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On a Certain Family of Meromorphic p- valent Functions with Negative Coefficients

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Abstract
In this paper, we introduce and study a new subclass $M_P(w.\alpha,\beta,\gamma)$ of meromorphically p-valent functions with negative coefficients. We first obtained necessary and sufficient conditions for a certain meromorphically p-valent function to be in the class$M_P(w.\alpha,\beta,\gamma)$, we then investigated the convex combination of certain meromorphic functions as well as the distortion theorems and convolution properties.
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1.Introduction

Let M_P denote the class of functions $f(z)$ of the form

$$\boldsymbol{f(z)} = \frac{\mathbf{1}}{\boldsymbol{z}} + \sum_{n=p}^{\infty} a_n\, z^n\ (a_n \geq 0, p \in N = \{1,2,3,\dots\} \qquad (1)$$

Which are analytic and univalent in the punctured unit disk

$$D = \{z : z \in C \text{ and } 0 < |z| < 1\}$$

an which have a simple pole at the origin $(z = 0)$ with residue 1 there. Altintas et all [1] defines the function $M(p,\alpha,\beta)$ as the function $f(z) \in M_p$ satisfying the inequality:

$$\text{Re}\,\{zf(z) - \alpha z^2 f'(z)\} > \beta \qquad (2)$$

For some $\alpha(\alpha > 1)$ and $\beta(0 \leq \beta < 1)$, for all $z \in D$. Other subclasses of the class M_p were studied recently by Cho et al [2, 3]. While Firas Ghanim and Maslina Darus [4] obtained various properties of the function of the form

$$f(z) = \frac{1}{z-p} + \sum_{n=1}^{\infty} a_n z^n$$

With fixed second coefficient. Also, M.Kamali et al [5] studied various properties of meromorphic P-valent function of the form:

$$f(z) = \frac{1}{z^p} - \sum_{k=p}^{\infty} a_k z^k \quad (a_k \geq 0; p \in N = \{1,2,\dots\})$$

Let $A_w(z)$ denote the set of functions analytic in D given by

$$f(z) = \frac{1-\alpha}{(z-w)^p} - \sum_{k=p}^{\infty} a_{n(z-w)^n} \quad (a_n \geq 0; p \in N = \{1,2,\dots\}) \qquad (3)$$

Which have a pole of order p at $(z = w), z \in D$ and w is an arbitrary fixed point in D. We define the function $f(z)$ in $A_w(z)$ to be in the class $M_P(w.\alpha,\beta,\gamma)$ if it satisfies the inequality

$$Re\{(z-w)f(z) - \beta(z-w)^2 f'(z)\} > \gamma \qquad (4)$$

and w is an arbitrary fixed point in D. We define the function $f(z)$ in $a_w(z)$ to be in the class
For some $\beta(\beta < 1), \alpha(0 \leq \alpha < 1)$and $\gamma(0 \leq \gamma < 1)$, for all $z, w \in D$.
The purpose of this paper is to investigate some properties of the functions belonging to the class $M_p(w,\alpha,\beta,\gamma)$.

2.Main Results

Theorem 1 Let the function $f(z)$ be in the class $A_w(z)$. Then $A_w(z)$ belong to the class $M_P(w.\alpha,\beta,\gamma)$ if and only if

$\sum_{n=1}^{\infty}(1-n\beta)a_n \leq (1-\alpha)(1+p\beta)-\gamma$ (5)

Proof: let $f(z)$ be as in (3), suppose that

$$f(z) \in M_P(w.\alpha,\beta,\gamma)$$

Then we have from (4) that

$\text{Re}\{(z-w)\left(\frac{1-\alpha}{(z-w)^p}-\sum_{n=p}^{\infty}a^n(z-w)^n\right)-\beta(z-w)^2[\left(\frac{-p(1-\alpha)}{(z-w)^{p+1}}-\sum_{n=p}^{\infty}na_n(z-w)^{n-1}\right)]\}$

$=\text{Re}\{1-\alpha-\sum_{n=p}^{\infty}a_n\,(z-w)^{n+p}+p\beta(1-\alpha)+\sum_{n=p}^{\infty}\beta na_n\,(z-w)^{n+1}\}$

$=\text{Re}\{(1-\alpha)(1+p\beta)-\sum_{n=p}^{\infty}(1-n\beta)a_n\,(z-w)^{n+p}\}>\gamma$ $(z\in D)$, w an arbitrary fixed point in D

If we choose $(z-w)$ to be real and let $(z-w)\rightarrow 1^-$, we get

$$(1-\alpha)(1+p\beta)-\sum_{n=p}^{\infty}(1-n\beta)a_n \leq \gamma(\beta<1;\leq\gamma<1,\beta n<1)$$

Which is equivalent to (5)

Conversely, suppose that the inequality (5) holds. Then we have:

$|(z-w)f(z)-\beta(z-w)^2f'(z)-(1-\propto)(1+p\beta)|$

$=|-\sum_{n=1}^{\infty}(1-n\beta)a_n\,(z-w)^{n+p}| \leq \sum_{n=p}^{\infty}(1-n\beta)a_n\,|(z-w)|^{n+p}$

$\leq (1-\alpha)(1+p\beta)-\gamma,$

$(z,\in D, w$ an arbitrary fixed point in $D\ \beta<1, O\leq\alpha<1, 0\leq\gamma<1)$ which implies that $f(z)\in M_p(w,\propto,\beta,\gamma)$.

