

# On a Certain Family of Meromorphic $p$ -valent Functions with Negative Coefficients

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**Abstract**

In this paper, we introduce and study a new subclass  $M_p(w, \alpha, \beta, \gamma)$  of meromorphically  $p$ -valent functions with negative coefficients. We first obtained necessary and sufficient conditions for a certain meromorphically  $p$ -valent function to be in the class  $M_p(w, \alpha, \beta, \gamma)$ , we then investigated the convex combination of certain meromorphic functions as well as the distortion theorems and convolution properties.

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*Key Words:* Meromorphic,  $p$ -valent, Distortion, Convolution.

**1. Introduction**

Let  $M_p$  denote the class of functions  $f(z)$  of the form

$$f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n \quad (a_n \geq 0, p \in N = \{1, 2, 3, \dots\}) \tag{1}$$

Which are analytic and univalent in the punctured unit disk

$$D = \{z : z \in C \text{ and } 0 < |z| < 1\}$$

an which have a simple pole at the origin ( $z = 0$ ) with residue 1 there. Altintas et al [1] defines the function  $M(p, \alpha, \beta)$  as the function  $f(z) \in M_p$  satisfying the inequality:

$$Re \{zf(z) - \alpha z^2 f'(z)\} > \beta \tag{2}$$

For some  $\alpha(\alpha > 1)$  and  $\beta(0 \leq \beta < 1)$ , for all  $z \in D$ . Other subclasses of the class  $M_p$  were studied recently by Cho et al [2, 3]. While Firas Ghanim and Maslina Darus [4] obtained various properties of the function of the form

$$f(z) = \frac{1}{z-p} + \sum_{n=1}^{\infty} a_n z^n$$

With fixed second coefficient. Also, M.Kamali et al [5] studied various properties of meromorphic  $P$ -valent function of the form:

$$f(z) = \frac{1}{z^p} - \sum_{k=p}^{\infty} a_k z^k \quad (a_k \geq 0; p \in N = \{1, 2, \dots\})$$

Let  $A_w(z)$  denote the set of functions analytic in  $D$  given by

$$f(z) = \frac{1-\alpha}{(z-w)^p} - \sum_{k=p}^{\infty} a_n(z-w)^n \quad (a_n \geq 0; p \in N = \{1, 2, \dots\}) \tag{3}$$

Which have a pole of order  $p$  at ( $z = w$ ),  $z \in D$  and  $w$  is an arbitrary fixed point in  $D$ . We define the function  $f(z)$  in  $A_w(z)$  to be in the class  $M_p(w, \alpha, \beta, \gamma)$  if it satisfies the inequality

$$Re\{(z-w)f(z) - \beta(z-w)^2 f'(z)\} > \gamma \tag{4}$$

and  $w$  is an arbitrary fixed point in  $D$ . We define the function  $f(z)$  in  $A_w(z)$  to be in the class For some  $\beta(\beta < 1)$ ,  $\alpha(0 \leq \alpha < 1)$  and  $\gamma(0 \leq \gamma < 1)$ , for all  $z, w \in D$ .

The purpose of this paper is to investigate some properties of the functions belonging to the class  $M_p(w, \alpha, \beta, \gamma)$ .

## 2. Main Results

**Theorem 1** Let the function  $f(z)$  be in the class  $A_w(z)$ . Then  $A_w(z)$  belong to the class  $M_p(w, \alpha, \beta, \gamma)$  if and only if

$$\sum_{n=1}^{\infty} (1 - n\beta)a_n \leq (1 - \alpha)(1 + p\beta) - \gamma \quad (5)$$

