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MATHEMUSIC – Numbers and Notes
A Mathematical Approach To Musical Frequencies

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Abstract
Mathematics and Music, the most sharply contrasted fields of scientific activity which can be found, and yet related, supporting each other, as if to show forth the secret connection which ties together all the activities of our mind [8]. Music theorists sometimes use mathematics to understand music. Mathematics is "the basis of sound" and sound itself "in its musical aspects... exhibits a remarkable array of number properties", simply because nature itself "is amazingly mathematical". In today's technology, without mathematics it is difficult to imagine anything feasible. In this paper we have discussed the relation between music and mathematics. How piano keys are interrelated with mathematics, frequencies are correlated and discussed. Frequencies of musical instrument (piano) are analyzed using regression and geometric progression. Comparisons between both the methods are done in this paper. This paper will also be helpful for music seekers and mathematician to understand easily and practicing of musical instruments.
Keywords: Musical notes, Regression analysis, Geometric progression.

1. Introduction

Mathematics and music are interconnected topics. "Music gives beauty and another dimension to mathematics by giving life and emotion to the numbers and patterns." Mathematical concepts and equations are connected to the designs and shapes of musical instruments, scale intervals and musical compositions, and the various properties of sound and sound production. This paper will allow exploring several aspects of mathematics related to musical concepts.

A musical keyboard is the set of adjacent depressible levers or keys on a musical instrument, particularly the piano. Keyboards typically contain keys for playing the twelve notes of the Western musical scale, with a combination of larger, longer keys and smaller, shorter keys that repeats at the interval of an octave. Depressing a key on the keyboard causes the instrument to produce sounds, either by mechanically striking a string or tine (piano, electric piano, clavichord); plucking a string (harpsichord); causing air to flow through a pipe (organ); or strike a bell (carillon). On electric and electronic keyboards, depressing a key connects a circuit (Hammond organ, digital piano, and synthesizer). Since the most commonly encountered keyboard instrument is the piano, the keyboard layout is often referred to as the "piano keyboard".

The twelve notes of the Western musical scale are laid out with the lowest note on the left; The longer keys (for the seven "natural" notes of the C major scale: C, D, E, F, G, A, B) jut forward. Because these keys were traditionally covered in ivory they are often called the white notes or white keys. The keys for the remaining five notes—which are not part of the C major scale—(i.e .,C♯, D♯, F♯, G♯, A♯) are raised and shorter. Because these keys receive less wear, they are often made of black colored wood and called the black notes or black keys. The pattern repeats at the interval of an octave.

1.1 Piano Keyboard [10]

For understanding of this paper, it is important to have some knowledge of the piano keyboard, which is illustrated in the following diagram. This keyboard has 88 keys of which 36 (the top of the illustration), striking each successive key produces a pitch with a particular frequency that is higher than the pitch produced by striking the previous key by a fixed interval called a semitone. The frequencies increase from left to right. Some examples of the names of the keys are A0, A0#, B0, C1, C1#. For the purposes of this paper, all the black keys will be referred to as sharps (#). In this paper different frequencies of piano are discussed, how they are produced periodically with the use of Regression Analysis and Geometric Progression.
Diagram illustrates different key numbers, key names and their corresponding frequencies in piano keyboard. From key numbers 1 to 12 frequencies are given, but from 13 to 24 they form the same pattern but double the initial values and from 25 to 36 values are thrice of initial values and so on.

