c_2 t}\right)^2 + \frac{p_2}{c_2}\left(\frac{c_2 t}{1+c_2 t}\right)^2\right\}\frac{p_0(t)}{2} \\ &= \left\{(p_1 t)^2 + 2\,p_1 t\ \frac{p_2}{c_2}\left(\frac{c_2 t}{1+c_2 t}\right) + \left(\frac{p_2}{c_2}\right)^2\left(\frac{c_2 t}{1+c_2 t}\right)^2 + \frac{p_2}{c_2}\left(\frac{c_2 t}{1+c_2 t}\right)^2\right\}\frac{p_0(t)}{2} \\ &= \frac{(p_1 t)^2}{2}\ p_0(t) + p_1 t\ \frac{p_2}{c_2}\left(\frac{c_2 t}{1+c_2 t}\right)\ p_0(t) + \left(\frac{p_2}{c_2}+1\right)\frac{p_2}{c_2}\left(\frac{c_2 t}{1+c_2 t}\right)^2\frac{p_0(t)}{2} \end{aligned} \tag{4.17}$$

$$\begin{aligned} p_2(t) &= e^{-p_1 t}\ \frac{(p_1 t)^2}{2}\left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} + e^{-p_1 t}\ p_1 t\ \frac{p_2}{c_2}\left(\frac{c_2 t}{1+c_2 t}\right)\left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} + \\ &\quad \frac{1}{2}e^{-p_1 t}\left(\frac{p_2}{c_2}+1\right)\frac{p_2}{c_2}\left(\frac{c_2 t}{1+c_2 t}\right)^2\left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} \\ &= \sum_{k=0}^{2} \frac{e^{-p_1 t}\ (p_1 t)^{2-k}}{(2-k)!}\binom{\frac{p_2}{c_2}+k-1}{k}\left(\frac{c_2 t}{1+c_2 t}\right)^k \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} \end{aligned}$$

For $n = 3$

$$3p_3(t) = \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right) p_2(t) + \frac{p_2}{c_2} \sum_{x=2}^{3} \left(\frac{c_2 t}{1+c_2 t}\right)^x p_{3-x}(t)$$

$$= \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right) p_2(t) + \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^2 p_1(t) + \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^3 p_0(t)$$

$$= \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right) \left\{\left(p_1 t + \frac{p_2 t}{1+c_2 t}\right)^2 + \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^2\right\} \frac{p_0(t)}{2} +$$

$$\frac{p_2}{c_2} \left(\left(\frac{c_2 t}{1+c_2 t}\right)^2 \left\{p_1(t) + \left(\frac{p_2 t}{1+c_2 t}\right)\right\} p_0(t)\right) + \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^3 p_0(t)$$

$$= \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right)^3 \frac{p_0(t)}{2} + \frac{p_2}{c_2} \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right) \left(\frac{c_2 t}{1+c_2 t}\right)^2 \frac{p_0(t)}{2} +$$

$$\frac{p_2}{c_2} \left(\left(\frac{c_2 t}{1+c_2 t}\right)^2 \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right) p_0(t) + \left(\frac{c_2 t}{1+c_2 t}\right)^3 p_0(t)\right)$$

$$= \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right)^3 \frac{p_0(t)}{2} + \frac{3}{2} \frac{p_2}{c_2} \left(p_1 t + \frac{p_2 t}{1+c_2 t}\right) \left(\frac{c_2 t}{1+c_2 t}\right)^2 p_0(t) + \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^3 p_0(t)$$

$$= (p_1 t)^3 \frac{p_0(t)}{2} + 3\,(p_1 t)^2 \left(\frac{p_2 t}{1+c_2 t}\right) \frac{p_0(t)}{2} + p_1 t \left\{3 \left(\frac{p_2 t}{1+c_2 t}\right)^2 + 3\frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^2\right\} \frac{p_0(t)}{2} +$$

$$\left\{\left(\frac{p_2 t}{1+c_2 t}\right)^3 + 3\left(\frac{p_2 t}{1+c_2 t}\right) \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^2 + 2\frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^3\right\} \frac{p_0(t)}{2}$$

$$= (p_1 t)^3 \frac{p_0(t)}{2} + 3\,(p_1 t)^2 \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right) \frac{p_0(t)}{2} + 3\,p_1 t \left\{\left(\frac{p_2}{c_2}\right)^2 \left(\frac{c_2 t}{1+c_2 t}\right)^2 + \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^2\right\}$$

$$\left\{\left(\frac{p_2}{c_2}\right)^3 \left(\frac{c_2 t}{1+c_2 t}\right)^3 + 3\left(\frac{p_2}{c_2}\right)^2 \left(\frac{c_2 t}{1+c_2 t}\right)^3 + 2\frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right)^3\right\} \frac{p_0(t)}{2}$$

$$= (p_1 t)^3 \frac{p_0(t)}{2} + 3\,(p_1 t)^2 \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right) \frac{p_0(t)}{2} + 3\,(p_1 t) \left(\frac{p_2}{c_2} + 1\right) \left(\frac{p_2}{c_2}\right) \left(\frac{c_2 t}{1+c_2 t}\right)^2 \frac{p_0(t)}{2} +$$

$$\left\{\left(\frac{p_2}{c_2}\right)^2 + 3\left(\frac{p_2}{c_2}\right) + 2\right\} \left(\frac{p_2}{c_2}\right) \left(\frac{c_2 t}{1+c_2 t}\right)^3 \frac{p_0(t)}{2}$$

$$p_3(t) = e^{-p_1(t)} \frac{(p_1 t)^3}{3!} \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} + e^{-p_1(t)} \frac{(p_1 t)^2}{2!} \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t}\right) \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} +$$

$$e^{-p_1(t)} \frac{(p_1 t)}{1!} \binom{\frac{p_2}{c_2}+1}{2} \left(\frac{c_2 t}{1+c_2 t}\right)^2 \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}}$$

$$= \sum_{k=0}^{3} \frac{e^{-p_1 t}\,(p_1 t)^{3-k}}{(3-k)!} \binom{\frac{p_2}{c_2}+k-1}{k} \left(\frac{c_2 t}{1+c_2 t}\right)^k \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}}$$

In general therefore,

$$p_n(t) = \sum_{k=0}^{n} \frac{e^{-p_1 t}\,(p_1 t)^{n-k}}{(n-k)!} \binom{\frac{p_2}{c_2}+k-1}{k} \left(\frac{c_2 t}{1+c_2 t}\right)^k \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} \qquad n = 1, 2, 3, \ldots \tag{4.18}$$

Method 2: The pgf technique

$$H(s,t) = \sum_{n=0}^{\infty} p_n(t)\, s^n$$

$$= e^{-p_1 t(1-s)} \left(\frac{\frac{1}{1+c_2 t}}{1 - \frac{ct}{1+c_2 t} s}\right)^{\frac{p_2}{c_2}}$$

$$= e^{-p_1 t} \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} e^{p_1 t s} \left(1 - \frac{ct}{1+c_2 t} s\right)^{-\frac{p_2}{c_2}}$$

