

Stability of convergence theorems of the Noor iteration method for an enumerable class of continuous hemi contractive mapping in Banach spaces

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Abstract: The purpose of this is to study the Noor iteration process for the sequence $\{x_n\}$ converges to a common fix point for enumerable class of continuous hemi contractive mapping in Banach spaces.

Key words: Stability, Noor iterations, Hemiccontractive mapping, Convergence theorem Continuous pseudocontractive mapping.

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Introduction: Let E be a real Banach space and let J denote the normalized duality mapping from E to E^* and defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\| \|f\|, \|x\| = \|f\|\}; \text{ for all } x \in E,$$

Where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalization duality pair.

It is well known that if E^* is strictly convex then J is single-valued. In the sequel, we shall denote the single-valued duality mapping by j . Let K be a nonempty closed convex subset of Banach space E and $T: K \rightarrow K$ be a self-mapping of K .

Definition 1.1 [1] (i) A mapping T with domain $D(T)$ and range $R(T)$ in a Banach space is called pseudocontractive mapping, if for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 \tag{1}$$

(ii) A mapping T with domain $D(T)$ and range $R(T)$ in E is called a hemicontractive mapping if

$F(T) \neq \emptyset$ and for all $x \in D(T)$ $x^* \in F(T)$ such that,

$$\langle Tx - x^*, j(x - x^*) \rangle \leq \|x - x^*\|^2$$

(iii) A mapping $T: K \rightarrow K$ is called L -Lipschitzian there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \text{ For all } x, y \in K$$

Definition 1.2 [3] If $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of real numbers in $[0,1]$. For arbitrary $x_0 \in E$, Let $\{x_n\}_{n=0}^{\infty}$ be the Noor iteration and defined by,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tq_n$$

$$q_n = (1 - \beta_n)x_n + \beta_n Tr_n$$

$$r_n = (1 - \beta_n)x_n + \beta_n Tr_n$$

Where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$ are sequences of real numbers in $[0, 1]$.

Lemma 1.3 [2] Let E be a real uniformly convex Banach space, K is nonempty closed convex subset of E and T a continuous pseudocontractive mapping of K , then $I - T$ is demiclosed at zero, that is, for all sequences $\{x_n\} \subset K$ with $x_n \rightarrow p$ and $x_n - Tx_n \rightarrow 0$ it follows that $p = Tp$

Lemma 1.4 [4,5] Let δ be a number satisfying $0 \leq \delta < 1$ and $\{\epsilon_n\}$ a positive sequence satisfying $\lim_{n \rightarrow \infty} \epsilon_n = 0$. Then, for any positive sequence $\{u_n\}$ satisfying:

$$u_{n+1} \leq \delta u_n + \epsilon_n, \text{ It follows that } \lim_{n \rightarrow \infty} u_n = 0$$

2. Main Results

Theorem 2. Let $\{T_n\}_{n=1}^{\infty}$ be defined as above and $F := \bigcap_{i=1}^{\infty} F(T_n) \neq \emptyset$ and let $(E, \|\cdot\|)$ be a Banach space, $T: E \rightarrow E$ a self map of E with a fixed point p , satisfying the contractive condition

$$\langle Tx - x^*, j(x - x^*) \rangle \leq \|x - x^*\|^2 \text{ For } x_0 \in E.$$

Let $\{x_n\}_{n=1}^{\infty}$ is converge to p and defined by the iteration (1.2) where $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $(0, 1)$ and define as $\epsilon_n = \|x_{n+1} - (1 - \alpha_n)x_n - \alpha_n Tq_n\|$ Then

- (i) $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists for $p \in F$;
- (ii) $\lim_{n \rightarrow \infty} d(x_n, F) = \{ \inf \|x_n - p\| : p \in F \}$;
- (iii) $\{x_n\}$ converges strongly to a common fixed point of $\{T_n\}_{n=1}^{\infty}$ if and only if $\lim_{n \rightarrow \infty} d(x_n, F) = 0$

Proof Let $p \in F$ and $n \geq 1$ by 1.1 we choose $j(x_n - p) \in J(x_n - p)$ such that

$$\|x_{n+1} - p\|^2 = \langle x_{n+1} - p, j(x_{n+1} - p) \rangle$$

$$\|x_{n+1} - p\| \leq \|x_{n+1} - (1 - \alpha_n)x_n - \alpha_n Tq_n\| + \|(1 - \alpha_n)x_n + \alpha_n Tq_n - p\|$$

$$= \epsilon_n + \|(1 - \alpha_n)x_n + \alpha_n Tq_n - ((1 - \alpha_n) + \alpha_n)p\|$$

$$= \epsilon_n + (1 - \alpha_n)\|x_n - p\| + \alpha_n\|Tq_n - p\|$$

$$\leq \epsilon_n + (1 - \alpha_n)\|x_n - p\| + \alpha_n\|Tq_n - p\|$$

$$\begin{aligned}
 &= \epsilon_n + (1 - \alpha_n) \|x_n - p\| + \alpha_n \|p - Tq_n\| \\
 &\leq \epsilon_n + (1 - \alpha_n) \|x_n - p\| + \alpha_n a \|p - q_n\| \\
 &= \epsilon_n + (1 - \alpha_n) \|x_n - p\| + \alpha_n a \|q_n - p\| \tag{1}
 \end{aligned}$$