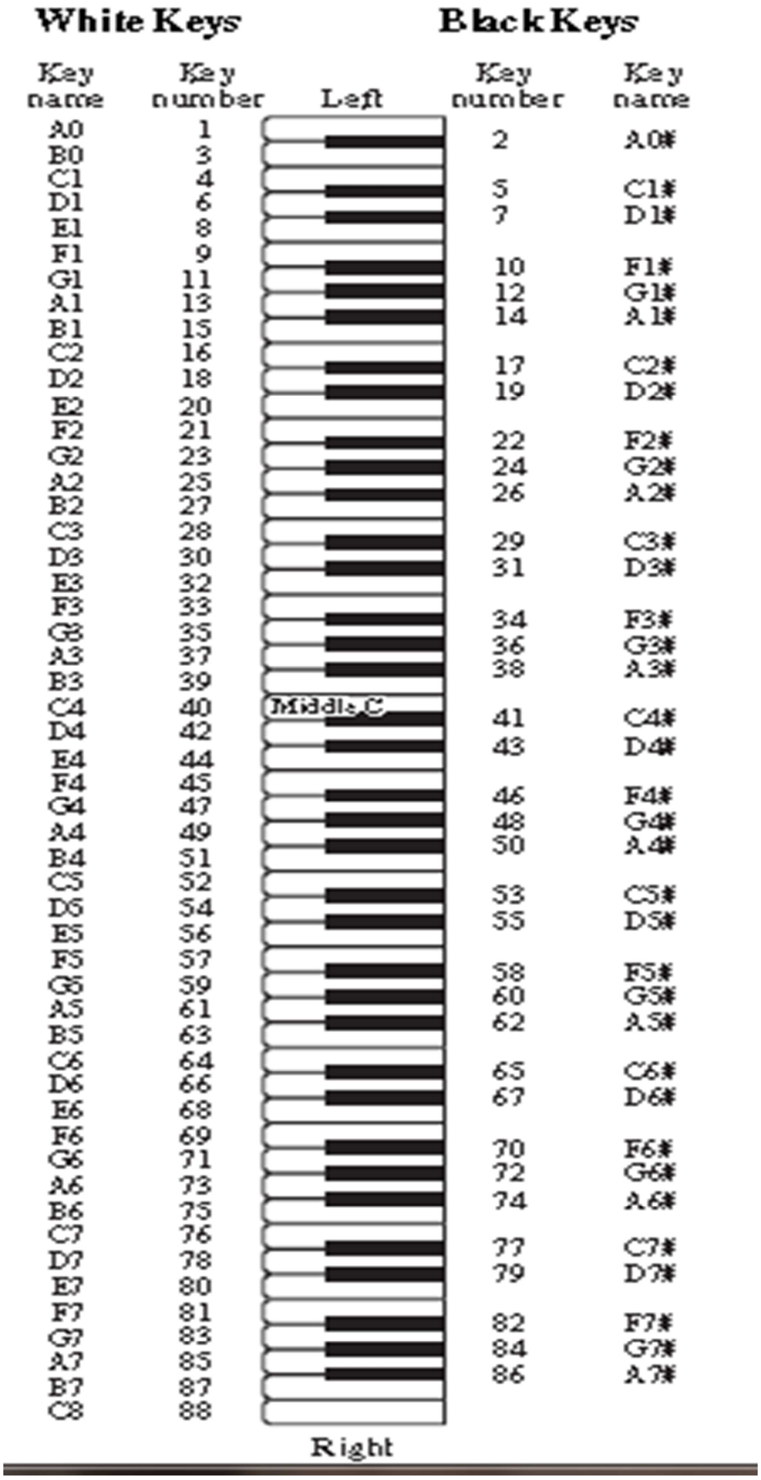

Figure 1.1 Piano Keyboard[10]

The frequencies of all successive pitches produced by striking the keys on a piano keyboard form a pattern. The diagram on the left shows the first 12 keys of a piano. The table down shows the frequency of the pitch produced by each key, to the nearest thousandth of a Hertz (Hz).