But

$$e^{p_1 ts} = 1 + \frac{p_1 t}{1}\, s + \frac{(p_1 t)^2}{2}\, s^2 + \frac{(p_1 t)^3}{3}\, s^3 + \ldots$$

$$= \sum_{k=0}^{n} \left\{ \frac{(p_1 t)^{n-k}}{(n-k)!} \right\} s^{n-k}$$

$$\left(1 - \frac{ct}{1+c_2 t}\, s\right)^{-\frac{p_2}{c_2}} = 1 + \binom{-\frac{p_2}{c_2}}{1} \frac{-c_2 t}{1+c_2 t}\, s + \binom{-\frac{p_2}{c_2}}{2} \left(\frac{-c_2 t}{1+c_2 t}\right)^2 s^2 + \ldots$$

$$= \sum_{k=0}^{n} \left\{ \binom{-\frac{p_2}{c_2}}{k} \left(\frac{-c_2 t}{1+c_2 t}\right)^k \right\} s^k$$

$$\therefore \quad H(s,t) = \sum_{n=0}^{\infty} \left\{ e^{-p_1 t} \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} \sum_{k=0}^{n} \frac{(p_1 t)^{n-k}}{(n-k)!} \binom{-\frac{p_2}{c_2}}{k} \left(\frac{-c_2 t}{1+c_2 t}\right)^k \right\} s^n$$

$$\therefore \quad p_n(t) = \sum_{k=0}^{n} e^{-p_1 t} \frac{(p_1 t)^{n-k}}{(n-k)!} \binom{-\frac{p_2}{c_2}}{k} \left(\frac{-c_2 t}{1+c_2 t}\right)^k \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}}$$

$$= \sum_{k=0}^{n} e^{-p_1 t} \frac{(p_1 t)^{n-k}}{(n-k)!} (-1)^k \binom{-\frac{p_2}{c_2}}{k} \left(\frac{c_2 t}{1+c_2 t}\right)^k \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}}$$

$$= \sum_{k=0}^{n} e^{-p_1 t} \frac{(p_1 t)^{n-k}}{(n-k)!} \binom{\frac{p_2}{c_2}+k-1}{k} \left(\frac{c_2 t}{1+c_2 t}\right)^k \left(\frac{1}{1+c_2 t}\right)^{\frac{p_2}{c_2}} \qquad (4.19)$$

which is a convolution of a Poisson distribution and negative binomial distribution.

4.1.3 *The case of* $a_1 = 0$ *and* $a_2 = 2$

$$h(t) = \theta'(t) = p_1 + \frac{p_2}{(1+c_2 t)^2} \qquad p_1 > 0, \quad p_2 > 0, \quad c_2 > 0$$

where the second hazard function is what we have called Polya-Aeppli hazard function (Wakoli and Ottieno 2015, p. 234)

$$\therefore \quad \theta_1(t) = p_1 t \quad ; \quad \theta_2(t) = \frac{p_2}{c_2}\left(1 - (1+c_2 t)^{-1}\right)$$

$$\theta_1(t-ts) = p_1 * (t-ts) \quad ; \quad \theta_2(t-ts) = \frac{p_2}{c_2}\left(1 - (1+c_2 t - c_2 ts)^{-1}\right)$$

The pgf of the convolution is

$$H(s,t) = e^{-p_1 t(1-s)}\, e^{\frac{p_2}{c_2}\left(1-(1+c_2 t - c_2 ts)^{-1}\right)} \qquad (4.20)$$

Therefore the sum of hazard functions of the exponential distribution and that of Polya-Aeppli distribution gives rise to the convolution of the Poisson distribution and Polya-Aeppli distribution.

The pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$G(s,t) = 1 - \frac{\theta(t-ts)}{\theta(t)} = 1 - \frac{1}{\theta(t)}\left(p_1 t - p_1 ts + \frac{p_2}{c_2}\left(1 - (1+c_2 t - c_2 ts)^{-1}\right)\right)$$

$$G'(s,t) = -\frac{1}{\theta(t)}\left(-p_1 t - \frac{p_2}{c_2}(-1)\,(-c_2 t)^1\,(1+c_2 t - c_2 ts)^{-2}\right)$$

$$G''(s,t) = -\frac{1}{\theta(t)}\left(-\frac{p_2}{c_2}(-1)(-2)\,(-c_2 t)^2\,(1+c_2 t - c_2 ts)^{-3}\right)$$

$$G'''(s,t) = -\frac{1}{\theta(t)}\left\{-\frac{p_2}{c_2}(-1)(-2)(-3)\,(-c_2 t)^3\,(1+c_2 t - c_2 ts)^{-4}\right\}$$

$$\therefore \quad G^{(x)}(s,t) = \frac{1}{\theta(t)}\,\frac{p_2}{c_2}\, x!\,(c_2 t)^x\,(1+c_2 t - c_2 ts)^{-(x+1)}$$

$$= \frac{1}{\theta(t)}\,\frac{p_2}{c_2}\, x!\,\left(\frac{c_2 t}{(1+c_2 t - c_2 ts)}\right)^x \frac{1}{(1+c_2 t - c_2 ts)}, \qquad x = 2, 3, \ldots \qquad (4.21)$$

The pmfs of the iid random variables are

$$g_0(t) = 0 \tag{4.22a}$$

$$g_1(t) = \frac{1}{\theta(t)} \left(p_1 t + \frac{p_2}{c_2} \frac{c_2 t}{(1+c_2 t)^2} \right) \tag{4.22b}$$

$$g_x(t) = \frac{1}{\theta(t)} \frac{p_2}{c_2} \left(\frac{c_2 t}{(1+c_2 t}\right)^x \frac{1}{(1+c_2 t)} \qquad x = 2, 3, ... \tag{4.22c}$$

And so

$$\frac{g_x(t)}{g_{x-1}(t)} = \left(\frac{c_2 t}{1+c_2 t}\right)^x \left(\frac{(1+c_2 t)}{(c_2 t)}\right)^{x-1} = \frac{c_2 t}{1+c_2 t}$$
$$= \left(\frac{c_2 t}{1+c_2 t} + \frac{0}{x}\right) \qquad x = 2, 3, ...$$

which is Panjer's recursive form with

$$a = \frac{c_2 t}{1+c_2 t} \quad and \quad b = 0$$

The recursive form for the convolution of compound Poisson distribution is:

$$np_n(t) = \theta(t) \sum_{x=1}^{n} x \, g_x(t) \, p_{n-x}(t)$$
$$= \theta(t) \, g_1(t) \, p_{n-1}(t) + \theta(t) \sum_{x=2}^{n} x \, g_x(t) \, p_{n-x}(t)$$
$$= \left(p_1 t + \frac{p_2}{c_2} \frac{c_2 t}{(1+c_2 t)^2} \right) p_{n-1}(t) + \sum_{x=2}^{n} x \, \frac{p_2}{c_2} \left(\frac{c_2 t}{(1+c_2 t}\right)^x \frac{1}{(1+c_2 t)} \, p_{n-x}(t)$$
$$= \left(p_1 t + \frac{p_2}{c_2} \frac{c_2 t}{(1+c_2 t)^2} \right) p_{n-1}(t) +$$
$$\frac{1}{(1+c_2 t)} \frac{p_2}{c_2} \sum_{x=2}^{n} x \left(\frac{c_2 t}{(1+c_2 t}\right)^x p_{n-x}(t), \qquad n = 1, 2, ... \tag{4.23}$$

4.1.4 *The case of* $a_1 = 0$ *and* $a_2 \to \infty$

$$h(t) = \theta'(t) = p_1 + \lim_{a_2 \to \infty} p_2 \, (1+c_2 t)^{-a_2} \qquad p_1 > 0, \quad p_2 > 0, \quad c_2 > 0$$
$$= p_1 + \lim_{a_2 \to \infty} p_2 \sum_{k=0}^{\infty} \binom{-a_2}{k} (c_2 t)^k$$
$$h(t) = p_1 + p_2 \, e^{-bt} \tag{4.24}$$

where

$$b = \lim_{a_2 \to \infty} a_2 \, c_2$$

The second hazard function is that of the Gompertz distribution and the sum is known as the Gompertz-Makeham hazard function.

$$\therefore \qquad \theta_1(t) = p_1 t \quad ; \quad \theta_2(t) = \frac{p_2}{b} \left(1 - e^{-bt}\right)$$
$$\theta_1(t - ts) = p_1 * (t - ts) \quad ; \quad \theta_2(t - ts) = \frac{p_2}{b} \left(1 - e^{-bt(1-s)}\right)$$

The pgf of the convolution is:

$$H(s,t) = e^{-p_1 t(1-s)} \, e^{-\frac{p_2}{b} \left(1 - e^{-bt(1-s)}\right)} \tag{4.25}$$

Thus the sum of hazard functions of exponential distribution and that of the Gompertz distribution gives rise to a convolution of the Poisson distribution and the Neyman type A (Poisson-Poisson) distribution.

