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Stability of convergence theorems of the Noor iteration method for an enumerable class of continuous hemi contractive mapping in Banach spaces

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Abstract: The purpose of this is to study the Noor iteration process for the sequence $\{x_n\}$ converges to a common fix point for enumerable class of continuous hemi contractive mapping in Banach spaces.

Key words: Stability, Noor iterations, Hemicontractive mapping, Convergence theorem Continuous pseudocontractive mapping.

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Introduction: Let E be a real Banach space and let J denote the normalized duality mapping from E to E^* and defined by

$J(x) = \{f \in E^*: \langle x, f \rangle = \|x\|\|f\|, \|x\| = \|f\|\}$; for all $x \in E$,

Where E^*denotes the dual space of E and $\langle \cdot , \cdot \rangle$ denotes the generalization duality pair.

It is well known that if E^* is strictly convex then J is single–valued. In the sequel, we shall denote the single–valued duality mapping by j. Let K be a nonempty closed convex subset of Banach space E and T: $K \to K$ be a self-mapping of K.

Definition 1.1 [1] (i) A mapping T with domain D(T) and range R(T) in a Banach space is called pseudocontrative mapping, if for all x, y ∈ D(T), there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \le \|x-y\|^2 \tag{1}$$

(ii) A mapping T with domain D(T) and range R(T) in E is called a hemicontrative mapping if

F (T)≠ ∅ and for all x ∈ D(T) $x^* \in$ F (T) such that,

$\langle Tx - x^*, j(x - x^*) \rangle \le \|x - x^*\|^2$

(iii) A mapping T: K→ K is called L-Lipschitizan there exists L>0 such that

$\|Tx - Ty\| \le L\|x - y\|$ For all x, y ∈ K

Definition 1.2 [3] If $\{\propto_n\}_{n=0}^{\infty}$ and are sequences of real numbers in [0,1]. For arbitrary $x_0 \in$ E, Let $\{x_n\}_{n=0}^{\infty}$ be the Noor iteration and defined by,

$$x_{n+1=}(1-\propto_n)x_n + \propto_n Tq_n$$

$$q_n = (1-\beta_n)x_n + \beta_n Tr_n$$

$$r_n = (1-\beta_n)x_n + \beta_n Tr_n$$

Where $\{\propto_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$ are sequences of real numbers in [0, 1].

Lemma 1.3 [2] Let E be a real uniformly convex Banach space, K is nonempty closed convex subset of E and T a continuous pseudocontrative mapping of K, then $I-T$ is demiclosed at zero, that is, for all sequences $\{x_n\} \subset$K with $x_n \rightharpoonup p$ and $x_n - Tx_n \to 0$ it follows that p = Tp

Lemma 1.4 [4,5] Let δ be a number satisfying $0 \leq \delta < 1$ and $\{\epsilon_n\}$ a positive sequence satisfying $\lim_{n\to\infty} \epsilon_n = 0$. Then, for any positive sequence $\{u_n\}$ satisfying:

$u_{n+1} \leq \delta u_n + \epsilon_n$,It follows that $\lim_{n\to\infty} u_n = 0$

2. Main Results

Theorem 2. Let $\{T_n\}_{n=1}^{\infty}$ be defined as above and F: $= \bigcap_{i=1}^{\infty} F(T_{n)\neq} \phi$ and let (E,‖.‖) be a Banach space, T: E → E a self map of E with a fixed point p, satisfying the contractive condition

$\langle Tx - x^* , j(x-x^*) \rangle \leq \|x - x^*\|^2$ For $x_0 \in E$.

Let $\{x_n\}_{n=1}^{\infty}$ is converge to p and defined by the iteration (1.2) where $\{\propto_n\}_{n=1}^{\infty}$ is a real sequence in (0, 1) and define as $\epsilon_n = \| x_{n+1} - (1-\propto_n)x_n - \propto_n Tq_n \|$ Then

(i) $\lim_{n\to\infty} \| x_n - p \|$ exists for $p \in F$;
(ii) $\lim_{n\to\infty} d(x_n, F) = \{inf \| x_n - p \| : p \in F\}$;
(iii) $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}_{n=1}^{\infty}$ if and only if $\lim_{n\to\infty} d(x_n, F) = 0$

Proof Let $p \in F$ and n≥ 1 by 1.1 we choose $j(x_n - p) \in J(x_n - p)$ such that

$\| x_{n+1} - p \|^2 = \langle x_{n+1} - p , j(x_{n+1} - p) \rangle$

$\| x_{n+1} - p \| \leq \| x_{n+1} - (1-\propto_n)x_n - \propto_n Tq_n \| + \| (1-\propto_n)x_n + \propto_n Tq_n - p \|$

$= \epsilon_n + \| (1-\propto_n)x_n + \propto_n Tq_n - ((1-\propto_n) + \propto_n)p \|$

$= \epsilon_n + \| (1-\propto_n) \| x_n - p \| + \propto_n (Tq_n - p) \|$

$\leq \epsilon_n + (1-\propto_n) \| x_n - p \| + \propto_n \| Tq_n - p \|$

$= \epsilon_n + (1 - \propto_n) \parallel x_n - p \parallel + \propto_n \parallel p - Tq_n \parallel$

