An Explicit and Semi – Implicit Rational Runge – Kutta Schemes for Approximation of Second Order Ordinary Differential Equations

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Abstract
In this paper, two schemes of one stage are derived, an Explicit Rational Runge – Kutta and a Semi – Implicit Rational Runge – Kutta method were derived using Taylor and Binomial series expansion for the solution of general second order initial value problems of ordinary differential equations with constant step size. The analysis of the develop methods was carried out and found to be consistent and convergent and A – stable. The accuracy of the methods were tested on some examples and found to give better approximations.
Keywords: Explicit, Semi – Implicit Ration Runge – Kutta, Taylor series, Binomial Expansion, A – Stable.

1. Introduction

Consider the numerical approximation of second order initial value problem of the form:

$$y''= f(x,y,y'), \quad y(x_0)=y_0, \quad y'(x_0)=y_0, \qquad a \le x \le b \tag{1}$$

The general s – stage of Runge – Kutta scheme for second order initial value problems of ordinary differential equations of the form (1) as defined by Jain (1984) is

$$y_{n+1} = y_n + hy_n' + \sum_{r=1}^{s} c_r k_r \tag{2}$$

and

$$y'_{n+1} = y_n' + \frac{1}{h}\sum_{r=1}^{s} c_r k_r \tag{3}$$

where

$$K_r = \frac{h^2}{2} f\left(x_n + c_i h, y_n + hc_i y_n' + \sum_{j=1}^{r} a_{ij} k_j, y_n' + \tfrac{1}{h}\sum_{j=1}^{r} b_{ij} k_j \right), \quad i = 1(1)s \tag{4}$$

with

$$c_i = \sum_{j=i}^{i} a_{ij} = \tfrac{1}{2}\sum_{j=1}^{i} b_{ij}, \quad i(1)r \tag{5}$$

where c_i, a_{ij}, b_{ij}, c_r, c'_r are constants to be determined.
The rational form of (3) and (4) can defined as

$$y_{n+1} = \frac{y_n + hy_n' + \sum_{r=1}^{s} w_r K_r}{1 + y_n' \sum_{r=1}^{s} v_r H_r} \tag{6}$$

$$y_{n+1}' = \frac{y_n + \frac{1}{h}\sum_{r=1}^{s} w_r' K_r}{1 + \frac{1}{h} y_n' \sum_{r=1}^{s} v_r' H_r} \tag{7}$$

where

$$K_r = \frac{h^2}{2} f\left(x_n + c_i h, y_n + hc_i y_n' + \sum_{j=1}^{s} a_{ij} K_j, \frac{1}{h}\sum_{j=1}^{s} b_{ij} K_j \right), \qquad i = 1(1)s \tag{8}$$

$$H_r = \frac{h^2}{2} g\left(x_n + d_i h, z_n + hd_i z_n' + \sum_{j=1}^{s} \alpha_{ij} H_j, z_n' + \frac{1}{h}\sum_{j=1}^{s} \beta_{ij} H_j \right), \qquad i = 1(1)s \tag{9}$$

with constraits

$$c_i = \sum_{j=1}^{i} a_{ij} = \frac{1}{2}\sum_{j}^{i} bij,\ i = 1(1)s$$

$$d_i = \sum_{j=1}^{i} \alpha_{ij} = \frac{1}{2}\sum_{j}^{i} \beta ij,\ i = 1(1)s$$

in which

$$g(x_n, z_n, z_n') = -z_n^2 f(x_n, y_n, y_n') \text{ and } z_n = \tfrac{1}{y_n}$$

The constraint equations are to ensure consistency of the method, h is the step size and the parameters a_{ij}, b_{ij}, c_i, d_i, α_{ij}, β_{ij} are constants called the parameters of the method.
The scheme above is said to be Semi – Implicit if $b_{ij} = 0$, $j \geq 1$.

2. Derivation of the Schemes

Following Abhulimen and Uloku (2012), Bolarinwa *etal* (2012) and Usman *etal* (2013), the following procedures are adopted.

i. Obtain the Taylor series expansion of K_r and H_r about the point $(x_n, y_n, y^1{}_n)$ and binomial series expansion of the right side of (6) and (7).
ii. Insert the Taylor series expansion in to (6) and (7) respectively.
iii. Compare the final expansion of K_r and H_r about the point $(x_n, y_n, y^1{}_n)$ to the Taylor series expansion of y_{n+1} and $y^1{}_{n+1}$ about $(x_n, y_n, y^1{}_n)$ in the powers of h.