For the estimate of $\|q_n - p\|$ in (1) we get

$$\begin{aligned}
 \|q_n - p\| &= \|(1 - \beta_n)x_n + \beta_n Tr_n - p\| \\
 &= \|(1 - \beta_n)x_n + \beta_n Tr_n - ((1 - \beta_n) + \beta_n)p\| \\
 &= \|(1 - \beta_n)(x_n - p) + \beta_n(Tr_n - p)\| \\
 &\leq (1 - \beta_n) \|x_n - p\| + \beta_n \|Tr_n - p\| \\
 &= (1 - \beta_n) \|x_n - p\| + \beta_n \|p - Tr_n\| \\
 &\leq (1 - \beta_n) \|x_n - p\| + \beta_n a \|p - r_n\| \\
 &= (1 - \beta_n) \|x_n - p\| + \beta_n a \|r_n - p\| \tag{2}
 \end{aligned}$$

Substituting (2) into (1) gives

$$\|x_{n+1} - p\| \leq \epsilon_n + (1 - (1 - a)\alpha_n - \alpha_n \beta_n a) \|x_n - p\| + \alpha_n \beta_n a^2 \|r_n - p\| \tag{3}$$

For $\|r_n - p\|$ in (3) we have,

$$\begin{aligned}
 \|r_n - p\| &= \|(1 - \gamma_n)x_n + \gamma_n Tx_n - p\| \\
 &= \|(1 - \gamma_n)x_n + \gamma_n Tx_n - ((1 - \gamma_n) + \gamma_n)p\| \\
 &= \|(1 - \gamma_n)(x_n - p) + \gamma_n(Tx_n - p)\| \\
 &\leq (1 - \gamma_n) \|x_n - p\| + \gamma_n \|Tx_n - p\| \\
 &= (1 - \gamma_n) \|x_n - p\| + \gamma_n \|p - Tx_n\| \\
 &\leq (1 - \gamma_n) \|x_n - p\| + \gamma_n a \|p - x_n\| \\
 &= (1 - \gamma_n + \gamma_n a) \|x_n - p\| \tag{4}
 \end{aligned}$$

Substituting (4) into (3) and using lemma 1.3

$$\begin{aligned}
 &= \epsilon_n + (1 - (1 - a)\alpha_n - \alpha_n \beta_n a) \|x_n - p\| + \alpha_n \beta_n a^2 (1 - \gamma_n + \gamma_n a) \|x_n - p\| \\
 &= \epsilon_n (1 - (1 - a)\alpha_n - (1 - a)\alpha_n \beta_n a - (1 - a)\alpha_n \beta_n \gamma_n a^2) \|x_n - p\| \\
 &\leq (1 - (1 - a)\alpha - (1 - a)\alpha\beta a - (1 - a)\alpha\beta\gamma a^2) \|x_{n-1} - p\| + \epsilon_n
 \end{aligned}$$

Observe that

$$0 \leq (1 - (1 - a)\alpha - (1 - a)\alpha\beta a - (1 - a)\alpha\beta\gamma a^2) < 1 \quad (5)$$

Therefore, taking the limit as $n \rightarrow \infty$ of both sides of the inequality (5) and using lemma 1.6 we get

$$\lim_{n \rightarrow \infty} \|x_n - p\| = 0, \text{ That is } \lim_{n \rightarrow \infty} x_n = p$$

$$\text{By theorem 1.2 } \|x_n - p\| \leq \|x_{n-1} - p\|$$

Taking infimum over all $p \in F$, we have,

$$d(x_n, F) = \inf_{p \in F} \|x_n - p\| \leq \inf_{p \in F} \|x_{n-1} - p\| = d(x_{n-1}, F),$$

Thus $\lim_{n \rightarrow \infty} d(x_n, F)$ exist. We finally prove (iii). suppose that $x_n \rightarrow p \in F$ from (ii) and

$$d(x_n, F) \leq \|x_n - p\| \rightarrow 0, \text{ We have } \lim_{n \rightarrow \infty} d(x_n, F) = 0 \text{ for } n, m \in \mathbb{N} \text{ and } p \in F, \text{ it follows}$$

From (1.3) that

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p\| + \|x_n - p\| \leq 2 \|x_n - p\|$$

Consequently,

$$\|x_{n+m} - x_n\| \leq 2 \|x_n - p\| \rightarrow 0$$

Therefore $\{x_n\}$ is a Cauchy sequence. Suppose $\lim_{n \rightarrow \infty} x_n = u$ for some $u \in E$. then

$$d(u, F) = \lim_{n \rightarrow \infty} d(x_n, F) = 0$$

Since F is closed set, $u \in F$

So, Noor iteration process is T -stable.

Thus, the stability of Noor iteration considerable for finding fixed point for enumerable class of continuous hemi contractive mapping in Banach spaces.

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