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Solution of Fractional Mathieu Equation by Reproducing Kernel Hilbert Space Method

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Abstract: In this paper we use reproducing kernel Hilbert space method (RKHSM) and fractional power series to solve fractional Mathieu equation. A comparison tables are given.

Key words: Mathieu equation, Reproducing kernel Hilbert space method, fractional power series.

1- Introduction

Mathieu equation is a linear second order homogeneous differential equation, [6]. It was first discussed by Emile L ́eonard Mathieu in 1868 in connection with the problem of vibrations of an elliptic membrane, [5].

Mathieu equation arises in separation of variables of the Helmholtz differential equation in elliptic cylindrical coordinates, [1].

Mathieu differential equations arise as models in many contexts, including the stability of a floating body, stability of railroad rails as trains drive over them, vibrations in an elliptic drum, the inverted pendulum, the radio frequency quadrupole, frequency modulation, alternating gradient focusing, the Paul trap for charged particles, and the mirror trap for neutral particles, seasonally forced population dynamics, ion motion in quadrupole ion traps, propagation of light in periodic media and the phenomenon of parametric resonance in forced oscillators, [5], [6].

The canonical form for Mathieu differential equation is [4]

$$y''(x) + [a - 2q\cos(2x)]y(x) = 0 \qquad (1.1)$$

If we add the term $\varepsilon D^{\frac{1}{2}}y(x)$ to the equation (1.1), we obtain to the fractional Mathieu equation

$$y''(x) + \varepsilon D^{\frac{1}{2}}y(x) + [a - 2q\cos(2x)]y(x) = 0 \qquad (1.2)$$

with the boundary conditions

$$y(0) = 1, \qquad y'(0) = 0$$

where ε, a and q are constants and $D^{\frac{1}{2}}$ denotes the Caputo fractional derivative of order $\frac{1}{2}$.

The theory of reproducing kernels was used for the first time at the beginning of the 20th century by S. Zaremba in his work on boundary value problems for harmonic and biharmonic functions but he did not develop any theory and did not give any particular name to the kernels he introduced [9].

J. Mercer, Bergman, E. H. Moore and S. Bochner respectively proposed a special function in their different research fields, J. Mercer called the functions which satisfy reproducing property in the theory of integral equations as "positive definite kernels", S. Bergman introduced reproducing kernels and he called them "kernel functions" it is also called Bergman kernel function, [9], [11].

In 1948, the general theory of reproducing kernels was introduced by N. Aronszajn and since that time, the reproducing kernel Hilbert space has played a major role in operator theory and applications [9].

Reproducing kernel theory has important applications in numerical analysis, differential equations, integral equations, integro-differential equations, probability and statistics [7], [10].

2- Preliminaries

In this section, we give basic definitions and theorems of reproducing kernel Hilbert space method.

Definition (2.1) (Roughly Speaking) [3]: A reproducing kernel Hilbert space builds on a Hilbert space $\mathcal{H}$ and requires that all Dirac evaluation functional in $\mathcal{H}$ are bounded and continuous.

Definition (2.2) [3]: A Dirac functional at x is a functional $\delta_x \in \mathcal{H}$ such that $\delta_x(f) = f(x)$.

Note that δ_x is bounded if $\exists \lambda > 0$ such that $\|\delta_x f\|_{\mathbb{R}} \leq \lambda \|f\|_{\mathcal{H}}, \forall f \in \mathcal{H}$.

Definition (2.3) [10]: A function $K: X \times X \to \mathbb{R}$ is called kernel on X if $\exists$ a Hilbert space $\mathcal{H}$ and a map $\psi: X \to \mathcal{H}$, such that $K(x,y) =< \psi(x), \psi(y) > \forall x, y \in X$.

Definition (2.4) [9]: Let X be an arbitrary set. A symmetric function $K: X \times X \to \mathbb{C}$ is called positive definite kernel if $\sum_{i,j=1}^{n} \bar{\alpha}_i \alpha_j K(x_i, x_j) \geq 0,\ \forall n \in \mathbb{N}, x_1, x_2, \ldots, x_n \in X$ and $\alpha_i \in \mathbb{C}, i = 1,2, \ldots, n$.

Now, we will try to define a reproducing kernel Hilbert space.

Definition (Reproducing Kernel) (2.5) [9], [8]: Let $\mathcal{H}$ be a Hilbert space of functions $f: X \to F$ on a set X. A function $K: X \times X \to \mathbb{C}$ is a reproducing kernel of $\mathcal{H}$ iff:

1) $K(.,x) \in \mathcal{H},\ \forall x \in X$.

2) $f(x) = \langle f, K(.,x) \rangle,\ \forall f \in \mathcal{H}, \forall x \in X$.