Normally the number of parameters exceed the number of equations, these parameters are chosen to ensure that (one or more of) the following conditions are satisfied.

i. Minimum bound of local truncation error exist.
ii. The method has maximized interval of absolute stability.
iii. Minimized computer storage facilities are utilized.

In this paper, we shall consider the semi – implicit scheme where a ≠ 0 and b = 0 for at least one j= 1.

In equations (6), (7), (8) and (9) setting s = 1 we have

$$y_{n+1} = \frac{y_n + hy_n' + \sum_{r=1}^{1} w_r K_r}{1 + y_n' \sum_{r=1}^{1} v_r H_r} \tag{2.1}$$

$$y_{n+1}' = \frac{y_n + \frac{1}{h}\sum_{r=1}^{1} w_r' K_r}{1 + \frac{1}{h} y_n' \sum_{r=1}^{1} v_r' H_r} \tag{2.2}$$

where

$$K_r = \frac{h^2}{2} f\left(x_n + c_i h, y_n + hc_i y_n' + \sum_{j=1}^{1} a_{ij} K_j, \frac{1}{h}\sum_{j=1}^{1} b_{ij} K_j \right), \qquad i = 1 \tag{2.3}$$

$$H_r = \frac{h^2}{2} g\left(x_n + d_i h, z_n + hd_i z_n' + \sum_{j=1}^{1} \alpha_{ij} H_j, z_n' + \frac{1}{h}\sum_{j=1}^{1} \beta_{ij} H_j \right), \qquad i = 1 \tag{2.4}$$

with constraints

$$c_1 = a_{11} = \tfrac{1}{2} b_{11}$$

$$d_1 = \alpha_{11} = \tfrac{1}{2}\beta_{11}$$

Since we are considering the semi – implicit scheme, $b_{ij} = 0$ for $j > i$.
By adopting a binomial expansion on equation (6) gives

$$y_{n+1} = y_n + hy_n' - \left(y_n^2 v_1 + hy_n y_n' v_1\right)H_1 + \left(w_1 - y_n v_1 H_1 w_1\right)K_1 \tag{2.5}$$

Similarly the binomial expansion of (7) gives

$$y_{n+1}' = y_n' + \tfrac{1}{h} w_1' K_1 - \left(\tfrac{1}{h} y_n'^2 v_1' + \tfrac{1}{h^2} y_n' w_1' v_1' K_1\right)H_1 \tag{2.6}$$

Expanding using Taylor's series of function of three variables gives

$$
\begin{aligned}
K_1 = {} & \frac{h^2}{2}\left(f_n + a_{11}K_1 f_y + c_1 b_{11} K_1 f_{xy'} + \tfrac{1}{2}a_{11}^2 K_1^2 f_{yy} + c_1 y_n' b_{11} K_1 f_{yy}\right) \\
& + \frac{h^3}{2}\left(c_1 f_x + c_1 y_n' f_y + c_1 a_{11} K_1 f_{xy} + c_1 a_{11} y_n' K_1 f_{yy}\right) \\
& + \frac{h^4}{4}\left(c_1^2 f_{xx} + 2c_1^2 y_n' f_{xy} + c_1^2 y_n'^2 f_{yy}\right) \\
& + 0(h^5)
\end{aligned} \tag{2.7}
$$

Equation (2.7) is implicit, which cannot be proceeded by successive substitutions, we assume a solution of K_1 which may be express as

$$K_1 = h^2 B_1 + h^3 C_1 + h^4 D_1 + O(h^5) \tag{2.8}$$

Substituting the values of K_1 of (2.8) into (2.7) expanding and re – arranging in the powers of h gives