The pgf of the iid random variables of the convolution of the compound Poisson distribution is:

$$\begin{aligned}
G(s,t) &= 1 - \frac{\theta(t-ts)}{\theta(t)} = 1 - \left\{ \frac{p_1 * (t-ts) + \frac{p_2}{b}\left(1 - e^{-bt(1-s)}\right)}{\theta(t)} \right\} \\
&= 1 - \left\{ \frac{p_1 t - p_1 ts + \frac{p_2}{b}\left(1 - e^{-bt}\, e^{bts}\right)}{\theta(t)} \right\} \\
&= 1 - \left\{ \frac{p_1 t - p_1 ts + \frac{p_2}{b}\left(1 - e^{-bt}\, e^{bts}\right)}{\theta(t)} \right\} \\
&= 1 - \left\{ \frac{p_1 t - p_1 ts + \frac{p_2}{b} - \frac{p_2}{b}\, e^{-bt} \sum_{x=0}^{\infty} \frac{(bts)^x}{x!}}{\theta(t)} \right\} \\
&= \frac{\theta(t) - p_1 t + p_1 ts - \frac{p_2}{b} + \frac{p_2}{b}\, e^{-bt} \sum_{x=0}^{\infty} \frac{(bts)^x}{x!}}{\theta(t)}
\end{aligned}$$

But

$$\theta(t) = p_1 t + \frac{p_2}{b}\left(1 - e^{-bt}\right)$$

Therefore

$$\begin{aligned}
G(s,t) &= \frac{p_1 t + \frac{p_2}{b}\left(1 - e^{-bt}\right) - p_1 t + p_1 ts - \frac{p_2}{b} + \frac{p_2}{b}\, e^{-bt} \sum_{x=0}^{\infty} \frac{(bts)^x}{x!}}{\theta(t)} \\
&= \frac{-\frac{p_2}{b}\, e^{-bt} + p_1 ts + \frac{p_2}{b}\, e^{-bt} \sum_{x=0}^{\infty} \frac{(bts)^x}{x!}}{\theta(t)} \\
&= \frac{p_1 ts + \frac{p_2}{b}\, e^{-bt} \sum_{x=1}^{\infty} \frac{(bts)^x}{x!}}{\theta(t)}
\end{aligned} \tag{4.26}$$

By differentiating G(s,t)

$$\begin{aligned}
G'(s,t) &= \frac{p_1 t + \frac{p_2}{b}\, e^{-bt} \sum_{x=1}^{\infty} \frac{(bt)^x\, s^{x-1}}{(x-1)!}}{\theta(t)} \\
&= \frac{p_1 t + \frac{p_2}{b}\, e^{-bt}\, (bt) \sum_{x=1}^{\infty} \frac{(bt)^{x-1}\, s^{x-1}}{(x-1)!}}{\theta(t)} \\
G''(s,t) &= \frac{\frac{p_2}{b}\, e^{-bt}\, (bt)^2 \sum_{x=2}^{\infty} \frac{(bt)^{x-2}\, s^{x-2}}{(x-2)!}}{\theta(t)} \\
G^x(s,t) &= \frac{\frac{p_2}{b}\, e^{-bt}\, (bt)^x \sum_{x=2}^{\infty} \frac{(bt)^{x-x}\, s^{x-x}}{(x-x)!}}{\theta(t)} \\
&= \frac{\frac{p_2}{b}\, e^{-bt}\, (bt)^x}{\theta(t)}
\end{aligned} \tag{4.27}$$

and the pmfs of the iid random variables are

$$g_0(t) = 0 \tag{4.28a}$$

$$g_1(t) = \frac{p_1 t + \frac{p_2}{b}\, e^{-bt}\, bt}{\theta(t)}$$

$$= \frac{1}{\theta(t)}\, \left\{p_1 t + p_2 t\, e^{-bt}\right\} \tag{4.28b}$$

$$g_x(t) = \frac{1}{\theta(t)}\, \frac{p_2\, e^{-bt}}{b}\, \frac{(bt)^x}{x!} \qquad x = 2, 3, \ldots \tag{4.28c}$$

By power series expansion and using (4.22), the pmfs of the iid random variables are

$$g_0 = 0$$

the coefficient of s^0

$$g_1(t) = \frac{1}{\theta(t)}\, \left\{p_1 t + p_2 t\, e^{-bt}\right\}$$

the coefficient of s^1

$$g_x(t) = \frac{1}{\theta(t)}\, \frac{p_2\, e^{-bt}}{b}\, \frac{(bt)^x}{x!} \qquad x = 2, 3, \ldots$$

the coefficient of s^x

$$\therefore \qquad \frac{g_x(t)}{g_{x-1}(t)} = \frac{1}{x}\, bt$$

$$= 0 + \frac{bt}{x} \qquad x = 1, 2, \ldots$$

which is in Panjer's recursive form, with parameters 0 and bt

The recursive form for the convolution of compound Poisson distribution is:

$$(n+1)\, p_{n+1}(t) = \theta(t) \sum_{i=0}^{n} (i+1)\, g_{i+1}(t)\ \ p_{n-i}(t) \qquad n = 0, 1, 2, \ldots$$

$$= \theta(t) g_1(t)\, p_n(t) + \theta(t) \sum_{i=1}^{n} (i+1)\, g_{i+1}(t)\ \ p_{n-i}(t)$$

$$= \theta(t) \left\{ \frac{1}{\theta(t)} \left(p_1 t + p_2 t e^{-bt}\right) \right\} p_n(t) + \theta(t) \sum_{i=1}^{n} (i+1) \left\{ \frac{1}{\theta(t)}\, \frac{p_2 e^{-bt}}{b}\, \frac{(bt)^{i+1}}{(i+1)!} \right\}\ p_{n-i}(t)$$

$$= \left(p_1 t + p_2 t e^{-bt}\right) p_n(t) + \sum_{i=1}^{n} (i+1) \left\{ \frac{p_2 e^{-bt}}{b}\, \frac{(bt)^{i+1}}{(i+1)!} \right\}\ p_{n-i}(t)$$

$$= \left(p_1 t + p_2 t e^{-bt}\right) p_n(t) + \frac{p_2}{b}\ bt \sum_{i=1}^{n} e^{-bt} \left\{ \frac{(bt)^i}{i!} \right\}\ p_{n-i}(t)$$

$$= \left(p_1 t + p_2 t e^{-bt}\right) p_n(t) + p_2 t \sum_{i=1}^{n} e^{-bt}\, \frac{(bt)^i}{i!}\ p_{n-i}(t) \qquad n = 0, 1, 2, \ldots \tag{4.29}$$

4.2 *When the first hazard function is that of a Pareto*

4.2.1 *The case of* $a_1 = 1$ *and* $a_2 = 1$

$$h(t) = \frac{p_1}{(1 + c_1 t)} + \frac{p_2}{(1 + c_2 t)} \qquad p_1 > 0, \quad p_2 > 0, \quad c_1 > 0, \quad c_2 > 0$$

The second hazard function is also that of a Pareto.

$$\therefore \qquad \theta_1(t) = \frac{p_1}{c_1}\, In\,(1 + c_1 t) \qquad ; \qquad \theta_2(t) = \frac{p_2}{c_2}\, In\,(1 + c_2 t)$$

$$\theta_1(t - ts) = \frac{p_1}{c_1}\, In\,(1 + c_1 t - c_1 ts) \qquad ; \qquad \theta_2(t - ts) = \frac{p_2}{c_2}\, In\,(1 + c_2 t - c_2 ts)$$

The pgf of the convolution is

$$H(s,t) = e^{-\frac{p_1}{c_1} In(1+c_1t-c_1ts)} \, e^{-\frac{p_2}{c_2} In(1+c_2t-c_2ts)}$$
$$= \left(\frac{1}{1+c_1t-c_1ts}\right)^{\frac{p_1}{c_1}} \left(\frac{1}{1+c_2t-c_2ts}\right)^{\frac{p_2}{c_2}} \tag{4.30}$$

Therefore the sum of hazard functions of two Pareto distributions gives rise to the convolution of two negative binomial distributions.