The last condition in definition (2.6), called "the reproducing property," means that the value of the function f at the point x is reproduced by the inner product of f with $K(.,x)$.

Definition (2.6) [9], [10]: A Hilbert space $\mathcal{H}$ of functions on a set X is called a reproducing kernel Hilbert space if there exists a reproducing kernel K of $\mathcal{H}$.

Remark (2.1) [9]: Given a reproducing kernel Hilbert space $\mathcal{H}$ and its kernel $K(.,x)$ on X, $\forall x, y \in X$ we have the followings:

1. The reproducing kernel $K(x,y)$ is unique.
2. The reproducing kernel $K(x,y)$ is symmetric.
3. $K_x(x) \geq 0$, for any fixed $x \in [a,b]$.
4. $|K(x,y)|^2 \leq K(x,x)K(y,y)$, (Schwarz Inequality).

Theorem (2.1) [9]: For a Hilbert space $\mathcal{H}$ of functions on X, there exists a reproducing kernel K for $\mathcal{H}$ if and only if for every $x \in X$, the evaluation linear functional $I: f \to f(x)$ is bounded linear functional.

Definition (2.7) [10]: $W_2^m[a,b] = \{u | u^{(j)}$ is absolutely continuous, $j = 1,2 \ldots, m-1$ and $u^{(m)} \in L^2[a,b]\}$.

Theorem (2.2) [9]: The function space $W_2^m[a,b]$ is a reproducing kernel space, that is, for each fixed $x \in [a,b]$ and for any $u(y) \in W_2^m[a,b]$, there exist $K_x(y) \in W_2^m[a,b], y \in [a,b]$ such that $\langle u(y), K_x(y) \rangle = u(x)$ and $K_x(y)$ is called reproducing kernel function of $W_2^m[a,b]$.

3- Evaluating Reproducing Kernel

In this section, we will try to construct reproducing kernel functions in the space $W_2^3[0,1]$ to solve fractional Mathieu equation (1.2).

At first, we have to represent the reproducing kernel function of the space $W_2^3[0,1]$ which is defined by: $W_2^3[0,1] = \{u: u, u', u''$ are absolutely continuous and $u''' \in L^2[0,1], u(0) = u'(0) = 0\}$ with the inner product defined by:

$$\langle u(y), v(y)\rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(0)v^{(i)}(0) + u(1)v(1) + \int_0^1 u'''(y)v'''(y)dy$$

and $\|u\|_{W_2^3} = \sqrt{< u, u >_{W_2^3}}$, where $u, v \in W_2^3[0,1]$.

Let $K_x(y)$ be the reproducing kernel of the space $W_2^3[0,1]$. Then for each $x \in [0,1]$ and $u(y) \in W_2^3[0,1], y \in [0,1]$, we have:

$$u(x) = \langle u(y), K_x(y)\rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(0)K_x^{(i)}(0) + u(1)K_x(1) + \int_0^1 u'''(y)K_x'''(y)dy$$

$$= u''(0)k''(0) + u(1)K_x(1) + \int_0^1 u'''(y)K_x'''(y)dy \tag{3.3}$$

Through three integrations by parts for (3.3) we have:

$$u(x) = \langle u(y), K_x(y)\rangle_{W_2^3} = u''(0)\left(K_x''(0) - K_x'''(0)\right) + u(1)\left(K_x(1) + K_x^{(5)}(1)\right) - u'(1)K_x^{(4)}(1)$$

$$+u''(1)K_x'''(1) + \int_0^1 u(y)K_x^{(6)}dy \tag{3.4}$$

Since $K_x(0) \in W_2^3[0,1]$, it follows that

$K_x(0) = 0, \qquad K'_x(0) = 0.$

If $K_x''(0) - K_x'''(0) = 0,\ K_x(1) + K_x^{(5)}(1) = 0,\ K_x^{(4)}(1) = 0$ and $K_x'''(1) = 0$.

Then (3.4) implies that

$$u(x) = \langle u(y), K_x(y)\rangle_{W_2^3} = -\int_0^1 u(y)K_x^{(6)}(y)dy.$$

For $\forall x \in [0,1]$, if $K_x(y)$ satisfies $-K_x^{(6)}(y) = \delta(x-y)$ (3.5)

then

$$u(x) = -\int_0^1 u(y)K_x^{(6)}(y)dy = \int_0^1 u(y)\delta(x-y)dy.$$

Let $x \neq y$, then $-K_x^{(6)}(y) = 0 \Rightarrow K_x^{(6)}(y) = 0$ (3.6)

The characteristic equation of (3.6) is $\lambda^6 = 0 \Rightarrow \lambda = 0$ with multiplicity 6

$$K_x(y) = \begin{cases} \sum_{i=0}^{5} c_i(x) y^i, & y \leq x \\ \sum_{i=0}^{5} d_i(x) y^i, & y > x \end{cases}$$

Also, since $-K_x^{(6)}(y) = \delta(x-y)$, then, $K_x{}^{(i)}(x+0) = K_x{}^{(i)}(x-0), i = 0,1,2,3,4$.