This completes the prove of the Theorem 1.

Corollary 1: Let $f(z)$ be in $A_w(z)$. If $(z)\in M_P(w,\propto,\beta,\gamma)$, then

$$a_n \leq \frac{(1-\alpha)(1+p\beta-\gamma)}{1-n\beta},\quad n\geq 1,\beta<1,n\beta<1$$

Theorem 2. Let the function $f(z)$ be in $A_w(z)$ and in the function $g(z)$ be defined by

$\frac{1-\alpha}{(z-w)^p}-\sum_{n=1}^{\infty}b_n\,(z-w)^n,\ b_n\geq 0$ (6)

Be in the same class $M_p(w,\alpha,\beta,\gamma)$. Then the function $h(z)$ defined by

$$h(z)=(1-\lambda)f(z)+\lambda g(z)=\frac{1-\alpha}{(z-w)^p}-\sum_{n=p}^{\infty}c_n\,(z-w)^n$$

is also in the class $M_P(w,\propto,\beta,\gamma)$, where $c_n=(1-\lambda)a_n+\lambda b_n;\ 0\leq\lambda\leq 1$

proof: Suppose that each of $f(z), g(z)$ is in the class $M_p(w,\alpha,\beta,\gamma)$. Then by (5) we have

$$\sum_{n=p}^{\infty}(1-n\beta)c_n=\sum_{n=p}^{\infty}(1-n\beta)[(1-\lambda)a_n+b_n]$$

$$=(1-\lambda)\sum_{n=p}^{\infty}(1-n\beta)a_n+\lambda\sum_{n=1}^{\infty}(1-n\beta)b_n$$

$$\leq(1-\lambda)(1-\alpha)\,(1+p\beta)-\gamma+\lambda(1-\alpha)(1+p\beta)-\gamma$$

$=(1-\alpha)(1+p\beta)-\gamma$ $(0\leq\alpha<1,\beta<1,0\leq\gamma<1,0\leq\lambda\leq 1)$

This complete the proof of Theorem 2.

3.Distortion Theorems

Theorem 3.: Let $f(z) \in M_P(w,\propto,\beta,\gamma)$, then

$$|f(z)| \leq \frac{1-\alpha}{|z-w|^p} + \frac{(1-\alpha)(1+p\beta)-\gamma}{1-n\beta}|z-w|^n \quad (7)$$

Proof : By Theorem 1,we have

$$\sum_{n=p}^{\infty} a_n \leq \frac{(1-\alpha)(1+p\beta)-\gamma}{1-n\beta}, (0 \leq \alpha < 1, \beta < 1, 0 \leq \gamma < 1, n \in N, n\beta < 1 \quad (8)$$

And

$$\sum_{n=p}^{\infty} na_n \leq n\frac{(1-\alpha)(1+p\beta)-\gamma}{1-n\beta}, (0 \leq \alpha < 1, \beta < 1, 0 \leq \gamma < 1, n \in N, n\beta < 1 \quad (9)$$

Let $f(z) \in M_P(w,\alpha,\beta,\gamma)$, we have

$$f(z) \leq \frac{1-\alpha}{|z-w|^p} + |z-w|^n \sum_{n=1}^{\infty} a_n$$

$$\leq \frac{1-\alpha}{|z-w|^p} + \frac{(1-\alpha)(1+p\beta)-\gamma}{1-n\beta}|z-w|^n$$

Which completes the proof of the Theorem 3.

Theorem 4: Let $f(z) \in A_w(z)$. If $f(z) \in M_P(w,\alpha,\beta,\gamma)$ then

$$\left|f'^{(z)}\right| \leq \frac{1-\alpha}{|z-w|^{p+1}} + \frac{n[(1-\alpha)(1+p\beta)-\gamma]}{1-n\beta}|z-w|^{n-1}$$

$$\textit{there is no } 10$$

$$\left|f'^{(z)}\right| \leq \frac{1-\alpha}{|z-w|^{p+1}} + |z-w|^{n-1}\sum_{n=p}^{\infty} na_n$$

$$\leq \frac{1-\alpha}{|z-w|^{p+1}} + \frac{n[(1-\alpha)(1+p\beta)-\gamma]}{1-n\beta}|z-w|^{n-1}$$

This completes the proof of the Theorem 4.