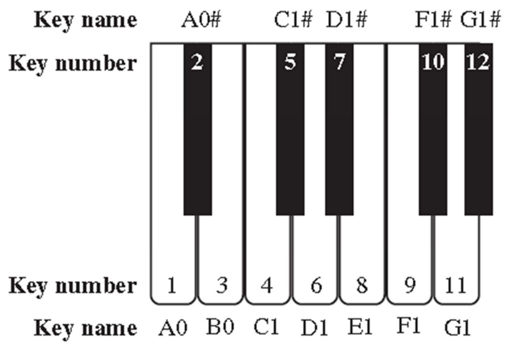

Table 1.1 – Different Frequencies on Piano Keyboard

Key Name	Key Number	Frequency (Hz)
A0	1	27.500
A0#	2	29.135
B0	3	30.868
C1	4	32.703
C1#	5	34.648
D1	6	36.708
D1#	7	38.891
E1	8	41.203
F1	9	43.654
F1#	10	46.249
G1	11	48.999
G1#	12	51.913

2. Regression

In statistics, regression analysis includes many techniques for modeling and analyzing several variables, when the focus is on the relationship between a dependent variable and one or more independent variables. More specifically, regression analysis helps one understand how the typical value of the dependent variable changes when any one of the independent variables is varied, while the other independent variables are held fixed. Regression are of two types,

Linear Regression and Exponential Regression

A linear regression produces the slope of a line that best fits a single set of data points. For example a linear regression could be used to help project the sales for next year based on the sales from this year.

An exponential regression produces an exponential curve that best fits a single set of data points. For example an exponential regression could be used to represent the growth of a population. This would be a better representation than using a linear regression.

Best fit associated with n points

$(x_1, y_1), (x_2, y_2), \ldots\ldots\ldots\ldots (x_n, y_n)$ has exponential formula,

$$y = ar^x$$

Taking log both sides

$$\log y = \log a + x \log r$$

Equating with Y= mx + b

Slope

$$m = \log r$$

Intercept

$$b = \log a$$

Best fit line using log y as a function of x.

$r = 10^m$, $a = 10^b$.

Key Name	Key Number	Frequency (Hz)
A0	1	27.500
A0#	2	29.135
B0	3	30.868
C1	4	32.703
C1#	5	34.648
D1	6	36.708
D1#	7	38.891
E1	8	41.203
F1	9	43.654
F1#	10	46.249
G1	11	48.999
G1#	12	51.913

Equating above table with Regression analysis, taking key number as x and frequencies as y = log (frequency). Developing the table with these attributes .

Table 2.1 – For Regression analysis

x	z	$y=\log z$	xy	x^2
1	27.500	1.4393	1.4393	1
2	29.135	1.4644	2.9288	4
3	30.868	1.4895	4.4685	9
4	32.703	1.51458	6.0583	16
5	34.648	1.5396	7.698	25
6	36.708	1.5647	9.3882	36
7	38.891	1.5898	11.1286	49
8	41.203	1.6149	12.9192	64
9	43.654	1.6400	14.760	81
10	46.249	1.6651	16.651	100
11	48.999	1.6901	18.5911	121
12	51.913	1.7152	20.5824	144
$\Sigma x=78$	$\Sigma z=462.471$	$\Sigma y=18.92718$	$\Sigma xy=126.61342$	$\Sigma x^2=650$

Equations for exponential regression

Slope =

$$m=\frac{n\Sigma(xy)-(\Sigma x)(\Sigma y)}{n\Sigma x^2-(\Sigma x)^2}$$

$$m=\frac{12\times 126.61342-(78)\times(18.92718)}{12\times 650-(78)^2}$$

$$m=.0250821$$

Intercept =

$$b=\frac{(\Sigma y)-m(\Sigma x)}{n}$$

$$b=1.41423135$$

$$y=mx+b$$

$$y=.0250821x+1.41423135$$

For x = 1, y is y = .0250821x1 + 1.41423135, y = 27.49878

For x = 8, y is y = .0250821x8 + 1.41423135, y = 41.19914

For x=13, y is y = .0250821 x 13 + 1.41423135 y = 1.74029865

As y = log z so z = 54.99189

For x = 14, y is y = .0250821 x 14+ 1.41423135

y = 1.76538075

As y = log z so z = 58.26137

For x = 18, y is y = .0250821x18 + 1.41423135, y = 73.40221

Following Table contains the full range of frequencies, with actual and calculated frequencies through regression analysis.