Integrating (3.5) from $x-\epsilon$ to $x+\epsilon$ with respect to y and let $\epsilon \to 0$, we have the jump degree of $K_x^{(5)}(y)$ at $x = y$,

$K_x^{(5)}(x-0) - K_x^{(5)}(x+0) = 1$.

To find $c_i(x), d_i(x), i = 0,1,\dots,5$ we will solve the following equations:

1) $K_x(0) = 0$ 2) $K'_x(0) = 0$
3) $K''_x(0) - K'''_x(0) = 0$ 4) $K_x(1) + K_x{}^{(5)}(1) = 0$
5) $K_x^{(4)}(1) = 0$ 6) $K'''_x(1) = 0$
7) $K_x^{(i)}(x+0) = K_x^{(i)}(x-0), \quad i = 0,1,2,3,4$
8) $K_x^{(5)}(x+0) - K_x^{(5)}(x-0) = -1$.

The reproducing kernel function of $W_2^3[0,1]$ is given by:

$$K_x(y) = \frac{-1}{18720} \begin{cases} f_1(x,y), & y \geq x \\ f_2(y,x), & y < x \end{cases}$$

$$f_1(x,y) = y^2(780xy^2 - 156y^3 + 30x^2(-126 - 42y - 5y^2 + y^3) + 10x^3(30 + 10y - 5y^2 + y^3) - 5x^4(30 + 10y - 5y^2 + y^3) + x^5(30 + 10y - 5y^2 + y^3))$$

$$f_2(y,x) = x^2\big(780x^2y - 156x^3 + 30y^2(-126 - 42x - 5x^2 + x^3) + 10y^3(30 + 10x - 5x^2 + x^3) - 5y^4(30 + 10x - 5x^2 + x^3) + y^5(30 + 10x - 5x^2 + x^3)\big).$$

By the same way we can define five inner products with their reproducing kernels:

1) $\langle u(y), K_{1x}(y)\rangle_{W_2^3} = u(0)K_{1x}(0) + u'(1)K'_{1x}(1) + u''(0)K''_{1x}(0) + \int_0^1 u'''(y)K'''_{1x}(y)\,dy$

$$K_{1x}(y) = \frac{1}{6720} \begin{cases} f_1(x,y), & y \geq x \\ f_2(y,x), & y < x \end{cases}$$

$$f_1(x,y) = y^2(56y^3 + 60x^2(16 - 4x + x^2) + 20x^2y(16 - 4x + x^2) - 5xy^2(56 - 12x - 4x^2 + x^3))$$

$$f_2(y,x) = x^2(56x^3 + 60y^2(16 - 4y + y^2) + 20xy^2(16 - 4y + y^2) - 5x^2y(56 - 12y - 4y^2 + y^3))$$

2) $\langle u(y), K_{2x}(y)\rangle_{W_2^3} = \sum_{i=0}^{1} u^{(i)}(0)K_{2x}^{(i)}(0) + \sum_{i=0}^{1} u^{(i)}(1)K_{2x}^{(i)}(1) + \int_0^1 u'''(y)K'''_{2x}(y)\,dy$

$$K_{2x}(y) = \frac{1}{72480} \begin{cases} f_1(x,y), & y \geq x \\ f_2(y,x), & y < x \end{cases}$$

$$f_1(x,y) = y^2(5x^2(3987 - 1208x + 367x^2 - 26x^3) + 5xy^2(-604 + 367x - 5x^3 + 2x^4) + 2y^3(302 - 65x^2 + 5x^4 - 2x^5))$$

$$f_2(y,x) = x^2(5y^2(3987 - 1208y + 367y^2 - 26y^3) + 5x^2y(-604 + 367y - 5y^3 + 2y^4) + 2x^3(302 - 65y^2 + 5y^4 - 2y^5))$$