When $c_1 = c_2 = c$, then we have a single hazard function

$$h(t) = \frac{p_1+p_2}{1+ct} \qquad where \qquad p_1+p_2 > 0 \qquad and \qquad c > 0$$

Its associated Poisson mixture is a negative binomial distribution with parameters $\frac{p_1+p_2}{c}$ *and* $\frac{1}{1+ct}$

.

whose pgf is

$$H(s,t) = \left(\frac{\frac{1}{1+ct}}{1-\frac{ct}{1+ct}s}\right)^{\frac{p_1+p_2}{c}}$$

The pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$G(s,t) = 1 - \frac{1}{\theta(t)}\left\{\frac{p_1}{c_1} In(1+c_1t-c_1ts) + \frac{p_2}{c_2} In(1+c_2t-c_2ts)\right\}$$

and

$$G^x(s,t) = \frac{1}{\theta(t)}\left\{(x-1)!\,\frac{p_1}{c_1}\left(\frac{c_1t}{1+c_1t}\right)^x + (x-1)!\,\frac{p_2}{c_2}\left(\frac{c_2t}{1+c_2t}\right)^x\right\}$$

refer to (4.11)

The pmfs of the iid random variables are

$$g_0(t) = 0 \tag{4.31a}$$
$$g_x(t) = \frac{1}{x}\frac{1}{\theta(t)}\left\{\frac{p_1}{c_1}\left(\frac{c_1t}{1+c_1t}\right)^x + \frac{p_2}{c_2}\left(\frac{c_2t}{1+c_2t}\right)^x\right\} \tag{4.31b}$$

where

$$\theta(t) = In\left((1+c_1t)^{\frac{p_1}{c_1}}(1+c_2t)^{\frac{p_2}{c_2}}\right)$$

$$\frac{g_x(t)}{g_{x-1}(t)} = \left\{\frac{\frac{p_1}{c_1}\left(\frac{c_1}{1+c_1}\right)^x + \frac{p_2}{c_2}\left(\frac{c_2t}{1+c_2t}\right)^x}{\frac{p_1}{c_1}\left(\frac{c_1}{1+c_1}\right)^{x-1} + \frac{p_2}{c_2}\left(\frac{c_2t}{1+c_2t}\right)^{x-1}}\right\}\frac{x-1}{x}$$

which is not in Panjer's recursive form, since the term in the curled bracket is not a constant.

However, in the case where $c_1 = c_2 = c$

$$\frac{g_x(t)}{g_{x-1}(t)} = \left\{\frac{x-1}{x}\,\frac{\frac{p_1+p_2}{c}\left(\frac{ct}{1+ct}\right)^x}{\frac{p_1+p_2}{c}\left(\frac{ct}{1+ct}\right)^{x-1}}\right\}$$
$$= \left\{\frac{x-1}{x}\left(\frac{ct}{1+ct}\right)\right\}$$

which is Panjer's recursive model with

$$a = \frac{ct}{1+ct} \qquad and \qquad b = -\frac{ct}{1+ct}$$

The recursive form for the convolution of compound Poisson distribution is:

$$\begin{aligned} np_n &= \theta(t) \sum_{x=1}^{n} x \, g_x(t) \, p_{n-x}(t) \\ &= \theta(t) \sum_{x=1}^{n} x \, \frac{1}{x} \, \frac{1}{\theta(t)} \left\{ \frac{p_1}{c_1} \, (\frac{c_1 t}{1+c_1 t})^x + \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t} \right)^x \right\} p_{n-x}(t) \\ &= \sum_{x=1}^{n} \left\{ \frac{p_1}{c_1} \, (\frac{c_1 t}{1+c_1 t})^x + \frac{p_2}{c_2} \left(\frac{c_2 t}{1+c_2 t} \right)^x \right\} p_{n-x}(t) \\ &= \frac{p_1}{c_1} \sum_{x=1}^{n} (\frac{c_1 t}{1+c_1 t})^x \, p_{n-x}(t) + \frac{p_2}{c_2} \sum_{x=1}^{n} \left(\frac{c_2 t}{1+c_2 t} \right)^x p_{n-x}(t) \qquad n = 1, 2, 3, ... \end{aligned} \tag{4.32}$$

4.2.2 *The case of* $a_1 = 1$ *and* $a_2 = \frac{1}{2}$

$$h(t) = \theta'(t) = \frac{p_1}{(1+c_1 t)} + \frac{p_2}{(1+c_2 t)^{\frac{1}{2}}} \qquad p_1 > 0, \quad p_2 > 0, \quad c_1 > 0, \quad c_2 > 0$$

where the second hazard function is that of an exponential-inverse Gaussian distribution.

This sum of hazard functions can be obtained by considering reciprocal inverse Gaussian as the a mixing distribution in an exponential mixture as described below:

Let $\mathrm{X} = \frac{1}{\Lambda}$ where X is a random variable from an inverse-Gaussian distribution. We wish to determine the distribution of Λ.

The pdf of Λ is

$$\begin{aligned} g(\lambda) &= f(x)|J| \\ &= f(x)|\frac{dx}{d\lambda}| \\ \therefore \qquad g(\lambda) &= f(x)|-\frac{1}{\lambda^2}| \end{aligned}$$

The pdf of an inverse Gaussian distribution is

$$f(x) = \left(\frac{\phi}{2\,\pi\,x^3} \right)^{\frac{1}{2}} \exp\left\{ -\frac{\phi\,(x-\mu)^2}{2\,\mu^2\,x} \right\} \qquad x > 0, \; \phi > 0, \; -\infty < \mu < \infty$$

The pdf of the reciprocal inverse Gaussian distribution is

$$\begin{aligned} g(\lambda) &= \left(\frac{\phi \lambda^3}{2\pi} \right)^{\frac{1}{2}} \exp\left\{ -\frac{\phi(\frac{1}{\lambda} - \mu)^2}{2\,\mu^2\,\frac{1}{\lambda}} \right\} \frac{1}{\lambda^2} \\ &= \left(\frac{\phi}{2\pi\lambda} \right)^{\frac{1}{2}} \exp\left\{ -\frac{\phi\lambda}{2\mu^2} \left(\frac{1}{\lambda^2} - \frac{2\mu}{\lambda} + \mu^2 \right) \right\} \\ &= \left(\frac{\phi}{2\pi\lambda} \right)^{\frac{1}{2}} \exp\left\{ -\frac{\phi}{2\mu^2\lambda} \left(1 - 2\mu\lambda + (\mu\lambda)^2 \right) \right\} \\ &= \left(\frac{\phi}{2\,\pi\,\lambda} \right)^{\frac{1}{2}} \exp\left\{ -\frac{\phi(1-\mu\,\lambda)^2}{2\,\mu^2\,\lambda} \right\} \qquad \lambda > 0 \end{aligned}$$

The survival function of an exponential mixture is the Laplace transform of the mixing distribution.

