Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.5, No.10, 2015

Jordan Triple Higher Derivations on Prime Rings

Salah Mehdi Salih(1) and Claus Haetinger(2)

(1) Mathematic Department College of Education Al – Mustansirya University, Iraq

(2) Center of Exact and Technological Sciencess, UNIVATEES University Center, Brazil

Abstract

In this paper, we develop some important results relating to the concepts of triple higher derivation and Jordan triple higher derivation on ring R. We show that under certain conditions on R, every Jordan triple higher derivation on R is triple higher derivation on R.

Keywords: derivation, higher derivation, Jordan triples higher derivation.

Mathematics subject classification: 16W25, 16N60, 16Y30

1. Introduction:

Through the present paper R will denote an associative ring, the set of natural numbers including 0 will be denoted by N. [, ,] denotes the usual commutator operator such that [a,b,c]= abc-cba , for all a,b,c∈R.

A ring R is said to be prime if aRb= (0) implies that a=0 or b=0 where a,b∈R, and R is semiprime in case aRa =(0) implies a=0 [5]. A ring R is called n-torsion free, where n is an integer number in case na=0, for a∈R, implies a=0 [5]. An additive mapping d: R→ R is called a derivation (resp. Jordan derivation) on R if d(ab) = d(a)b + ad(b) (resp. $d(a^2)$ = d(a) a + a d(a)) holds for all a,b∈R. Obviously, every derivation is a Jordan derivation on R but the converse need not be true in general. However, in 1957 I.N.Herstein [6] proved that on a prime ring with char.(R)≠ 2, every Jordan derivation is a derivation. It turns out that every Jordan derivation of a 2-torsion free ring is a Jordan triple derivation [5]. We recall that an additive mapping d: R→ R is said to be triple derivation (resp. Jordan triple derivation) on R if d(abc) =d(a)bc +ad(b)c +abd(c) (d(aba)=d(a)ba +ad(b)a +abd(a)) for all a,b,c ∈R.

The concept of derivation was extended to higher derivation by F.K.Schmidt [4] also see [1] as follow, let D =$(d_n)_{n\in N}$ be a family of additive mappings d_n: R→ R , D is said to be a higher derivation (resp. Jordan higher derivation) on R if $d_0=Id_R$ (the identity map on R) and

$$d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b) \ (resp.\ d_n(a^2) = \sum_{i+j=n} d_i(a)d_j(a))$$

For all a,b,c∈R and n∈N.

In an attempt to generalize Hersteins result for higher derivations, C.Haetinger [6] proved that on a prime ring with 2-torsion free every Jordan higher derivation is a higher derivation.

Now the main purpose of this paper is to extend this result for triple higher derivations in rings.

We need the following lemma

Lemma 1.1:[2]

Let R be a semmiprime ring. If a,b∈R are such that axb+bxa=0, for all x∈R then axb=bxa=0 , for all x∈R .axb=0 , for all x∈R implies that bxa=ab=ba=0.

2. Triple Higher Derivations:

In this section we present the definitions of triple higher derivation and Jordan triple higher derivation also we introduce some properties of them.

Definition 2.1: Let R be a ring and D= $(d_i)_{i\in N}$ be a family of additive mappings of R such that $d_0=Id_R$. Then D is called a triple higher derivation of R if

$$d_n(abc) = \sum_{i+j+k=n} d_i(a)d_j(b)d_k(c)$$

For all a,b,c ϵR, nϵN

And D is called Jordan triple higher derivation of R if [2]

$$d_n(aba) = \sum_{i+j+k=n} d_i(a)d_j(b)d_k(a)$$

For all a,bϵR

Lemma 1: Let R be a ring and D= $(d_i)_{i\in N}$ be Jordan triple higher derivation of R. Then for all a,b,c ϵR, nϵN

$$d_n(abc+cba) = \sum_{i+j+k=n} d_i(a)d_j(b)d_k(c) + d_i(c)d_j(b)d_k(a)$$

Proof:

$$d_n\big((a+c)b(a+c)\big) = \sum_{i+j+k=n} d_i(a+c)d_j(b)d_k(a+c)$$

$$= \sum_{i+j+k=n} d_i(a)d_j(b)d_k(a) + d_i(a)d_j(b)d_k(c) + d_i(c)d_j(b)d_k(a) + d_i(c)d_j(b)d_k(c) \quad \text{... (1)}$$