Proof: let  $f(z)$  be as in (3), suppose that

$$f(z) \in M_p(w, \alpha, \beta, \gamma)$$

Then we have from (4) that

$$\begin{aligned} & \operatorname{Re}\left\{(z-w) \left(\frac{1-\alpha}{(z-w)^p} - \sum_{n=p}^{\infty} a_n (z-w)^n\right) - \beta(z-w)^2 \left[\frac{-p(1-\alpha)}{(z-w)^{p+1}} - \sum_{n=p}^{\infty} n a_n (z-w)^{n-1}\right]\right\} \\ &= \operatorname{Re}\{1 - \alpha - \sum_{n=p}^{\infty} a_n (z-w)^{n+p} + p\beta(1-\alpha) + \sum_{n=p}^{\infty} \beta n a_n (z-w)^{n+1}\} \\ &= \operatorname{Re}\{(1-\alpha)(1+p\beta) - \sum_{n=p}^{\infty} (1-n\beta)a_n (z-w)^{n+p}\} > \gamma \quad (z \in D), w \text{ an arbitrary fixed point in } D \end{aligned}$$

If we choose  $(z-w)$  to be real and let  $(z-w) \rightarrow 1^-$ , we get

$$(1-\alpha)(1+p\beta) - \sum_{n=p}^{\infty} (1-n\beta)a_n \leq \gamma (\beta < 1; \leq \gamma < 1, \beta n < 1)$$

Which is equivalent to (5)

Conversely, suppose that the inequality (5) holds. Then we have:

$$\begin{aligned} & |(z-w)f(z) - \beta(z-w)^2 f'(z) - (1-\alpha)(1+p\beta)| \\ &= |\sum_{n=1}^{\infty} (1-n\beta)a_n (z-w)^{n+p}| \leq \sum_{n=p}^{\infty} (1-n\beta)a_n |(z-w)^{n+p}| \\ & \leq (1-\alpha)(1+p\beta) - \gamma, \end{aligned}$$

$(z \in D, w \text{ an arbitrary fixed point in } D, \beta < 1, 0 \leq \alpha < 1, 0 \leq \gamma < 1)$  which implies that  $f(z) \in M_p(w, \alpha, \beta, \gamma)$ .

This completes the prove of the Theorem 1.

**Corollary 1:** Let  $f(z)$  be in  $A_w(z)$ . If  $(z) \in M_p(w, \alpha, \beta, \gamma)$ , then

$$a_n \leq \frac{(1-\alpha)(1+p\beta-\gamma)}{1-n\beta}, \quad n \geq 1, \beta < 1, n\beta < 1$$

**Theorem 2.** Let the function  $f(z)$  be in  $A_w(z)$  and in the function  $g(z)$  be defined by

$$\frac{1-\alpha}{(z-w)^p} - \sum_{n=1}^{\infty} b_n (z-w)^n, \quad b_n \geq 0 \quad (6)$$

Be in the same class  $M_p(w, \alpha, \beta, \gamma)$ . Then the function  $h(z)$  defined by

$$h(z) = (1-\lambda)f(z) + \lambda g(z) = \frac{1-\alpha}{(z-w)^p} - \sum_{n=p}^{\infty} c_n (z-w)^n$$

is also in the class  $M_p(w, \alpha, \beta, \gamma)$ , where  $c_n = (1-\lambda)a_n + \lambda b_n; 0 \leq \lambda \leq 1$

proof: Suppose that each of  $f(z), g(z)$  is in the class  $M_p(w, \alpha, \beta, \gamma)$ . Then by (5) we have

$$\sum_{n=p}^{\infty} (1-n\beta)c_n = \sum_{n=p}^{\infty} (1-n\beta)[(1-\lambda)a_n + b_n]$$

$$= (1-\lambda) \sum_{n=p}^{\infty} (1-n\beta)a_n + \lambda \sum_{n=1}^{\infty} (1-n\beta)b_n$$

$$\leq (1-\lambda)(1-\alpha)(1+p\beta) - \gamma + \lambda(1-\alpha)(1+p\beta) - \gamma$$

$$= (1-\alpha)(1+p\beta) - \gamma \quad (0 \leq \alpha < 1, \beta < 1, 0 \leq \gamma < 1, 0 \leq \lambda \leq 1)$$

This complete the proof of Theorem 2.