$$
\begin{aligned}
K_1 = {} & \frac{h}{2}\left[b_{11}(hA_1 h^2 B_1 + h^3 C_1) f_{y'} + a_{11} b_{11} (hA_1 + h^2 B_1)^2 f_{yy'}\right] \\
& + \frac{h^2}{2}\begin{pmatrix} f_n + a_{11}(hA_1 + h^2 B_1) f_y + c_1 b_{11}(hA_1 + h^2 B_1) f_{xy'} \\ + \tfrac{1}{2} a_{11}^2 (hA)^2 f_{yy} + c_1 y_n' b_{11}(hA_1 + h^2 B_1) f_{yy'} \end{pmatrix} \\
& + \frac{h^3}{2}\left(c_1 f_x + c_1 y_n' f_y + c_1 a_{11}(hA_1) f_{xy} + c_1 a_{11} y_n' (hA_1) f_{yy}\right) \\
& + \frac{h^4}{4}\left(c_1^2 f_{xx} + 2c_1 y_n' f_{xy'} + c_1^2 y_n'^2 f_{yy}\right) \\
& + 0(h^5)
\end{aligned} \tag{2.9}
$$

On equating the powers of h from (2.8) and (2.9) gives

$$
\left.\begin{aligned}
A_1 &= 0 \\
B_1 &= \tfrac{1}{2} f_n \\
C_1 &= \tfrac{1}{2}\left(c_1 f_x + c_1 y_n' f_y + b_{11} f_n f_{y'}\right) = \tfrac{1}{2} C_1 \Delta f_n \\
D_1 &= \tfrac{1}{4}\left(c_1^2 \Delta^2 f_n + b_{11} \Delta f_n f_{y'} + a_{11} f_n f_y\right)
\end{aligned}\right\} \tag{2.10}
$$

Then

$$K_1 = \frac{h^2}{2} f_n + \frac{h^3}{2} c_1 \Delta f_n + \frac{h^4}{4}\left(c_1^2 \Delta^2 f_n + b_{11} \Delta f_n f_{y'} + a_{11} f_n f_y\right) \tag{2.11}$$

In a similar manner

$$H_1 = h^2 M_1 + h^3 N_1 + h^4 R_1 + 0(h^3) \tag{2.12}$$

where

$$
\left.\begin{aligned}
M_1 &= \tfrac{1}{2} g_n \\
N_1 &= \tfrac{1}{2} d_1 \Delta g_n \\
R_1 &= \tfrac{1}{4}\left(d_1^2 \Delta^2 g_n + \beta_{11} \Delta g_n g_{z'} + \alpha_{11} g_n g_z\right)
\end{aligned}\right\} \tag{2.13}
$$

and also

$$H_1 = \frac{h^2}{2} g_n + \frac{h^3}{2} d_1 \Delta g_n + \frac{h^4}{4}\left(d_1^2 \Delta^2 g_n + \beta_{11} \Delta g_n g_{z'} + \alpha_{11} g_n g_z\right) \tag{2.14}$$

Substituting equations (2.8) and (2.12) in to equations (2.5) and (2.6) re – arranging and comparing the result with the Taylor's series expansion of y_{n+1} about x_n gives the following

$$\frac{1}{2} w_1 f_n - \frac{1}{2} y_n^2 v_1 g_n = \frac{1}{2} f_n$$

$$\frac{1}{2} w_1 c_1 \Delta f_n - \frac{1}{2} y_n^{'2} v_1 d_1 g_n - \frac{1}{2} y_n y_n^{'} v_1 g_n = \frac{1}{6} \Delta f_n$$

$$\frac{1}{2} w_1^{'} f_n - \frac{1}{2} y_n^{'2} g_n = f_n$$

$$\frac{1}{2} w_1^{'} f_n - \frac{1}{2} y_n^{'2} v_1^{'} d_1 \Delta g_n - \frac{1}{2} y_n^{'} w_1^{'} v_1^{'} f_n \left(\tfrac{1}{2} g_n\right) = \frac{1}{2} \Delta f_n$$

From above we get the following sets of linear differentiation equations

$$w_1 + v_1 = 1$$

$$w_1 c_1 + v_1 d_1 = \frac{1}{3}$$

$$w_1^{'} + v_1^{'} = 2$$

$$w_1^{'} c_1 b_{11} + v_1^{'} \beta_{11} d_1 = 1$$

with constraints

$$c_1 = a_{11} = \tfrac{1}{2} b_{11}$$

$$d_1 = \alpha_{11} = \tfrac{1}{2} \beta_{11}$$

There are four equations with eight unknowns; that is, there will be no unique solution, for the scheme to be semi – implicit, we choose d_1 to be zero and there will be a one – stage scheme.