Table 2.2- Calculated Frequencies through Regression Analysis

Key Name	Key No.	Frequency (Actual) Hz	Frequency (Regression) Hz	Key Name	Key No.	Frequency (Actual) Hz	Frequency (Regression) Hz
A0	1	27.500	27.49878	A1	13	2*27.500= 55.000	54.99189
A0#	2	29.135	29.13369	A1#	14	2*29.135= 58.270	58.26137
B0	3	30.868	30.86580	B1	15	2*30.868= 61.736	61.725249
C1	4	32.703	32.70090	C2	16	2*32.703= 65.406	65.39511
C1#	5	34.648	34.645102	C2#	17	2*34.648= 69.296	69.28306
D1	6	36.708	36.704891	D2	18	2*36.708= 73.416	73.40221
D1#	7	38.891	38.88714	D2#	19	2*38.891= 77.782	77.76627
E1	8	41.203	41.19914	E2	20	2*41.203= 82.406	82.10991
F1	9	43.654	43.64859	F2	21	2*43.654= 87.308	87.2882
F1#	10	46.249	46.24367	F2#	22	2*46.249= 92.498	92.477821
G1	11	48.999	48.99305	G2	23	2*48.999= 97.998	97.97599
G1#	12	51.913	51.90588	G2#	24	2*51.913= 103.826	103.80135

3. Geometric Progression

The frequencies of pitches produced by striking the piano keys can also be modeled by a geometric sequence. The model can be determined by using a pair of keys with the same letter and consecutive numbers; for example, A0 and A1, or B1 and B2, or G2# and G3#. Each pair of consecutive keys with the same letter has frequencies with a ratio of 2:1. In other words, the frequency of A1 (55.000 Hz) is double the frequency of A0 (27.500 Hz), the frequency of A2 (110.000 Hz) is double the frequency of A1 (55.000 Hz), and so on. In mathematics, a geometric progression, also known as a geometric sequence, is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed non-zero number called the common ratio. For example, the sequence 2, 6, 18, 54 ... is a geometric progression with common ratio 3. Similarly 10, 5, 2.5, 1.25, is a geometric sequence with common ratio 1/2. The sum of the terms of a geometric progression, or of an initial segment of a geometric progression, is known as a geometric series.

Thus, the general form of a geometric sequence is

$a_0, a_1, a_2 \ldots\ldots\ldots\ldots a_{n-1}$ Or

$a, a_1, a_2 \ldots\ldots\ldots\ldots a_{n-1}$

nth term will be $a_n = ar^{n-1}$

Where a is the first term and r is the common ratio.

$$r = \frac{a_1}{a}$$

Referring table no. 2.1

So if we take frequencies of key numbers in geometric progression then first term will be 27.500 and second term will be 29.135, so r = 29.135/27.500 = 1.05945

$a_n = ar^{n-1}$, $a_n = 27.500(1.05945)^{n-1}$

for n = 2 , $a_2 = ar^1$, $a_2 = 27.500 \times (1.05945)^{2-1}$

$a_2 = 27.500 \times (1.05945)$ = 29.13487

for n =9, $a_9 = ar^8$, $a_9 = 27.500 \times (1.05945)^8$ =43.64921

for n = 14, $a_{14} = ar^{13}$ =58.2611

for n = 24, $a_{24} = ar^{23}$, $a_{24} = 27.500 \times (1.05945)^{23}$ =103.79666

Table followed contains Key name, Key number with the actual frequencies and frequencies calculated through geometric progression.

Table 3.1 – Calculation of Frequencies through Geometric Progression.