3) $\langle u(y), K_{3x}(y)\rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(0)K_{3x}^{(i)}(0) + \sum_{i=0}^{1} u^{(i)}(1)K_{3x}^{(i)}(1) + \int_0^1 u'''(y)K_{3x}'''(y)\,dy$

$$K_{3x}(y) = \frac{1}{913320}\begin{cases} f_1(x,y), & y \geq x \\ f_2(y,x), & y < x \end{cases}$$

$$f_1(x,y) = y^2(7611y^3 - 38055xy^2 + 30x^2(3987 + 1329y + 367y^2 - 26y^3) - 10x^3(3624 + 1208y - 367y^2 + 26y^3) + 5x^4(2202 + 734y - 286y^2 + 41y^3) - x^5(780 + 260y - 205y^2 + 56y^3))$$

$$f_2(y,x) = x^2(7611x^3 - 38055x^2y + 30y^2(3987 + 1329x + 367x^2 - 26x^3) - 10y^3(3624 + 1208x - 367x^2 + 26x^3) + 5y^4(2202 + 734x - 286x^2 + 41x^3) - y^5(780 + 260x - 205x^2 + 56x^3))$$

4) $\langle u(y), K_{4x}(y)\rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(0)K_{4x}^{(i)}(0) + \sum_{i=0}^{2} u^{(i)}(1)K_{4x}^{(i)}(1) + \int_0^1 u'''(y)K_{4x}'''(y)\,dy$

$$K_{4x}(y) = \frac{1}{913320}\begin{cases} f_1(x,y), & y \geq x \\ f_2(y,x), & y < x \end{cases}$$

$$f_1(x,y) = y^2(-185065xy^2 + 37013y^3 + 5x^4(2736 + 912y + 164y^2 - 279y^3) + 10x^3(-16767 - 5589y + 456y^2 + 127y^3) + x^5(3810 + 1270y - 1395y^2 + 342y^3) + 10x^2(60738 + 20246y + 1368y^2 + 381y^3))$$

$$f_2(y,x) = x^2(37013x^3 - 185065x^2y + 10(60738 + 20246x + 1368x^2 + 381x^3)y^2 + 10(-16767 - 5589x + 456x^2 + 127x^3)y^3 + 5(2736 + 912x + 164x^2 - 279x^3)y^4 + (3810 + 1270x - 1395x^2 + 342x^3)y^5)$$

5) $\langle u(y), K_{5x}(y)\rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(0)K_{5x}^{(i)}(0) + \int_0^1 u'''(y)K_{5x}'''(y)\,dy$

$$K_x(y) = \frac{-1}{120}\begin{cases} y^2\left(5xy^2 - y^3 - 10x^2(3+y)\right) & y \leq x \\ x^2\left(5x^2y - x^3 - 10y^2(3+x)\right) & y > x \end{cases}.$$

Also, we need to find the reproducing kernel of $W_2^1[0,1]$ which is defined by: $W_2^1[0,1] = \{u: u \text{ is absolutely continuous}, u, u' \in L^2[0,1]\}$ with the inner product

$$\langle u(y), R_x(y)\rangle_{W_2^1} = \int_0^1 \left(u(y)R_x(y) + u'(y){R_x}'(y)\right)dy$$

and norm $\|u\|_{W_2^1} = \sqrt{\langle u,u\rangle_{W_2^1}}$.

In [2], Li and Cui proved that $W_2^1[0,1]$ is complete reproducing kernel Hilbert space and its reproducing kernel is given by

$$R_x(y) = \frac{1}{2\sinh(1)}[\cosh(x+y-1) + \cosh|x-y| - 1].$$

4- Solution of fractional Mathieu equation

In this section, comparisons of results have been made through reproducing kernel Hilbert space method and fractional power series.

1. Solution by using fractional power series

Suppose the series solution of equation (1.2) is:

$$y(x) = \sum_{n=0}^{\infty} C_n\, x^{n/2} \tag{4.7}$$

Substituting equation (4.7) into equation (1.2), it then follows

$$\sum_{n=0}^{\infty} \frac{n}{2}(\frac{n}{2}-1)C_n\, x^{n/2-2} + \varepsilon \sum_{n=0}^{\infty} C_n \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)}\, x^{\frac{n-1}{2}} + \left[(a-2q)+4qx^2-\frac{4}{3}\,qx^4+\cdots\right] \left(1+\sum_{n=0}^{\infty} C_n\, x^{n/2}\right) = 0 \tag{4.8}$$

Note that $y(0) = 1 \Longrightarrow C_0 = 1$, by comparing the coefficients of the resulting fractional power series in equation (4.8) we obtain

$C_1 = C_2 = C_3 = 0$, $C_4 = \frac{-a+2q}{2}$, $C_5 = C_6 = 0$, $C_7 = \frac{(a-2q)\varepsilon}{8.75\,\Gamma(2.5)}$, $C_8 = \frac{(a-2q)^2-8q}{24}$, $C_9 = 0$,

$C_{10} = -0.00833333(a-2q)\varepsilon^2$, $C_{11} = \frac{8q-2(a-2q)^2}{216.5625\,\Gamma(2.5)}\varepsilon$,

$C_{12} = 0.0444444\,q + \frac{(56q-(a-2q)^2)(a-2q)}{720}$, $C_{13} = 0.0005344(a-2q)\varepsilon^3$,