### 3. Distortion Theorems

**Theorem 3.:** Let  $f(z) \in M_p(w, \alpha, \beta, \gamma)$ , then

$$|f(z)| \leq \frac{1-\alpha}{|z-w|^p} + \frac{(1-\alpha)(1+p\beta)-\gamma}{1-n\beta} |z-w|^n \quad (7)$$

Proof: By Theorem 1, we have

$$\sum_{n=p}^{\infty} a_n \leq \frac{(1-\alpha)(1+p\beta)-\gamma}{1-n\beta}, (0 \leq \alpha < 1, \beta < 1, 0 \leq \gamma < 1, n \in N, n\beta < 1) \quad (8)$$

And

$$\sum_{n=p}^{\infty} n a_n \leq n \frac{(1-\alpha)(1+p\beta)-\gamma}{1-n\beta}, (0 \leq \alpha < 1, \beta < 1, 0 \leq \gamma < 1, n \in N, n\beta < 1) \quad (9)$$

Let  $f(z) \in M_p(w, \alpha, \beta, \gamma)$ , we have

$$\begin{aligned} f(z) &\leq \frac{1-\alpha}{|z-w|^p} + |z-w|^n \sum_{n=1}^{\infty} a_n \\ &\leq \frac{1-\alpha}{|z-w|^p} + \frac{(1-\alpha)(1+p\beta)-\gamma}{1-n\beta} |z-w|^n \end{aligned}$$

Which completes the proof of the Theorem 3.

**Theorem 4:** Let  $f(z) \in A_w(z)$ . If  $f(z) \in M_p(w, \alpha, \beta, \gamma)$  then

$$|f'(z)| \leq \frac{1-\alpha}{|z-w|^{p+1}} + \frac{n[(1-\alpha)(1+p\beta)-\gamma]}{1-n\beta} |z-w|^{n-1}$$

there is no 10

$$\begin{aligned} |f'(z)| &\leq \frac{1-\alpha}{|z-w|^{p+1}} + |z-w|^{n-1} \sum_{n=p}^{\infty} n a_n \\ &\leq \frac{1-\alpha}{|z-w|^{p+1}} + \frac{n[(1-\alpha)(1+p\beta)-\gamma]}{1-n\beta} |z-w|^{n-1} \end{aligned}$$

This completes the proof of the Theorem 4.

### 4. Convolution Properties

Let the convolution of two complex-valued meromorphic functions

$$f_1(z) = \frac{1-\alpha}{(z-w)^p} + \sum_{n=1}^{\infty} a_{n1} (z-w)^n$$

And

$$f_2(z) = \frac{1-\alpha}{(z-w)^p} + \sum_{n=1}^{\infty} a_{n2} (z-w)^n$$

Be defined by

$$F(z) = (f_1(z) * f_2(z)) = \frac{1-\alpha}{(z-w)^p} + \sum_{n=1}^{\infty} a_{n1} a_{n2} (z-w)^n$$

**Theorem 5.** Let the function  $F(z)$  be in the class  $A_w(z)$ . Then  $F(z)$  belong to the class  $M_p(w, \alpha, \beta, \gamma)$  if and only if

$$\sum_{n=p}^{\infty} (1-n\beta) a_{n1} a_{n2} \leq (1-\alpha)(1+p\beta)-\gamma$$