4. Comparison of Frequencies through Regression Analysis an Geometric Progression.

Key Name	**Key No.**	**Frequency (Actual) Hz**	**Frequency (GP) Hz**	**Key Name**	**Key No.**	**Frequency (Actual) Hz**	**Frequency (GP) Hz**
A0	1	**27.500**	**27.500**	**A1**	13	2*27.500= **55.000**	**54.99184**
A0#	2	**29.135**	**29.13487**	**A1#**	14	2*29.135= **58.270**	**58.26110**
B0	3	**30.868**	**30.86700**	**B1**	15	2*30.868= **61.736**	**61.72473**
C1	4	**32.703**	**32.70090**	**C2**	16	2*32.703= **65.406**	**65.39426**
C1#	5	**34.648**	**34.64611**	**C2#**	17	2*34.648= **69.296**	**69.28196**
D1	6	**36.708**	**36.70583**	**D2**	18	2*36.708= **73.416**	**73.40077**
D1#	7	**38.891**	**38.88798**	**D2#**	19	2*38.891= **77.782**	**77.76444**
E1	8	**41.203**	**41.19988**	**E2**	20	2*41.203= **82.406**	**82.38754**
F1	9	**43.654**	**43.64921**	**F2**	21	2*43.654= **87.308**	**87.28548**
F1#	10	**46.249**	**46.24416**	**F2#**	22	2*46.249= **92.498**	**92.47460**
G1	11	**48.999**	**48.99337**	**G2**	23	2*48.999= **97.998**	**97.97222**
G1#	12	**51.913**	**51.90603**	**G2#**	24	2*51.913= **103.826**	**103.79666**

KN = Key Name, GP = Geometric Progression

Table 4.1 – Comparison of Frequencies

K.N	Frequency (Actual) Hz	Frequency (Regression) Hz	Frequency (GP) Hz	K.N	Frequency (Actual) Hz	Frequency (Regression) Hz	Frequency (GP) Hz
A0	27.500	27.49878	27.500	A1	2*27.500= 55.000	54.99189	54.99184
A0#	29.135	29.13369	29.13487	A1#	2*29.135= 58.270	58.26137	58.26110
B0	30.868	30.86580	30.86700	B1	2*30.868= 61.736	61.725249	61.72473
C1	32.703	32.70090	32.70090	C2	2*32.703= 65.406	65.39511	65.39426
C1#	34.648	34.645102	34.64611	C2#	2*34.648= 69.296	69.28306	69.28196
D1	36.708	36.704891	36.70583	D2	2*36.708= 73.416	73.40221	73.40077
D1#	38.891	38.88714	38.88798	D2#	2*38.891= 77.782	77.76627	77.76444
E1	41.203	41.19914	41.19988	E2	2*41.203= 82.406	82.10991	82.38754
F1	43.654	43.64859	43.64921	F2	2*43.654= 87.308	87.2882	87.28548
F1#	46.249	46.24367	46.24416	F2#	2*46.249= 92.498	92.477821	92.47460
G1	48.999	48.99305	48.99337	G2	2*48.999= 97.998	97.97599	97.97222
G1#	51.913	51.90588	51.90603	G2#	2*51.913= 103.826	103.80135	103.79666

As we can observe from the table, moving from key numbers 1 to 12, Geometric progression is more effective in determining values of frequencies near to actual frequencies. But as we proceed further towards 13, onwards to higher numbers, Regression Analysis is the method to count upon in determining frequencies quite close to actual frequencies.

If a single key produces a frequency of 783.5Hz, than which is this key.

From Regression analysis y = .0250821x + 1.41423135

$$10^{y} = f = 783.5$$

After solving x= 58.99.

From geometric progression analysis

$$a_n = 27.500(1.05945)^{n-1}, \quad 783.5 = 27.500(1.05945)^{n-1}$$

After solving n= 59.00

So 59 = 12x5 -1 = equal to 11 = G1 Key

Or 783.5 = 48.99(G1) x16 =783.9 = a multiple of frequency of key G1.

So any frequencies produced by the piano can be related to given key.

5. Conclusion and Further work

In this paper we have elaborated the fact that music and mathematics are interrelated. The different frequencies used in music are based on mathematical calculations. Paper discusses two methods, Regression Analysis and Geometric Progression Analysis. Both the methods are effective and have produced desired results. For lower key numbers geometric analysis and for higher key numbers regression analysis is more effective to produce desired results. Further work in determining frequencies and their pattern can be done through Fourier Transform. This paper will help both music seekers as well as mathematical intellectuals a belief that both mathematics and music are interconnected.

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