$\leq \epsilon_n + (1 - \propto_n) \parallel x_n - p \parallel + \propto_n a \parallel p - q_n \parallel$

$= \epsilon_n + (1 - \propto_n) \parallel x_n - p \parallel + \propto_n a \parallel q_n - p \parallel$ (1)

For the estimate of $\parallel q_n - p \parallel$ in (1) we get

$\parallel q_n - p \parallel = \parallel (1 - \beta_n)x_n + \beta_n Tr_n - p \parallel$

$= \parallel (1 - \beta_n)x_n + \beta_n Tr_n - ((1 - \beta_n) + \beta_n)p \parallel$

$= \parallel (1 - \beta_n)(x_n - p) + \beta_n(Tr_n - p) \parallel$

$\leq (1 - \beta_n) \parallel x_n - p \parallel + \beta_n \parallel Tr_n - p \parallel$

$= (1 - \beta_n) \parallel x_n - p \parallel + \beta_n \parallel p - Tr_n \parallel$

$\leq (1 - \beta_n) \parallel x_n - p \parallel + \beta_n a \parallel p - r_n \parallel$

$= (1 - \beta_n) \parallel x_n - p \parallel + \beta_n a \parallel r_n - p \parallel$ (2)

Substituting (2) into (1) gives

$\parallel x_{n+1} - p \parallel \leq \epsilon_n + (1 - (1 - a) \propto_n - \propto_n \beta_n a) \parallel x_n - p \parallel + \propto_n \beta_n a^2 \parallel r_n - p \parallel$ (3)

For $\parallel r_n - p \parallel$ in (3) we have,

$\parallel r_n - p \parallel = \parallel (1 - \gamma_n)x_n + \gamma_n Tx_n - p \parallel$

$= \parallel (1 - \gamma_n)x_n + \gamma_n Tx_n \ ((1 - \gamma_n) + \gamma_n) - p \parallel$

$= \parallel (1 - \gamma_n)(x_n - p) + \gamma_n(Tx_n - p) \parallel$

$\leq (1 - \gamma_n) \parallel x_n - p \parallel + \gamma_n \parallel Tx_n - p \parallel$

$= (1 - \gamma_n) \parallel x_n - p \parallel + \gamma_n \parallel p - Tx_n \parallel$

$\leq (1 - \gamma_n) \parallel x_n - p \parallel + \gamma_n a \parallel p - x_n \parallel$

$= (1 - \gamma_n + \gamma_n a) \parallel x_n - p \parallel$ (4)

Substituting (4) into (3) and using lemma 1.3

$= \epsilon_n + (1 - (1 - a) \propto_n - \propto_n \beta_n a) \parallel x_n - p \parallel + \propto_n \beta_n a^2 (1 - \gamma_n + \gamma_n a) \parallel x_n - p \parallel$

$= \epsilon_n (1 - (1 - a) \propto_n - (1 - a) \propto_n \beta_n a - (1 - a) \propto_n \beta_n \gamma_n a^2) \parallel x_n - p \parallel$

$\leq (1 - (1 - a)\alpha - (1 - a)\alpha\beta a - (1 - a)\alpha\beta\gamma a^2) \parallel x_{n-1} - p \parallel + \epsilon_n$

Observe that

$$0\leq (1-(1-a)\alpha-(1-a)\alpha\beta a-(1-a)\alpha\beta\gamma a^2)<1 \quad (5)$$

Therefore, taking the limit as $n\to\infty$ of both sides of the inequality (5) and using lemma 1.6 we get

$\lim_{n\to\infty}\parallel x_n-p\parallel=0$, That is $\lim_{n\to\infty}x_{n=}p$

By theorem 1.2 $\parallel x_n-p\parallel\leq\parallel x_{n-1}-p\parallel$

Taking infimum over all $p\in F$,we have,

$d(x_n,F)=\inf_{p\in F}\parallel x_n-p\parallel\leq\inf_{p\in F}\parallel x_{n-1}-p\parallel=d(x_{n-1},F)$,

Thus $\lim_{n\to\infty} d(x_n,F)$exist.We finally prove (iii).suppose that $x_n\to p\in F$ from (ii) and

$d(x_n,F)\leq\parallel x_n-p\parallel\to 0$, We have $\lim_{n\to\infty} d(x_n,F)=0$ for $n,m\in\mathbb{N}$ and $p\in F$,it follows

From (1.3) that

$\parallel x_{n+m}-x_n\parallel\leq\parallel x_{n+m}-p\parallel+\parallel x_n-p\parallel\leq 2\parallel x_n-p\parallel$

Consequently,

$\parallel x_{n+m}-x_n\parallel\leq 2\parallel x_n-F\parallel\to 0$

Therefore $\{x_n\}$ is a Cauchy sequence. Suppose $\lim_{n\to\infty}x_n=u$ for some $u\in E$.then

$d(u,F)=\lim_{n\to\infty}d(x_n,F)=0$

Since F is closed set, $u\in F$

So, Noor iteration process is T–stable.

Thus, the stability of Noor iteration considerable for finding fixed point for enumerable class of continuous hemi contractive mapping in Banach spaces.

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