$C_{14} = (-0.0015873q + 0.0005952(a-2q)^2)\varepsilon^2$

The results obtained above lead to a series solution of equation (1.2) as

$$\begin{aligned} y(x) = \sum_{n=0}^{14} C_n\, x^{\frac{n}{2}} &= 1 + \frac{-a+2q}{2}x^2 + \frac{\varepsilon(a-2q)}{8.75\,\Gamma(2.5)}x^{\frac{7}{2}} + \frac{(a-2q)^2-8q}{24}x^4 \\ &- 0.00833333(a-2q)\varepsilon^2x^5 + \frac{8q-2(a-2q)^2}{216.5625\,\Gamma(2.5)}\varepsilon x^{\frac{11}{2}} \\ &+ \left(0.0444444\,q + \frac{(56q-(a-2q)^2)(a-2q)}{720}\right)x^6 + 0.0005344(a-2q)\varepsilon^3x^{\frac{13}{2}} \\ &+ (-0.0015873q + 0.0005952(a-2q)^2)\varepsilon^2x^7 \end{aligned}$$

2. Solution by using reproducing kernel method

To solve equation (1.2) by using reproducing kernel method we must homogenization the boundary conditions as follows:

let $u(x) = y(x) + b$

$u(0) = 0 = y(0) + b \Longrightarrow b = -1$

$u(x) = y(x) - 1 \Longrightarrow y(x) = u(x) + 1$

Then $u''(x) + \varepsilon D^{\frac{1}{2}}u(x) + (a - 2q\cos(2x))u(x) = -a + 2q\cos(2x)$

with the homogenize boundary conditions

$u(0) = 0, \qquad u'(0) = 0.$

Define the operator L: $W_2^3[0,1] \to W_2^1[0,1]$ such that:

$$Lu(x) = u''(x) + \varepsilon D^{\frac{1}{2}}u(x) + [a - 2q\cos(2x)]u(x) = f(x) \tag{3.9}$$

where $f(x) = -a + 2q\cos(2x)$, $x \in [0,1], u(x) \in W_2^3[0,1]$ and $f(x) \in W_2^1[0,1]$ with the boundary conditions $u(0) = 0, u'(0) = 0$.

Theorem (4.3) [8]: The operator L defined by (3.9) is a bounded linear operator.

Let $\phi_i(x) = R_{x_i}(x)$ and $\psi_i(x) = L^*\phi_i(x) = \langle L^*\phi_i(x), K_x(y)\rangle_{W_2^3} = \langle \phi_i(x), LK_x(y)\rangle_{W_2^1}$

$= \langle R_{x_i}(x), LK_x(y)\rangle_{W_2^1} = L_{x_i}K_x(x_i)$, where L^* is the adjoint operator of L.

Theorem (4.4) [8], [10]: Assume that the inverse operator L^{-1} exists. Then if $\{x_i\}_{i=1}^{\infty}$ is dense on [0,1], then $\{\psi_i\}_{i=1}^{\infty}$ is the complete function system of the space $W_2^3[0,1]$.

Now, we will form an orthonormal function $\{\overline{\psi_\iota}(x)\}_{i=1}^{\infty}$ of the space W_2^3 by Gram- Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ as follows:

$$\overline{\psi_\iota}(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad i = 1,2,\ldots$$

where $\beta_{11} = \frac{1}{\|\psi_1\|}, \beta_{ii} = \frac{1}{\sqrt{\|\psi_i\|^2 - \sum_{j=1}^{i-1}(\overline{C_{\iota k}})^2}}, \beta_{ik} = \frac{-\sum_{j=k}^{i-1}\overline{C_{\iota J}}\beta_{jk}}{\sqrt{\|\psi_i\|^2 - \sum_{j=1}^{i-1}(\overline{C_{\iota J}})^2}}, k < i$ and

$$\overline{C_{\iota k}} = \langle \psi_i, \overline{\psi_k}\rangle_{W_2^3} = \sum_{j=1}^{k} \beta_{jk}\, c_{ik}$$

$\overline{C_{\iota k}}$ does not mean the conjugate of c_{ik}

Theorem (4.5) [8], [10]: $\forall u(x) \in W_2^3[0,1]$, the series $\sum_{i=1}^{\infty}\langle u(x), \overline{\psi_\iota}(x)\rangle\overline{\psi_\iota}(x)$ is convergent in sense of the norm of $W_2^3[0,1]$. Moreover, if $\{x_i\}_{i=1}^{\infty}$ is a countable dense set in [0,1], then the solution of (1) is unique and given by:

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi_\iota}(x).$$

The n[th] term of the solution is given by:

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(x_k)\overline{\psi_\iota}(x).$$

Theorem (4.6) [8], [10]: The approximate solution $u_n(x)$ and its derivative $u_n'(x), u_n''(x)$ are uniformly convergent.