Choosing the parameters

$$w_1 = \frac{2}{3},\ v_1 = \frac{1}{3},\ c_1 = \frac{1}{2},\ a = \frac{1}{2},\ b_{11} = 1,\ d_1 = \alpha_{11} = \beta_{11} = 0,\ w_1^{'} = 2 \text{ and } v_1^{'} = 0.$$

then equations (2.1) and (2.2) becomes

$$y_{n+1} = \frac{y_n + h y_n^{'} + \frac{2}{3} K_1}{1 + \frac{1}{3} y_n H_1} \tag{2.15}$$

and

$$y_{n+1}^{'} = y_n^{'} + \frac{2}{h} K_1 \tag{2.16}$$

where

$$K_1 = \frac{h^2}{2} f\left(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}hy_n' + \tfrac{1}{2}K_1,\ y_n' + \tfrac{1}{h}K_1\right)$$

and

$$H_1 = \frac{h^2}{2} g(x_n, z_n, z_n')$$

Also if

$$w_1' + v_1' = 1$$

And choosing the parameters

$$w_1 = v_1 = w_1' = v_1' = \tfrac{1}{2}$$

Then from equations (2.1) and (2.2) we have the following explicit equations

$$y_{n+1} = \frac{y_n + hy_n' + \frac{1}{2}K_1}{1 + \frac{1}{2}y_n H_1} \qquad 2.17$$

$$y_{n+1}' = \frac{y_n' + \frac{1}{2}hK_1}{1 + \frac{1}{2}hy_n' H_1} \qquad 2.18$$

where

$$K_1 = \frac{h^2}{2} f\left(x_n, y_n, y_n'\right)$$

$$H_1 = \frac{h^2}{2} g\left(x_n, z_n, z_n'\right)$$

Note that the parameters were chosen arbitrarily, but making sure that they satisfy the equations (2.1) and (2.2).

3. STABILITY

The parameters in the scheme (6) are chosen to ensure that y_{n+1} is Pade's approximation to e^h. The resulting schemes are A – stable because in all, $|y_{n+1}| < 1$.

4. CONVERGENCE

Since a numerical method is said to be convergent if the numerical solution approaches the exact solution as the step size tent to zero. Mathematically,

$$\text{Convergent} = \lim_{h\to 0}\left|y(x_{n+1}) - y_{n+1}\right| \text{ or } e_{n+1} = \left|y(x_{n+1}) - y_{n+1}\right| \to 0 \text{ as } n \to \infty$$

And going by the definition above, the schemes were tested and found to be convergent.

5. CONSISTENCY

A scheme is said to be consistent, if the difference equation of the computation formula exactly approximates the differential equation it intends to solve as the step size tends to zero. Then from (2.15)

$$y_{n+1} - y_n = \frac{y_n + hy_n' + \frac{2}{3}K_1}{1 + \frac{1}{3}y_n H_1} - y_n$$

$$= \frac{hy_n' + \frac{2}{3}K_1 - \frac{1}{3}y_n^2 H_1}{1 + \frac{1}{3}y_n H_1} \qquad (2.27)$$

substituting the values of K_1 and H_1, (2.27) be come

$$y_{n+1} - y_n = \frac{hy_n' + \frac{2}{3}\left[\frac{h^2}{2} f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hy_n' + \frac{1}{2}K_1, y_n' + \frac{1}{h}K_1\right)\right] - \frac{1}{3}y_n'\left[\frac{h^2}{2} g(x_n, z_n, z_n')\right]}{1 + \frac{1}{3}y_n\left[\frac{h^2}{2} g(x_n, z_n, z_n')\right]}$$

Dividing through by h and taking the limit as h tends to zero on both sides gives

$$\lim_{h \to 0} \frac{y_{n+1} - y_n}{h} = y_n'$$

Hence, the scheme is consistent, and by the same procedure also (2.17) is consistent.

Numerical Example 1.

Consider the equation $y'' = (1 + x^2)y, \; y(0) = 1, \; y'(0) = 0, \; x \in [0,1]$

The exact solution is $y(x) = e^{\frac{x^2}{2}}$

Example 2

Consider the equation $y'' + 5y' + 6y = 0, \; y(0) = 2, \; y'(0) = 3$

The exact solution is given by

$$y(x) = 9e^{-2x} - 7e^{-3x}, \quad y'(x) = -18e^{-2x} + 21e^{-3x}$$

6. DISCUSSIONS

The result obtained by applying the scheme to the equation above shows that the result performed well and approximate the exact solution better as the step size goes to *h = 0.001*.