That is

$$\begin{aligned}
S(t) &= L_\Lambda(t) \\
&= \int_0^\infty e^{-t\,\lambda}\, g(\lambda)\, d\lambda \\
&= \int_0^\infty e^{-t\,\lambda}\, (\frac{\phi}{2\,\pi\,\lambda})^{\frac{1}{2}}\, \exp\left\{-\frac{\phi(1-\mu\,\lambda)^2}{2\,\mu^2\,\lambda}\right\} d\lambda \\
&= (\frac{\phi}{2\,\pi})^{\frac{1}{2}} \int_0^\infty \lambda^{-\frac{1}{2}} \left\{-t\,\lambda - \frac{\phi(1-\mu\,\lambda)^2}{2\,\mu^2\,\lambda}\right\} d\lambda \\
&= (\frac{\phi}{2\,\pi})^{\frac{1}{2}} \int_0^\infty \lambda^{-\frac{1}{2}}\, \exp\left\{-t\,\lambda - \frac{\phi(1-2\mu\,\lambda+\mu^2\,\lambda^2)}{2\,\mu^2\,\lambda}\right\} d\lambda \\
&= (\frac{\phi}{2\,\pi})^{\frac{1}{2}} \int_0^\infty \lambda^{-\frac{1}{2}}\, \exp\left\{-t\,\lambda - \frac{\phi}{2\mu^2\,\lambda} + \frac{\phi}{\mu} - \frac{\phi\,\lambda}{2}\right\} d\lambda \\
&= (\frac{\phi}{2\,\pi})^{\frac{1}{2}} \int_0^\infty \lambda^{-\frac{1}{2}}\, \exp\left\{-(\frac{\phi}{2}+t)\,\lambda - \frac{\phi}{2\mu^2\,\lambda} + \frac{\phi}{\mu}\right\} d\lambda \\
&= (\frac{\phi}{2\,\pi})^{\frac{1}{2}}\, e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{-\frac{1}{2}}\, \exp\left\{-(\frac{\phi}{2}+t)\,\lambda - \frac{\phi}{2\mu^2}\,\frac{1}{\lambda}\right\} d\lambda \\
&= (\frac{\phi}{2\,\pi})^{\frac{1}{2}}\, e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{-\frac{1}{2}} \left\{-(\frac{\phi}{2}+t)\,(\lambda + \frac{\phi}{2\mu^2\,(\frac{\phi}{2}+t)}\,\frac{1}{\lambda})\right\} d\lambda \\
&= (\frac{\phi}{2\,\pi})^{\frac{1}{2}}\, e^{\frac{\phi}{\mu}} \int_0^\infty \lambda^{-\frac{1}{2}}\, \exp\left\{-(\frac{\phi}{2}+t)\,(\lambda + \frac{\phi}{\mu^2\,(\phi+2t)}\,\frac{1}{\lambda})\right\} d\lambda
\end{aligned}$$

Let $\lambda = \sqrt{\frac{\phi}{\mu^2\,(\phi+2t)}}\, z \qquad \therefore \qquad d\lambda = \sqrt{\frac{\phi}{\mu^2\,(\phi+2t)}}\, dz$

$$\begin{aligned}
L_\lambda(t) &= \left(\frac{\phi}{2\,\pi}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}} \left(\sqrt{\frac{\phi}{\mu^2\,(\phi+2t)}}\right)^{\frac{1}{2}} \int_0^\infty z^{\frac{1}{2}-1} \exp\left\{-\frac{1}{2}\sqrt{\frac{\phi(\phi+2t)}{\mu^2}}\,(z+\frac{1}{z})\right\} dz \\
&= \left(\frac{\phi}{2\,\pi}\right)^{\frac{1}{2}} \left(\sqrt{\frac{\phi}{\mu^2\,(\phi+2t)}}\right)^{\frac{1}{2}} e^{\frac{\phi}{\mu}}\, 2\, K_{\frac{1}{2}}\left(\sqrt{\frac{\phi(\phi+2t)}{\mu^2}}\right)
\end{aligned}$$

where $K_v(\omega)$ is the modified Bessel function of the third kind of order v.

But $\qquad K_{\frac{1}{2}}(\omega) = \sqrt{\frac{\pi}{2\,\omega}}\, e^{-\omega}$ (Watson, 1952).

In this case $\omega = \sqrt{\sqrt{\frac{\phi(\phi+2t)}{\mu^2}}}$

Therefore

$$L_\lambda(t) = e^{\frac{\phi}{\mu}} \left(\frac{\phi}{2\,\pi} \sqrt{\frac{\phi}{\mu^2\,(\phi+2t)}} \right)^{\frac{1}{2}} \; 2 \sqrt{\frac{\pi}{2\,(\sqrt{\frac{\phi(\phi+2t)}{\mu^2}})}} \exp\left\{ -(\sqrt{\frac{\phi(\phi+2t)}{\mu^2}}) \right\}$$

$$= 2\, e^{\frac{\phi}{\mu}} \left(\frac{\phi}{4} \sqrt{\frac{\phi}{\mu^2\,(\phi+2t)} \quad \frac{1}{\frac{\phi(\phi+2t)}{\mu^2}}} \right)^{\frac{1}{2}} \exp\left\{ -(\sqrt{\frac{\phi(\phi+2t)}{\mu^2}}) \right\}$$

$$= 2\, e^{\frac{\phi}{\mu}} \left(\frac{\phi}{4} \sqrt{\frac{1}{(\phi+2t)^2}} \right)^{\frac{1}{2}} \exp\left\{ -(\sqrt{\frac{\phi(\phi+2t)}{\mu^2}}) \right\}$$

$$= 2\, e^{\frac{\phi}{\mu}} \left(\frac{\phi}{4\,(\phi+2t)} \right)^{\frac{1}{2}} \exp\left\{ -(\sqrt{\frac{\phi(\phi+2t)}{\mu^2}}) \right\}$$

$$= e^{\frac{\phi}{\mu}} \left(\frac{\phi}{\phi+2t} \right)^{\frac{1}{2}} \exp\left\{ -(\sqrt{\frac{\phi(\phi+2t)}{\mu^2}}) \right\}$$

$$= (1+\frac{2}{\phi}\,t)^{-\frac{1}{2}} \exp\left\{ -(\frac{\phi^2}{\mu^2}\,(\frac{\phi+2t}{\phi}))^{\frac{1}{2}} \right\} e^{\frac{\phi}{\mu}} = (1+\frac{2}{\phi}\,t)^{-\frac{1}{2}} \quad \exp\left\{ \frac{\phi}{\mu} - \frac{\phi}{\mu}\,(\frac{\phi+2t}{\phi})^{\frac{1}{2}} \right\}$$

$$= (1+\frac{2}{\phi}\,t)^{-\frac{1}{2}} \exp\left\{ \frac{\phi}{\mu} - \frac{\phi}{\mu}\,(1+\frac{2}{\phi}\,t)^{\frac{1}{2}} \right\}$$

$$= (1+\frac{2}{\phi}\,t)^{-\frac{1}{2}} \quad \exp\left\{ \frac{\phi}{\mu}\,(1-(1+\frac{2}{\phi}\,t)^{\frac{1}{2}}) \right\}$$

$$L'_\lambda(t) = (1+\frac{2}{\phi}\,t)^{-\frac{1}{2}} \quad \exp\left\{ \frac{\phi}{\mu}\,(1-(1+\frac{2}{\phi}\,t)^{\frac{1}{2}}) \right\} \,.\, \frac{\phi}{\mu}\,\frac{1}{2}\,(1+\frac{2}{\phi}\,t)^{-\frac{1}{2}}\,(-\frac{2}{\phi}) +$$

$$\exp\left\{ \frac{\phi}{\mu}\,(1-(1+\frac{2}{\phi}\,t)^{\frac{1}{2}}) \right\} \,(-\frac{1}{2})\,(1+\frac{2}{\phi}\,t)^{-\frac{3}{2}}\,(\frac{2}{\phi})$$

$$= (1+\frac{2}{\phi}\,t)^{-\frac{1}{2}} \quad \exp\left\{ \frac{\phi}{\mu}\,(1-(1+\frac{2}{\phi}\,t)^{\frac{1}{2}}) \right\} \,.\, \left(-\frac{1}{\mu}\,(1+\frac{2}{\phi}\,t)^{-\frac{1}{2}} \; - \frac{1}{\phi}\,(1+\frac{2}{\phi}\,t)^{-1} \right)$$