On the other hand

$$d_n((a+c)b(a+c)) = d_n(aba+abc+cba+cbc)$$

$$= \sum_{i+j+k=n} d_i(a)d_j(b)d_k(a) + d_i(c)d_j(b)d_k(c) + d_n(abc+cba) \quad \dots (2)$$

Comparing (1) and (2) we get

$$d_n(abc+cba) = \sum_{i+j+k=n} d_i(a)d_j(b)d_k(c) + d_i(c)d_j(b)d_k(a)$$

Definition 2.2: Let R be a ring and D= (d_i)$_{i\in N}$ be Jordan triple higher derivation of R, then for all a,b,c ϵR, nϵN we define

$$\Psi_n(a,b,c) = d_n(abc) - \sum_{i+j+k=n} d_i(a)d_j(b)d_k(c)$$

Lemma 2: Let D= (d_i)$_{i\in N}$ be Jordan triple higher derivation of a ring R. Then for all a,b,c ϵR and nϵN

i)Ψ_n(a,b,c) = - $\Psi_n(c,b,a)$

ii) $\Psi_n(a+h,b,c) = \Psi_n(a,b,c) + \Psi_n(h,b,c)$

iii) $\Psi_n(a,b+h,c) = \Psi_n(a,b,c) + \Psi_n(a,h,c)$

iv) $\Psi_n(a,b,c+h) = \Psi_n(a,b,c) + \Psi_n(a,b,h)$

Prove : We prove for example (iv)

Ψ_n(a,b,c+h) $=d_n(ab(c+h)) - \sum_{i+j+k=n} d_i(a)d_j(b)d_k(c+h)$

$= d_n(abc+abh) - \sum_{i+j+k=n} d_i(a)d_j(b)(d_k(c)+d_k(h))$

$= d_n(abc) - \sum_{i+j+k=n} d_i(a)d_j(b)d_k(c) +$

$$d_n(abc) - \sum_{i+j+k=n} d_i(a)d_j(b)d_k(c)$$

= Ψ_n(a,b,c) + Ψ_n(a,b,h)

Remark 2.3: Note that D=(d_i)$_{i\in N}$ is a triple higher derivations of a ring R if and only if Ψ_n(a,b,c) =0, for all a,b,c ϵR and nϵN.

Now, we prove some lemmas which make us able to give the next results.

Lemma 3: Let D= (d_i)$_{i\in N}$ be Jordan triple higher derivation of a ring R , assume that nϵN, a,b,c,rϵR if Ψ_s(a,b,c) =0 for every s< n then

Ψ_n(a,b,c) r [a,b,c] + [a,b,c] r Ψ_n(a,b,c) =0

Proof: By using Definition 2.1 we can commute

$$d_n\,(abcrcba + cbarabc) = \sum_{i+j+k+p+q+s+t=n} d_i(a)d_j(b)d_k(c)d_p(r)d_q(c)d_s(b)d_t(a) + d_i(c)d_j(b)d_k(a)d_p(r)d_q(a)d_s(b)d_t(c)$$

$$= \sum_{i+j+k=n} d_i(a)d_j(b)d_k(c)rcba + \sum_{i+j+k+p+q+s+t=n}^{i+j+k<n} d_i(a)d_j(b)d_k(c)d_p(r)d_q(c)d_s(b)d_t(a)$$

$$+abcr \sum_{q+s+t=n} d_q(c)d_s(b)d_t(a) + \sum_{i+j+k+p+q+s+t=n}^{q+s+t<n} d_i(a)d_j(b)d_k(c)d_p(r)d_q(c)d_s(b)d_t(a)$$

$$+ \sum_{i+j+k=n} d_i(c)d_j(b)d_k(a)rabc + \sum_{i+j+k+p+q+s+t=n}^{i+j+k<n} d_i(c)d_j(b)d_k(a)d_p(r)d_q(a)d_s(b)d_t(c)$$

$$+cbar \sum_{q+s+t=n} d_q(a)d_s(b)d_t(c) + \sum_{i+j+k+p+q+s+t=n}^{q+s+t<n} d_i(c)d_j(b)d_k(a)d_p(r)d_q(a)d_s(b)d_t(c)$$

$$abcd_n(r)cba + \sum_{i+j+k+p+q+s+t=n}^{p<n} d_i(a)d_j(b)d_k(c)d_p(r)d_q(c)d_s(b)d_t(a)$$

$$cbad_n(r)abc + \sum_{i+j+k+p+q+s+t=n}^{p<n} d_i(c)d_j(b)d_k(a)d_p(r)d_q(a)d_s(b)d_t(c) \qquad \ldots (1)$$