Proof: supposed that

$$F(z) \in M_p(w, \alpha, \beta, \gamma)$$

Then we have from (4) that

$$\begin{aligned} & \operatorname{Re} \left\{ (z-w) \frac{1-\alpha}{(z-w)^p} - \sum_{n=1}^{\infty} a_{n1} a_{n2} (z-w)^n - \beta (z-w)^2 \left[ \frac{-p(1-\alpha)}{(z-w)^{p+1}} - \sum_{n=1}^{\infty} a_{n1} a_{n2} (z-w)^{n-1} \right] \right\} \\ & = \operatorname{Re} \{ 1 - \alpha - \sum_{n=1}^{\infty} a_{n1} a_{n2} (z-w)^{n+p} + p\beta(1-\alpha) + \sum_{n=1}^{\infty} \beta a_{n1} a_{n2} (z-w)^{n+1} \} \\ & = \operatorname{Re} \{ (1-\alpha)(1+p\beta) - \sum_{n=p}^{\infty} (1-n\beta) a_{n1} a_{n2} (z-w)^{n+p} \} > \gamma \quad (z \in D), w \text{ an arbitrary fixed point in } D \end{aligned}$$

If we choose  $(z-w)$  to be real and let  $(z-w) \rightarrow 1^-$ , we get

$$(1-\alpha)(1+p\beta) - \sum_{n=p}^{\infty} (1-n\beta) a_{n1} a_{n2} \leq \gamma \quad (\beta < 1; 0 \leq \gamma < 1, \beta n < 1)$$

which is equivalent to (5)

Conversely, suppose that the inequality (5) holds. Then we have:

$$\begin{aligned} & |(z-w) f(z) - \beta (z-w)^2 f'(z) - (1-\alpha)(1+p\beta)| \\ & = | - \sum_{n=1}^{\infty} (1-n\beta) a_{n1} a_{n2} (z-w)^{n+p} | \leq \sum_{n=p}^{\infty} (1-n\beta) a_{n1} a_{n2} |z-w|^{n+p} \\ & \leq (1-\alpha)(1+p\beta) - \gamma, \end{aligned}$$

$(z \in D), w$  an arbitrary fixed point in  $D$   $\beta < 1, 0 \leq \alpha < 1, 0 \leq \gamma < 1$  which implies that

$$f(z) \in M_p(w, \alpha, \beta, \gamma).$$

This completes the prove of Theorem 5.

**Theorem 6.** Let the function  $F(z)$  be in  $A_w(z)$  and the function  $G(z)$  defined by

$$\frac{(1-\alpha)}{(z-w)^p} + \sum_{n=1}^{\infty} b_{n1} b_{n2} (z-w)^n, \quad b_{n1} b_{n2} \geq 0$$

Be in the same class  $M_p(w, \alpha, \beta, \gamma)$ . Then the function  $H(z)$  define by

$$H(z) = (1-\lambda)F(z) + \lambda G(z) = \frac{(1-\alpha)}{(z-w)^p} + \sum_{n=1}^{\infty} c_n (z-w)^n$$

Is also in the class  $M_p(w, \alpha, \beta, \gamma)$ , where  $c_n = (1-\lambda)a_{n1}a_{n2} + \lambda b_{n1}b_{n2}; 0 \leq \lambda \leq 1$

proof: suppose that each of  $F(z), G(z)$  is in the class  $M_p(w, \alpha, \beta, \gamma)$ . Then by 5 we have

$$\begin{aligned} \sum_{n=1}^{\infty} (1-n\beta) c_n & = \sum_{n=1}^{\infty} (1-n\beta) (1-\lambda) a_{n1} a_{n2} + \lambda b_{n1} b_{n2} \\ & = (1-\lambda) \sum_{n=1}^{\infty} (1-n\beta) a_{n1} a_{n2} + \lambda \sum_{n=1}^{\infty} (1-n\beta) b_{n1} b_{n2} \\ & \leq (1-\lambda) (1-\alpha)(1+p\beta) - \gamma + \lambda(1-\alpha)(1+p\beta) - \gamma \\ & = (1-\alpha)(1+p\beta) - \gamma \quad (0 \leq \alpha < 1, \beta < 1, 0 \leq \gamma < 1, 0 \leq \lambda \leq 1) \end{aligned}$$

This completes the proof of Theorem 6.

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