The approximate results of reproducing kernel method and fractional power series for fractional Mathieu equation are given in tables

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X_i	RK $\frac{x_i-1}{n-1}, n=5, a=\varepsilon=1$ $q=0$	RK_1 $\frac{x_i-1}{n-1}, n=5, a=\varepsilon=1$ $q=0$	RK_2 $\frac{x_i-1}{n-1}, n=5, a=\varepsilon=1$ $q=0$
0	1	1	1
0.25	0.9700890056845057	0.9700963239349192	0.9701353312393597
0.5	0.8849391499815973	0.8848559023663441	0.8849097120741788
0.75	0.7478238831383045	0.7469961216762038	0.7471651995864138
1	0.558284653231984	0.5531214426896718	0.5551008445564383

RK_{3x} $\frac{x_i-1}{n-1}, n=5, a=\varepsilon=1$ $q=0$	RK_{4x} $\frac{x_i-1}{n-1}, n=5, a=\varepsilon=1$ $q=0$	Power Series $n=5, a=\varepsilon=1$ $q=0$
1	1	1
0.9701353762653322	0.9700454996894771	0.969572574323299
0.8849088844283476	0.8848248655154767	0.8847686914304639
0.7471633601052606	0.7459395143927888	0.7596942166368739
0.5551983591214592	0.55144908936128	0.6109082120390124

Table (4.1) Comparison between RK_x's and Power Series

X_i	RK $\frac{x_i-1}{n-1}, n=11, a=\varepsilon=1$ $q=0$	RK_1 $\frac{x_i-1}{n-1}, n=11, a=\varepsilon=1$ $q=0$	RK_2 $\frac{x_i-1}{n-1}, n=11, a=\varepsilon=1$ $q=0$
0	1	1	1
0.1	0.9950494301561579	0.9950511512043337	0.9950499154028478
0.2	0.9803796479453707	0.9803792320811204	0.9803793172686684
0.3	0.9562093875178921	0.9561812629555241	0.9562088950758565
0.4	0.9226424912188542	0.9225434083950091	0.9226586985648821
0.5	0.8796261364962504	0.8793985252431019	0.8796937527433081
0.6	0.8269488716877154	0.8265062109420713	0.8270979876870527
0.7	0.7642963369629775	0.7634886894840042	0.7645107869813359
0.8	0.6929949121099505	0.6899371158678548	0.6915388299706251
0.9	0.6109992938863179	0.6055921817341279	0.6079468217424137
1	0.5197031559263903	0.5106434732733808	0.5141064761583451

RK_3 $\frac{x_{i-1}}{n-1}, n = 11, a = \varepsilon = 1$ $q = 0$	RK_4 $\frac{x_{i-1}}{n-1}, n = 11, a = \varepsilon = 1$ $q = 0$	power series $n = 11, a = \varepsilon = 1, q = 0$
1	1	1
0.9950511512870329	0.9950479281002164	0.9950374967384358
0.9803792335213024	0.980379633875172	0.9803705064639671
0.9561812664351571	0.9562392576588177	0.9565784682347266
0.9225433796168679	0.9227687493552158	0.9244109530649598
0.8793987208804022	0.8799637293998934	0.8847686914304639
0.8265087803182025	0.8276746106462751	0.8386555853877838
0.7635026151058214	0.7656659455420907	0.7871385625372038
0.6899858235634965	0.6937362317747948	0.7313067490711174
0.6057060606251693	0.6118940413536317	0.6722288687438513
0.5107477564819802	0.5205910748453355	0.6109082120390124