On the other hand

$$d_n\,(abcrcba + cbarabc) = \sum_{i+j+k=n} d_i(abc)d_j(r)d_k(cba) + d_i(cba)d_j(r)d_k(abc)$$

$$= d_n(abc)rcba + \sum_{i+j+k+p+q+s+t=n}^{i+j+k<n} d_i(a)d_j(b)d_k(c)d_p(r)d_q(c)d_s(b)d_t(a)$$

$$+abcrd_n(cba) + \sum_{i+j+k+p+q+s+t=n}^{q+s+t<n} d_i(a)d_j(b)d_k(c)d_p(r)d_q(c)d_s(b)d_t(a)$$

$$+d_n(cba)rabc + \sum_{i+j+k+p+q+s+t=n}^{i+j+k<n} d_i(c)d_j(b)d_k(a)d_p(r)d_q(a)d_s(b)d_t(c)$$

$$+cbard_n(cba) + \sum_{i+j+k+p+q+s+t=n}^{q+s+t<n} d_i(c)d_j(b)d_k(a)d_p(r)d_q(a)d_s(b)d_t(c)$$

$$+abcd_n(r)cba \sum_{i+j+k+p+q+s+t=n}^{p<n} d_i(a)d_j(b)d_k(c)d_p(r)d_q(c)d_s(b)d_t(a)$$

$$+cbad_n(r)abc + \sum_{i+j+k+p+q+s+t=n}^{p<n} d_i(c)d_j(b)d_k(a)d_p(r)d_q(a)d_s(b)d_t(c) \quad \dots (2)$$

Compare (1) , (2) and by assumption we get

$$0 = (d_n(abc) - \sum_{i+j+k=n} d_i(a)d_j(b)d_k(c))rcba + (d_n(cba) - \sum_{i+j+k=n} d_i(c)d_j(b)d_k(a)$$
$$+abcr(d_n(cba) - \sum_{i+j+k=n} d_i(c)d_j(b)d_k(a) + cbar(d_n(abc) - \sum_{i+j+k=n} d_i(a)d_j(b)d_k(c)$$

$= \Psi_n(a,b,c)rcba + \Psi_n(c,b,a)rabc + abcr\,\Psi_n(c,b,a) + cbar\,\Psi_n(a,b,c)$

Hence

$\Psi_n(a,b,c)r\,[a,b,c] + [a,b,c]\,r\,\Psi_n(c,b,a) = 0$

Lemma 4: Let D= (d_i)$_{i\in N}$ be Jordan triple higher derivation of a ring R. Then for all a,b,c,r ϵR and n∈N

$\Psi_n(a,b,c)r\,[a,b,c] = [a,b,c]\,r\,\Psi_n(c,b,a) = 0$

Proof: By Lemma 3 we get
$\Psi_n(a,b,c)r\,[a,b,c] + [a,b,c]\,r\,\Psi_n(c,b,a) = 0$
by Lemma1.1 we get

$\Psi_n(a,b,c)r\,[a,b,c] = [a,b,c]\,r\,\Psi_n(c,b,a) = 0$

3. The Main Results

In this section we present the main results of this paper.

Lemma 5: Let $D = (d_i)_{i \in N}$ be Jordan triple higher derivation of a 2-torsion free prime ring R. Then for all a,b,c,r,x,y,z ϵR and n∈N

$\Psi_n(a,b,c)r\,[x,y,z] = 0$

Proof: Replace a by a+x in Lemma 4 we get

$\Psi_n(a+x,b,c)r\,[a+x,b,c] = 0$

$\Psi_n(a,b,c)r\,[a,b,c] + \Psi_n(x,b,c)r\,[a,b,c] + \Psi_n(a,b,c)r\,[x,b,c] + \Psi_n(x,b,c)r\,[x,b,c] = 0$

By Lemma 4 we get

$\Psi_n(x,b,c)r\,[a,b,c] + \Psi_n(a,b,c)r\,[x,b,c] = 0$

Therefore we get

$\Psi_n(x,b,c)r\,[a,b,c]\,r\,\Psi_n(x,b,c)r\,[a,b,c] = 0$

$-\Psi_n(x,b,c)r\,[a,b,c]r\,\Psi_n(a,b,c)r\,[x,b,c] = 0$

Hence by primness of R

$\Psi_n(a,b,c)r\,[x,b,c] = 0$

Similarly, replacing b by b+y and c by c+z and use the same way we get

$\Psi_n(a,b,c)r\,[x,y,z] = 0$

Theorem 6: Every Jordan triple higher derivation of a 2-torsion free prime ring R is triple higher derivation of R.