and

$$h(t) = -\frac{L'_\lambda(t)}{L_\lambda}$$

$$= \frac{1}{\mu}\,(1+\frac{2}{\phi}\,t)^{-\frac{1}{2}} + \frac{1}{\phi}\,(1+\frac{2}{\phi}\,t)^{-1}$$

$$= \frac{\frac{1}{\mu}}{(1+\frac{2}{\phi}\,t)^{\frac{1}{2}}} + \frac{\frac{1}{\phi}}{(1+\frac{2}{\phi}\,t)}$$

$$\equiv \frac{p_1}{1+c_1 t} + \frac{p_2}{(1+c_2\,t)^{\frac{1}{2}}}$$

$$\theta_1(t) = \frac{p_1}{c_1}\,In(1+c_1 t) \quad ; \quad \theta_2(t) = \frac{2p_2}{c_2}\,\left((1+c_2 t)^{\frac{1}{2}} - 1\right)$$

$$\theta_1(t-ts) = \frac{p_1}{c_1}\,In(1+c_1 t - c_1 ts) \quad ; \quad \theta_2(t-ts) = \frac{2p_2}{c_2}\,\left((1+c_2 t - c_2 ts)^{\frac{1}{2}} - 1\right)$$

The pgf of the convolution is

$$H(s,t) = e^{-\frac{p_1}{c_1}\,In(1+c_1 t - c_1 ts)}\; e^{-\frac{2p_2}{c_2}\,\left((1+c_2 t - c_2 ts)^{\frac{1}{2}} - 1\right)}$$

$$= (\frac{\frac{1}{1+c_1 t}}{1-\frac{ct}{1+c_1 t}s})^{\frac{p_1}{c_1}}\; e^{-\frac{2p_2}{c_2}\,\left((1+c_2 t - c_2 ts)^{\frac{1}{2}} - 1\right)} \tag{4.33}$$

The sum of the hazard functions of the exponential-gamma and that of the exponential-inverse Gaussian distribution, therefore, gives rise to the convolution of the negative binomial and the Poisson-inverse Gaussian (Sichel) distributions.

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The pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$G(s,t) = 1 - \frac{1}{\theta(t)} \left(\frac{p_1}{c_1} \, In(1 + c_1 t - c_1 ts) + \frac{2p_2}{c_2} \, [(1 + c_2 t - c_2 ts)^{\frac{1}{2}} - 1] \right)$$

$$\therefore \quad G^x(s,t) = \frac{(x-1)!}{\theta(t)} \left\{ \frac{p_1}{c_1} \left(\frac{c_1 t}{1 + c_1 t - c_2 ts}\right)^x + \binom{\frac{1}{2} + x - 1 - 1}{x-1} \frac{p_2}{c_2} \left(\frac{c_2 t}{(1 + c_2 t - c_2 ts)}\right)^x \left(\frac{1}{(1 + c_2 t - c_2 ts)}\right)^{-\frac{1}{2}} \right\}$$

$$= \frac{(x-1)!}{\theta(t)} \left\{ \frac{p_1}{c_1} \left(\frac{c_1 t}{1 + c_1 t - c_2 ts}\right)^x \right\} + \frac{(x-1)!}{\theta(t)} \left\{ p_2 t \binom{\frac{1}{2} + x - 1 - 1}{x-1} \left(\frac{c_2 t}{(1 + c_2 t - c_2 ts)}\right)^{x-1} \left(\frac{1}{(1 + c_2 t - c_2 ts)}\right)^{\frac{1}{2}} \right\} \tag{4.34}$$

Therefore, the pmfs of the iid random variables are

$$g_0(t) = 0 \tag{4.35a}$$

$$g_x(t) = \frac{1}{x\,\theta(t)} \left\{ \frac{p_1}{c_1} \left(\frac{c_1 t}{1 + c_1 t}\right)^x + p_2 t \binom{\frac{1}{2} + x - 1 - 1}{x-1} \left(\frac{c_2 t}{(1 + c_2 t)}\right)^{x-1} \left(\frac{1}{(1 + c_2 t)}\right)^{\frac{1}{2}} \right\} \quad x = 1, 2, 3, \ldots \tag{4.35b}$$

where

$$\theta(t) = \frac{p_1}{c_1} \, In(1 + c_1 t) + \frac{2p_2}{c_2} \, \left((1 + c_2 t)^{\frac{1}{2}} - 1\right)$$

Panjer's recursive model does not hold in general case, unless $p_1 = p_2 = p$ say and $c_1 = c_2 = c$.

In this case the Panjer's model is

$$\frac{g_x(t)}{g_{x-1}(t)} = \frac{x-1}{x} \left(\frac{ct}{1 + ct}\right)$$

The recursive form for the convolution of compound Poisson distributions is:

$$n \, p_n(t) = \theta(t) \sum_{x=1}^{n} x \, g_x(t) \; p_{n-x}(t)$$

$$= \sum_{x=1}^{n} \left\{ \frac{p_1}{c_1} \left(\frac{c_1 t}{1 + c_1 t}\right)^x + p_2 t \binom{\frac{1}{2} + x - 1 - 1}{x-1} \left(\frac{c_2 t}{(1 + c_2 t)}\right)^{x-1} \left(\frac{1}{(1 + c_2 t)}\right)^{\frac{1}{2}} \right\} p_{n-x}(t) \quad n = 1, 2, 3, \ldots \tag{4.36}$$

4.2.3 *The case of* $a_1 = 1$ *and* $a_2 = 2$

$$h(t) = \theta'(t) = \frac{p_1}{(1 + c_1 t)} + \frac{p_2}{(1 + c_2 t)^2} \qquad p_1 > 0, \quad p_2 > 0, \quad c_1 > 0, \quad c_2 > 0$$

where the second hazard function will be referred to as Polya-Aeppli hazard function (Wakoli and Ottieno 2015, p. 234).

$$\therefore \qquad \theta_1(t) = \frac{p_1}{c_1} \, In(1 + c_1 t) \quad ; \quad \theta_2(t) = \frac{p_2}{c_2} \left(1 - (1 + c_2 t)^{-1}\right)$$

$$\theta_1(t - ts) = \frac{p_1}{c_1} \, In(1 + c_1 t - c_1 ts) \quad ; \quad \theta_2(t - ts) = \frac{p_2}{c_2} \left(1 - (1 + c_2 t - c_1 ts)^{-1}\right)$$

The pgf of the convolution is

$$H(s,t) = e^{-\left(\frac{p_1}{c_1} \, In(1 + c_1 t - c_1 ts)\right)} \, e^{\frac{-p_2}{c_2}\left(1 - (1 + c_2 t - c_2 ts)^{-1}\right)}$$

$$= \left(\frac{\frac{1}{1 + c_1 t}}{1 - \frac{ct}{1 + c_1 t} s} \right)^{\frac{p_1}{c_1}} e^{-\frac{p_2}{c_2}\left(1 - (1 + c_2 t - c_2 ts)^{-1}\right)} \tag{4.37}$$

The sum of hazard function of a Pareto distribution and the Polya-Aeppli hazard function, therefore, gives rise to the convolution of the negative binomial distribution and the Polya-Aeppli distribution.

The pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$\begin{aligned} G(s,t) &= 1-\frac{\theta(t-ts)}{\theta(t)} = 1-\frac{1}{\theta(t)}\left(\frac{p_1}{c_1}\,In(1+c_1t-c_1ts)+\frac{p_2}{c_2}\left(1-(1+c_2t-c_1ts)^{-1}\right)\right) \\ G^{(x)}(s,t) &= \frac{(x-1)!}{\theta(t)}\left\{\frac{p_1}{c_1}\,(\frac{c_1t}{1+c_1t-c_1ts})^x\right\}+\frac{x!}{\theta(t)}\left\{\frac{p_2}{c_2}(\frac{c_2t}{1+c_2t-c_2ts})^x\frac{c_2t}{1+c_2t-c_2ts}\right\} \\ &= \frac{1}{\theta(t)}\left((x-1)!\,\frac{p_1}{c_1}\,(\frac{c_1t}{1+c_1t-c_1ts})^x+\frac{p_2}{c_2}\,x!\,(\frac{c_2t}{(1+c_2t-c_2ts)})^x\,\frac{1}{(1+c_2t-c_2ts)}\right) \end{aligned} \tag{4.38}$$

and the pmfs of the iid random variables are

$$g_0(t) = 0 \tag{4.39a}$$

$$g_x(t) = \frac{1}{\theta(t)}\,\frac{1}{x}\,\frac{p_1}{c_1}\left(\frac{c_1t}{1+c_1t}\right)^x+\frac{1}{\theta(t)}\,\frac{p_2}{c_2}\left(\frac{c_2t}{1+c_2t}\right)^x\,\frac{1}{1+c_2t} \quad for \quad x=1,2,.. \tag{4.39b}$$

Again Panjer's recursive model is not satisfied, unless $c_1 = c_2 = c$, in which case

$$\begin{aligned} \frac{g_x(t)}{g_{x-1}(t)} &= \left\{\frac{x-1}{x}\,\frac{\frac{p_1+p_2}{c}\,(\frac{ct}{1+ct})^x}{\frac{p_1+p_2}{c}\,(\frac{ct}{1+ct})^{x-1}}\right\} \\ &= \frac{x-1}{x}\left(\frac{ct}{1+ct}\right) \end{aligned}$$

The recursive form for the convolution of compound Poisson distributions is

$$\begin{aligned} np_n(t) &= \theta(t)\sum_{x=1}^{n} x\,g_x(t)\,p_{n-x}(t) \\ &= \theta(t)\sum_{x=1}^{n} x\,\frac{1}{\theta(t)}\left(\frac{1}{x}\,\frac{p_1}{c_1}\,(\frac{c_1t}{1+c_1t})^x+\frac{p_2}{c_2}\,(\frac{c_2t}{(1+c_2t)})^x\,\frac{1}{(1+c_2t)}\right)p_{n-x}(t) \\ &= \frac{p_1}{c_1}\sum_{x=1}^{n}\left(\frac{c_1t}{1+c_1t}\right)^x p_{n-x}(t)+\frac{p_2}{c_2}\sum_{x=1}^{n} x\left(\frac{c_2t}{(1+c_2t)}\right)^x\frac{1}{(1+c_2t)})\,p_{n-x}(t) \qquad n=1,2,3,... \end{aligned} \tag{4.40}$$

4.3 *The case of* $a_1 = \frac{1}{2}$ *and* $a_2 = \frac{1}{2}$

$$h(t) = \theta'(t) = \frac{p_1}{(1+c_1t)^{\frac{1}{2}}}+\frac{p_2}{(1+c_2t)^{\frac{1}{2}}} \qquad p_1>0, \quad p_2>0, \quad c_1>0, \quad c_2>0$$

Both hazard functions belong to the exponential-inverse Gaussian distributions.

$$\begin{aligned} \therefore \quad \theta_1(t) &= \frac{2p_1}{c_1}\left((1+c_1t)^{\frac{1}{2}}-1\right) \quad ; \quad \theta_2(t) = \frac{2p_2}{c_2}\left((1+c_2t)^{\frac{1}{2}}-1\right) \\ \theta_1(t-ts) &= \frac{2p_1}{c_1}\left((1+c_1t-c_1ts)^{\frac{1}{2}}-1\right) \quad ; \quad \theta_2(t-ts) = \frac{2p_2}{c_2}\left((1+c_2t-c_1ts)^{\frac{1}{2}}-1\right) \end{aligned}$$

The pgf of the convolution is

$$H(s,t) = e^{-\frac{2p_1}{c_1}\left((1+c_1t-c_1ts)^{\frac{1}{2}}-1\right)}\,e^{-\frac{2p_2}{c_2}\left((1+c_2t-c_1ts)^{\frac{1}{2}}-1\right)} \tag{4.41}$$

and therefore the sum of two hazard functions of exponential-inverse Gaussian distributions gives rise to the convolution of two Poisson-inverse Gaussian distributions.

The pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$G(s,t) = 1 - \frac{1}{\theta(t)} \left\{ \frac{2p_1}{c_1} \left((1 + c_1 t - c_1 ts)^{\frac{1}{2}} - 1 \right) + \frac{2p_2}{c_2} \left((1 + c_2 t - c_2 ts)^{\frac{1}{2}} - 1 \right) \right\}$$

$$\begin{aligned} G^x(s,t) &= \frac{1}{\theta(t)} (x-1)! \{ p_1 t \binom{-\frac{1}{2}}{x-1} (-\frac{c_1 t}{(1 + c_1 t - c_1 ts})^{x-1} (1 + c_1 t - c_1 ts)^{-\frac{1}{2}} + \\ & \quad p_2 t \binom{-\frac{1}{2}}{x-1} (-\frac{c_2 t}{(1 + c_2 t - c_2 ts})^{x-1} (1 + c_2 t - c_2 ts)^{-\frac{1}{2}} \} \\ &= \frac{1}{\theta(t)} (x-1)! \binom{-\frac{1}{2}}{x-1} \{ p_1 t (-\frac{c_1 t}{(1 + c_1 t - c_1 ts})^{x-1} (1 + c_1 t - c_1 ts)^{-\frac{1}{2}} + \\ & \quad p_2 t (-\frac{c_2 t}{(1 + c_2 t - c_2 ts})^{x-1} (1 + c_2 t - c_2 ts)^{-\frac{1}{2}} \} \end{aligned} \tag{4.42}$$

and the pmfs of the iid random variables are

$$g_0(t) = 0 \tag{4.43a}$$

$$\begin{aligned} g_x(t) &= \frac{1}{\theta(t)} \frac{1}{x} \{ p_1 t \binom{-\frac{1}{2}}{x-1} (-\frac{c_1 t}{(1 + c_1 t})^{x-1} (1 + c_1 t)^{-\frac{1}{2}} + \\ & \quad p_2 t \binom{-\frac{1}{2}}{x-1} (-\frac{c_2 t}{(1 + c_2 t})^{x-1} (1 + c_2 t)^{-\frac{1}{2}} \} \end{aligned} \tag{4.43b}$$

In the case where $c_1 = c_2 = c$

$$\begin{aligned} \frac{g_x(t)}{g_{x-1}(t)} &= \frac{x-1}{x} \frac{\binom{-\frac{1}{2}}{x-1}}{\binom{-\frac{1}{2}}{x-2}} (-\frac{c_2 t}{(1 + c_2 t}) \\ &= (\frac{x-1}{x}) (\frac{\frac{3}{2} - x}{x-1}) (\frac{-c_2 t}{1 + c_2 t}) \\ &= \frac{c_2 t}{(1 + c_2 t)} - \frac{3\, c_2 t}{2\,(1 + c_2 t)} \frac{1}{x} \end{aligned}$$

which is in Panjer's recursive form with

$$a = \frac{c_2 t}{(1 + c_2 t)} \qquad \textit{and} \qquad b = -\frac{3\, c_2 t}{2\,(1 + c_2 t)}$$