4. Convolution Properties

Let the convolution of two complex-valued meromorphic functions

$$f_1(z) = \frac{1-\alpha}{(z-w)^p} + \sum_{n=1}^{\infty} a_{n1}(z-w)^n$$

And

$$f_2(z) = \frac{1-\alpha}{(z-w)^p} + \sum_{n=1}^{\infty} a_{n2}(z-w)^n$$

Be defined by

$$F(z) = \left(f_1(z) * f_2(z)\right) = \frac{1-\alpha}{(z-w)^p} + \sum_{n=1}^{\infty} a_{n1}\, a_{n2}(z-w)^n$$

Theorem 5. Let the function $F(z)$ be in the class $A_w(z)$. Then $F(z)$ belong to the class $M_P(w,\alpha,\beta,\gamma)$ if and only if

$$\sum_{n=p}^{\infty}(1-n\beta)a_{n1}\, a_{n2} \leq (1-\alpha)(1+p\beta)-\gamma$$

Proof: supposed that

$$F(z) \in M_P(w, \alpha, \beta, \gamma)$$

Then we have from (4) that

Re $\{(z-w)\frac{1-\alpha}{(z-w)^p} - \sum_{n=1}^{\infty} a_{n1}\, a_{n2}(z-w)^n) - \beta(z-w)^2[\left(\frac{-p(1-\alpha)}{(z-w)^{p+1}} - \sum_{n=1}^{\infty} a_{n1}\, a_{n2}(z-w)^{n-1}\right)]\}$

=Re $\{1-\alpha-\sum_{n=1}^{\infty} a_{n1}\, a_{n2}(z-w)^{n+p} + p\beta(1-\alpha) + \sum_{n=1}^{\infty} \beta a_{n1}\, a_{n2}(z-w)^{n+1}\}$

=Re $\{(1-\alpha)(1+p\beta) - \sum_{n=p}^{\infty}(1-n\beta)a_{n1}\, a_{n2}(z-w)^{n+p}\} > \gamma \quad (z \in D), w$ an arbitrary fixed point in D

If we choose $(z-w)$ to be real and let $(z-w) \to 1^-$, we get

$(1-\alpha)(1+p\beta) - \sum_{n=p}^{\infty}(1-n\beta)a_{n1}\, a_{n2} \le \gamma \; (\beta < 1; 0 \le \gamma < 1, \beta n < 1)$

which is equivalent to(5)

Conversely,suppose that the inequality (5) holds. Then we have:

$|(z-w)\, f(z) \text{-} \beta(z-w)^2 f'(z) - (1-\propto)(1+p\beta)|$

$=|\text{-}\sum_{n=1}^{\infty}(1-n\beta)a_{n1}\, a_{n2}(z-w)^{n+p}| \le \sum_{n=p}^{\infty}(1-n\beta)a_{n1}\, a_{n2}|(z-w)|^{n+p}$

$$\le (1-\alpha)(1+p\beta) - \gamma,$$

$(z \in D), w$ an arbitrary fixed point in $D\ \beta < 1, 0 \le \alpha < 1, 0 \le \gamma < 1)$ which implies that

$f(z) \in M_P(w, \alpha, \beta, \gamma)$.

This completes the prove of Theorem 5.

Theorem 6. Let the function $F(z)$ be in$A_w(z)$ and the function $G(z)$ defined by

$\frac{(1-\alpha)}{(z-w)^p} + \sum_{n=1}^{\infty} \mathrm{b}_{n1}\, b_{n2}(z-w)^n, \quad b_{n1}b_{n2} \ge 0$

Be in the same class $M_p(w, \alpha, \beta, \gamma)$.Then the function $H(z)$ define by

$$H(z) = (1-\lambda)F(z) + \lambda G(z) = \frac{(1-\alpha)}{(z-w)^p} + \sum_{n=1}^{\infty} c_n\,(z-w)^n$$

Is also in the class $M_P(w, \alpha, \beta, \gamma)$,where $c_n = (1-\lambda)a_{n1}a_{n2} + \lambda b_{n1}b_{n2}; 0 \le \lambda \le 1$

proof: suppose that each of $F(z), G(z)$ is in the class $M_P(w, \alpha, \beta, \gamma)$. Then by 5 we have

$$\sum_{n=1}^{\infty}(1-n\beta)c_n = \sum_{n=1}^{\infty}(1-n\beta)\,(1-\lambda)a_{n1}a_{n2} + \lambda b_{n1}b_{n2}$$

$$= (1-\lambda)\sum_{n=1}^{\infty}(1-n\beta)\, a_{n1}a_{n2} + \lambda\sum_{n=1}^{\infty}(1-n\beta)\mathrm{b}_{n1}\, b_{n2}$$

$$\le (1-\lambda)\ (1-\alpha)(1+p\beta) - \gamma + \lambda(1-\alpha)(1+p\beta) - \gamma$$

$$= (1-\alpha)(1+p\beta) - \gamma(0 \le \alpha < 1, \beta < 1, 0 \le \gamma < 1, 0 \le \lambda \le 1)$$

This completes the proof of Theorem6.

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