Proof: Let $D = (d_i)_{i \in N}$ be Jordan triple higher derivation of a 2-torsion free prime ring R. Since R is prime we get from Lemma 5 either $\Psi_n(a,b,c) = 0$ or $[x,y,z] = 0$ for all a,b,c,x,y,z ∈R and n∈N.

If $\Psi_n(a,b,c) = 0$ for all a,b,c ∈R and n∈N then by Remark 2.3 we get D is triple higher derivation on R. If $[x,y,z] = 0$ for all x,y,z ∈R and n∈N, then R is a commutative ring and by Lemma 1 we get

$$d_n(2abc) = 2 \sum_{i+j+k=n} d_i(a)d_j(b)d_k(c)$$

Since R is 2-torsion free we get D is triple higher derivation of R.

Theorem 7: Every Jordan triple higher derivation of a prime ring R is higher derivation of R.

Proof: Let $D = (d_i)_{i \in N}$ be Jordan triple higher derivation of a ring R then for all a,b,r ∈R and n∈N.

$d_n(abrab) = d_n(a(bra)b)$

$$= \sum_{i+j+k=n} d_i(a)d_j(bra)d_k(b))$$

$$= \sum_{i+j=n} d_i(a)d_j(b)rab + \sum_{i+j+k+s+t=n}^{i+j<n} d_i(a)d_j(b)d_k(r)d_s(a)d_t(b)$$

$$+abrd_n(r)ab + \sum_{i+j+k+s+t=n}^{k<n} d_i(a)d_j(b)d_k(r)d_s(a)d_t(b)$$

$$+abr\sum_{i+j=n} d_i(a)d_j(b) + \sum_{i+j+k+s+t=n}^{s+t<n} d_i(a)d_j(b)d_k(r)d_s(a)d_t(b)$$

…(1)

O n the other hand
$d_n(abrab) = d_n((ab)r(ab))$

$$= \sum_{i+j+k=n} d_i(ab)d_j(r)d_k(ab))$$

$$= d_n(ab)rab + \sum_{i+j+k+s=n}^{i<n} d_i(ab)d_j(r)d_k(a)d_s(b)$$

$$+abd_n(r)ab + \sum_{i+j+k+s=n}^{j<n} d_i(ab)d_j(r)d_k(a)d_s(b)$$

$$abr\sum_{i+j=n} d_i(a)d_j(b) + \sum_{i+j+k+s=n}^{k+s<n} d_i(ab)d_j(r)d_k(a)d_s(b)$$

...(2)

Comparing (1) and (2) we get

$$(d_n(ab) - \sum_{i+j=n} d_i(a)d_j(b))rab = 0$$

By primness of R we get

$$d_n(ab) - \sum_{i+j=n} d_i(a)d_j(b) = 0$$

Thus D is higher derivation on R.

References

1) M.Ashraf, A. Khan and C.Haetinger " On (σ,τ)-higher derivations in prime rings" International electronic Journal of Algebra vol.8,PP.65-79, 2010.
2) M. Ferro and C.Heatinger "Higher derivations and a theorem by Herstien" Quaestions Mathematica, 25, PP.1-9, 2002.
3) C.Haetinger " Derivaccoes de ordem superior em aneis primos e semiprimos" Ph.D. thesis univ. Federal do Rio Grand do sul, porto algebra, Brazil, 2000.
4) H .Hasse, Noch eine Begrundung " der hohere Differentiaquotienten in einem algebraischen Funktionenkorper einer unbestimmten" J. Reine Angew. Math.177 PP. 215-237, 1936.

5) I.N.Herstein " Topics in rings theory" (Ed. The university of Chicago press, Chicago) 1957.
6) I.N.Herstein " Jordan derivations of prime rings" Proc. Amer. Math. Soc.8 , PP.1104-1110, 1957.
7) A.K.Joardar and A.C.Paul "Jordan generalize tripal derivations of prime **Γ**-rings" J.Math. Comput. Soc.2, No.5,PP. 1161-1169 , 2012.