The recursive form for the convolution of the compound Poisson distribution is:

$$\begin{aligned} n\, p_n(t) &= \theta(t) \sum_{x=1}^{n} x\, g_x(t)\ p_{n-x}(t) \\ &= \sum_{x=1}^{n} \{ p_1 t \binom{-\frac{1}{2}}{x-1} (-\frac{c_1 t}{(1 + c_1 t})^{x-1} (1 + c_1 t)^{-\frac{1}{2}} + \\ & \quad p_2 t \binom{-\frac{1}{2}}{x-1} (-\frac{c_2 t}{(1 + c_2 t})^{x-1} (1 + c_2 t)^{-\frac{1}{2}} \}\ p_{n-x}(t) \\ &= p_1 t\, (1 + c_1 t)^{-\frac{1}{2}} \sum_{x=1}^{n} \binom{-\frac{1}{2}}{x-1} (-\frac{c_1 t}{(1 + c_1 t})^{x-1} p_{n-x}(t) \\ & \quad + p_2 t\, (1 + c_2 t)^{-\frac{1}{2}} \sum_{x=1}^{n} \binom{-\frac{1}{2}}{x-1} (-\frac{c_2 t}{(1 + c_2 t})^{x-1} p_{n-x}(t) \qquad n = 1, 2, \ldots \end{aligned} \tag{4.44}$$

4.4 *The case of* $a_1 = 2$ *and* $a_2 = 2$

$$h(t) = \theta'(t) = \frac{p_1}{(1 + c_1 t)^2} + \frac{p_2}{(1 + c_2 t)^2} \qquad p_1 > 0, \quad p_2 > 0, \quad c_1 > 0, \quad c_2 > 0$$

Both are the Polya-Aeppli hazard functions

.

$$\therefore \quad \theta_1(t) = \frac{p_1}{c_1}\left(1-(1+c_1t)^{-1}\right) \quad ; \quad \theta_2(t) = \frac{p_2}{c_2}\left(1-(1+c_2t)^{-1}\right)$$
$$\theta_1(t-ts) = \frac{p_1}{c_1}\left(1-(1+c_1t-c_1ts)^{-1}\right) \quad ; \quad \theta_2(t-ts) = \frac{p_2}{c_2}\left(1-(1+c_2t-c_1ts)^{-1}\right)$$

The pgf of the convolution is

$$H(s,t) = e^{-\frac{p_1}{c_1}\left(1-(1+c_1t-c_1ts)^{-1}\right)}\, e^{-\frac{p_2}{c_2}\left(1-(1+c_2t-c_1ts)^{-1}\right)} \tag{4.45}$$

and therefore the sum of two Polya-Aeppli hazard functions gives rise to the convolution of two Polya-Aeppli distributions.

The pgf of the iid random variables of the convolution of the compound Poisson distribution is

$$G(s,t) = 1 - \frac{\theta(t-ts)}{\theta(t)} = 1 - \frac{1}{\theta(t)}\left(\frac{p_1}{c_1}\{1-(1+c_1t-c_1ts)^{-1} + \frac{p_2}{c_2}\{1-(1+c_2t-c_1ts)^{-1}\right)$$
$$\therefore \quad G^{(x)}(s,t) = \frac{x!}{\theta(t)}\left\{\frac{p_1}{c_1}\,(\frac{c_1t}{1+c_1t-c_1ts})^x\,\frac{1}{1+c_1t-c_1ts}\right\} + \frac{x!}{\theta(t)}\left\{\frac{p_2}{c_2}\left(\frac{c_2t}{1+c_2t-c_2ts}\right)^x\frac{1}{(1+c_2t-c_2ts)}\right\} \tag{4.46}$$

and the pmfs of the iid random variables are

$$g_0(t) = 0 \tag{4.47a}$$
$$g_x(t) = \frac{1}{\theta(t)}\left\{\frac{p_1}{c_1}\,(\frac{c_1t}{1+c_1t})^x\,\frac{1}{1+c_1t}\right\} + \frac{1}{\theta(t)}\left\{\frac{p_2}{c_2}\left(\frac{c_2t}{1+c_2t}\right)^x\frac{1}{(1+c_2t)}\right\} \quad for \quad x = 1,2,... \tag{4.47b}$$

Panjer's recursive model does not hold, unless $c_1 = c_2 = c$, in which case

$$\frac{g_x(t)}{g_{x-1}(t)} = (\frac{ct}{1+ct})^x\,(\frac{(1+ct)}{(ct)})^{x-1} = \frac{ct}{1+ct} \qquad x = 2,3,...$$
$$= (\frac{ct}{1+ct} + \frac{0}{x}) \qquad for \quad x = 2,3,...$$

which is Panjer's form with

$$a = \frac{ct}{1+ct} \quad \text{and} \quad b = 0$$

The recursive form for the convolution of the compound Poisson distribution is

$$np_n(t) = \theta(t)\sum_{x=1}^{n} x\, g_x(t)\, p_{n-x}(t) \qquad n = 1,2...$$
$$= \theta(t)\sum_{x=1}^{n} x\,\frac{1}{\theta(t)}\left(\frac{p_1}{c_1}\,(\frac{c_1t}{1+c_1t})^x\,\frac{1}{1+c_1t} + \frac{p_2}{c_2}\left(\frac{c_2t}{1+c_2t}\right)^x\frac{1}{1+c_2t}\right) p_{n-x}(t)$$
$$= \sum_{x=1}^{n}\left(\frac{p_1}{c_1}\,x\,(\frac{c_1t}{1+c_1t})^x\,\frac{1}{1+c_1t} + \frac{p_2}{c_2}\,x\left(\frac{c_2t}{1+c_2t}\right)^x\frac{1}{1+c_2t}\right) p_{n-x}(t)$$
$$= \frac{p_1}{c_1}\,\frac{1}{1+c_1t}\sum_{x=1}^{n} x\left(\frac{c_1t}{1+c_1t}\right)^x p_{n-x}(t) + \frac{p_2}{c_2}\,\frac{1}{1+c_2t}\sum_{x=1}^{n} x\left(\frac{c_2t}{1+c_2t}\right)^x p_{n-x}(t) \qquad for \quad n = 1,2,3,... \tag{4.48}$$

5 Concluding Remarks

Sums of two hazard functions gives rise to convolutions of infinitely divisible mixed Poisson distributions which are also convolutions of compound Poisson distributions. The sums can be extended to more than two hazard functions. The sum of the hazard function of exponential distribution and that of Pareto distribution is the same as the hazard function of exponential-shifted gamma distribution. Similarly, the sum of the hazard function of Pareto and exponential-inverse Gaussian is the same as the hazard function of exponential-reciprocal inverse Gaussian.

It is easier to express the convolutions in terms of pgfs and recursive forms rather than obtaining pmfs explicitly. Panjer's recursive model holds for Hofamman hazard function when:

- one of the two hazard functions is a constant.
- when $a_1 = a_2$, $p_1 = p_2$ and $c_1 = c_2$

Further work is to identify other families of hazard functions of exponential mixtures, which are not necessarily members of the family of Hofmann distributions, and whose sums of hazard functions give rise to convolutions of Poisson mixtures.

References

[1] Feller, W. *An Introduction to probability Theory and Its Applications* Vol I, 3rd Edition, John Wiley and Sons (1968):.

[2] Feller, W. *An Introduction to probability Theory and Its Applications.* Vol II. John Wiley and Sons (1971):.

[3] Ospina,V and Gerber, M.U *"A Simple Proof of Feller's Characterizations of the Compound Poisson Distribution."* Insurance Mathematics and Economics, 6, (1987): 63-64.

[4] Wakoli, M.W. and Ottieno J.A.M. *"Mixed Poisson Distributions Associated with Hazard Functions of Exponential Mixtures* Mathematical Theory and Modeling Vol.5, No. 6 (2015): pages 209-244.

[5] Walhin, J.F and J. Paris *"Using Mixed Poisson Processes in Connection with Bonus – Malus Systems"* Astin Bulletin, Vol 29, No.1, (1999): pp 81 – 99.

[6] Walhin, J.F and J. Paris *"A General Family of Over-dispersed Probability Laws"* Belgian Actuarial Bulletin, Vol 2, No.1, (2002): pp 1 – 8.

[7] Watson, G. N. *"A Treatise on the Theory of Bessel Functions"* Cambridge: The University Press (1952).