Table (4.2) Comparison between RK_x's and Power Series

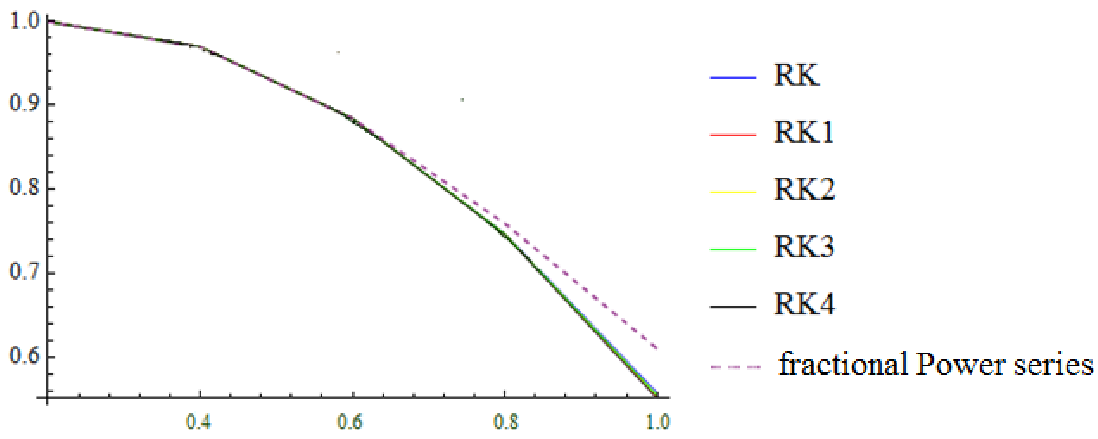

Figure (4.1): Plots of Comparison between RK_x's and Power Series at $n = 5, a = \varepsilon = 1$ and $q = 0$

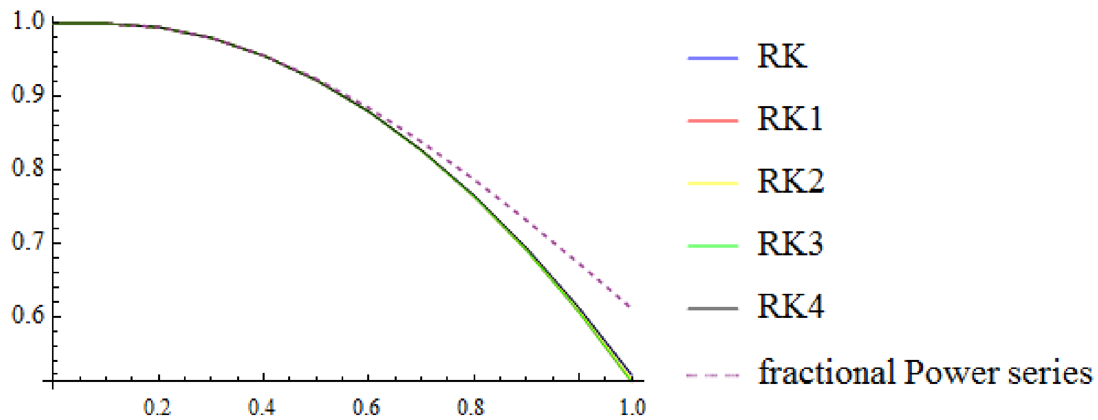

Figure (4.2): Plots of Comparison between RK_x's and Power Series at $n = 11, a = \varepsilon = 1$ and $q = 0$

X_i	RK $\frac{x_i-1}{n-1}, n=5, a=\varepsilon=1$ $q=1$	RK_1 $\frac{x_i-1}{n-1}, n=5, a=\varepsilon=1$ $q=1$	RK_2 $\frac{x_i-1}{n-1}, n=5, a=\varepsilon=1$ $q=1$	RK_3 $\frac{x_i-1}{n-1}, n=5, a=\varepsilon=1$ $q=1$	RK_4 $\frac{x_i-1}{n-1}, n=5, a=\varepsilon=1$ $q=1$	power series $n=5, a=\varepsilon=1, q=1$
0	1	1	1	1	1	1
0.25	1.0279299031011442	1.0279133574615325	1.0278727788820368	1.0278726347383391	1.0279720959614869	1.0294494126504399
0.5	1.0994497400846623	1.0998670383830123	1.0994508833688594	1.099451374365992	1.0994558357754924	1.0993801000465315
0.75	1.2082452995230955	1.212819256231625	1.2079285337533252	1.2079655175147175	1.2080017364781754	1.1579151860174877
1	1.3966470258426518	1.4183662618880148	1.3965043350533302	1.3901963044305	1.3956234119149498	1.1180655337895808