7. CONCLUSION

The new numerical schemes derived follows the techniques of rational form of Runge – Kutta methods proposed by Hong (1982) which was adopted by Okunbor (1987) and Ademiluyi and Babatola(2000) by using Taylor and Binomial expansion in stages evaluation.
The new schemes obtained performed well in approximating the exact solutions and the solutions gets better as the step size tends to zero. The analysis of the scheme prove to be consistent, convergent and A – stable. Thus, the schemes are effective and efficient; these suggest the wider application of the scheme since the method are used to solve the equation of both general and special type of second order differential equations.

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PRESENTATION OF RESULTS

Result of the numerical example1for equation (2.15) at $h = 0.001$

	Exact	Solutions	Approximated	Solutions	Error	
X	$y(x)$	$y^1(x)$	$y(x)$	$y^1(x)$	y	y^1
0.001	1.000000500	0.001000000	1.000000000	0.001000000	5.00E-07	0.00E00
0.002	1.000002000	0.002000000	1.000001000	0.002000000	1.00E-06	0.00E00
0.003	1.000004500	0.003000010	1.000001000	0.003000000	3.50E-06	1.00E-09
0.004	1.000008000	0.004000030	1.000003000	0.004000030	5.00E-06	0.00E00
0.005	1.000012500	0.005000060	1.000004000	0.005000060	8.50E-06	0.00E00
0.006	1.000018000	0.006000110	1.000006000	0.006000110	1.20E-05	0.00E00
0.007	1.000024500	0.007000170	1.000008000	0.007000170	1.65E-05	0.00E00
0.008	1.000032000	0.008000260	1.000011000	0.008000260	2.10E-05	0.00E00
0.009	1.000040500	0.009000370	1.000014000	0.009000370	2.65E-05	0.00E00
0.010	1.000000500	0.001000000	1.000017000	0.010000500	1.65E-05	9.00E-03
0.020	1.000002000	0.002000000	1.000067000	0.020004000	6.50E-05	1.80E-02
0.030	1.000004500	0.003000010	1.000150000	0.030013510	1.46E-04	2.70E-02
0.040	1.000008000	0.004000030	1.000267000	0.040032020	2.60E-04	3.60E-02
0.050	1.000012500	0.005000060	1.000418000	0.050062560	4.06E-04	4.51E-02
0.060	1.000018000	0.006000110	1.000602000	0.060108150	5.84E-04	5.41E-02
0.070	1.000024500	0.007000170	1.000820000	0.070171800	7.96E-04	6.32E-02
0.080	1.000032000	0.008000260	1.001072000	0.080256610	1.04E-03	7.23E-02
0.090	1.000040500	0.009000370	1.001359000	0.090365610	1.32E-03	8.14E-02
0.10	1.000050000	0.010000500	1.005019000	0.100501900	4.97E-03	9.05E-02

Result of numerical example 2 for equation (2.17) at *h = 0.001*

H	Exact solutions		Approximate solutions		Error	
	$y(x_n)$	$y^{'}(x_n)$	y_n	$y_n^{'}$	y	$y^{'}$
0.001	2.182849233	-0.820029047	2.182848907	-0.820029179	3.26E-07	1.05E-07
0.002	2.191198961	-0.540716472	2.191198640	-0.540716254	3.00E-08	2.18E-07
0.003	2.093317107	-1.340646595	2.093316869	-1.340646334	2.38E-07	2.61E-07
0.004	1.935601194	-1.762842904	1.935601037	-1.762842657	1.57E-07	2.47E-07
0.005	1.749003850	-1.936096578	1.749003745	-1.936096370	1.05E-07	2.08E-07
0.006	1.553655690	-1.950219162	1.553655607	-1.950218994	8.30E-08	1.68E-07
0.007	1.362177678	-1.867160358	1.362177592	-1.867160217	8.60E-08	1.29E-07
0.008	1.182042989	-1.729060305	1.182042885	-1.729060176	1.04E-07	1.29E-07
0.009	1.017251405	-1.560464220	1.017251273	-1.560640860	1.32E-07	1.34E07
0.010	0.869508070	-1.390506663	0.869507907	-1.390506510	1.63E-07	1.53E-07

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