Table (4.3) Comparison between RK_x's and Power Series

X_i	RK $\frac{x_i-1}{n-1}, n=11, a=\varepsilon=1$ $q=0$	RK_1 $\frac{x_i-1}{n-1}, n=11, a=\varepsilon=1$ $q=0$	RK_2 $\frac{x_i-1}{n-1}, n=11, a=\varepsilon=1$ $q=0$	RK_3 $\frac{x_i-1}{n-1}, n=11, a=\varepsilon=1$ $q=1$	RK_4 $\frac{x_i-1}{n-1}, n=11, a=\varepsilon=1$ $q=1$	power series $n=, a=\varepsilon=1, q=1$
0	1	1	1	1	1	1
0.1	1.0049003905846121	1.0048997058722409	1.0048984294240664	1.0048984294597205	1.0049019439951083	1.0049437637074818
0.2	1.0192214646429878	1.019221957769314	1.019221338918672	1.0192213373228078	1.0192216468819533	1.0192293282904594
0.3	1.0424455826185484	1.0424641973556024	1.0424808893728488	1.0424808683567808	1.0424182885012365	1.041390377593199
0.4	1.07421543192738	1.074289711726806	1.0743227333072443	1.0743226959506733	1.0741159726876328	1.069139897863303
0.5	1.1147479744982298	1.1149363371815921	1.1148845977082336	1.1148849253146853	1.1145325842861358	1.0993801000465315
0.6	1.1657605557674189	1.166232029121039	1.1657600151184508	1.1657557958578266	1.1653935510488205	1.128181676070746
0.7	1.2315571862219314	1.2327204439851729	1.231216647147115	1.2311143141272654	1.2309882662987994	1.1507378763461407
0.8	1.3190184091439656	1.321734737535771	1.3182991079633308	1.3174447975757986	1.3181393560597796	1.1612944846824496
0.9	1.4347708522284783	1.4407950901131303	1.4341043964655755	1.4298282604149115	1.433372265247914	1.1530556963476357
1	1.57902282009763	1.5917919652633299	1.597520240840496	1.5653845270185258	1.576820960927321	1.1180655337895808

Table (4.4) Comparison between RK_x's and Power Series

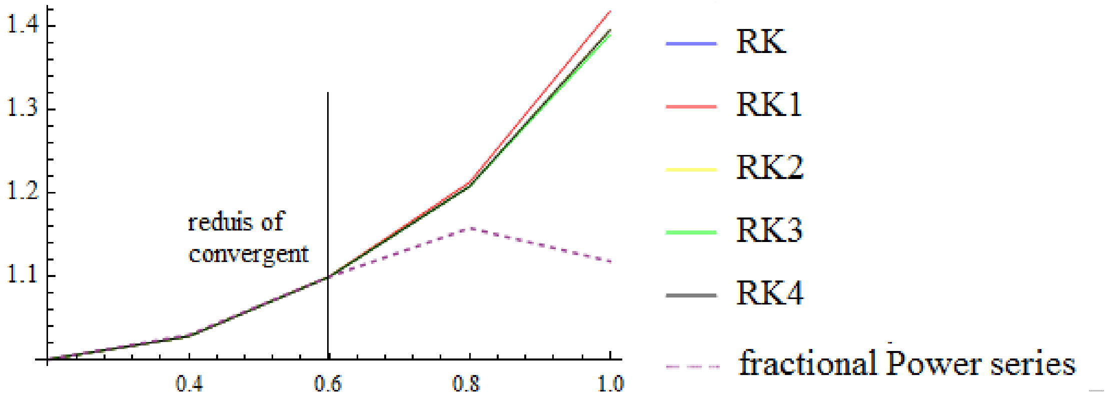

Figure (4.3): Plots of Comparison between RK_x's and Power Series at $n = 5, a = \varepsilon = q = 1$

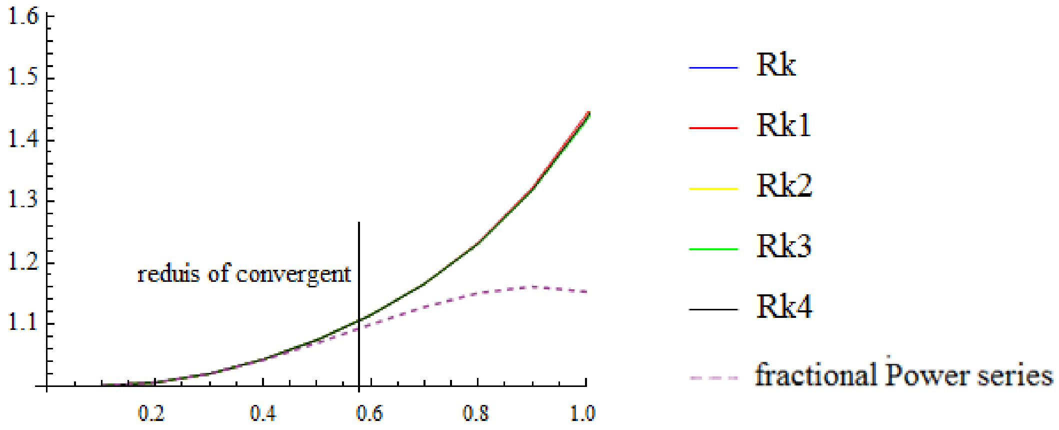

Figure (4.4): Plots of Comparison between RK_x's and Power Series at $n = 11, a = \varepsilon = q = 1$

1- Conclusion

It is known from the preview calculation that using reproducing kernel Hilbert space method is applicable with good accuracy. The fractional series method has radius of converges approximately 0.5, so one can notice the growing of the error in the tables. Also we get more than one approximate solution for the problem.

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