On An Extension of A $\mathbb{C}$*-Algebra

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Abstract

The objective of the authors in this paper is in two folds. First, concrete examples of $\mathbb{C}$*-Algebras are given. This is done so that $\mathbb{C}$*-Algebra which is an algebraic structure is made devoid of the usual abstraction common to pure mathematics at the higher research level, second, after describing what is meant by an extension of an $\mathbb{C}$*-Algebra, the authors shows that an extension of a $\mathbb{C}$*-Algebra can be given in a commutative diagram and a short exact sequence. Details of when a sequence is called short exact are also given in the paper.

Keywords: Banach Space, $\mathbb{C}$*-Algebra, Normed Linear Space, Short Exact Sequence

1. Introduction.

The objectives of the authors in this paper includes given concrete examples of $\mathbb{C}$*-Algebras and describing the extension of a $\mathbb{C}$*-Algebra. For the purpose of the paper, the following working definitions are given.

1.1 Definitions

1.1.1 Linear Space.

Let F be a field, a linear space X over F is an additive abelian group in which multiplication is related to addition in the following ways

$\alpha(x+y) = \alpha x+ \alpha y \ \forall\ x, y \in X, \alpha\in F$

$(\beta +\alpha)x = \alpha x+ \beta x \ \forall\ \alpha, \beta \in F, x\in X$

and

$(\alpha \beta)x = \alpha (\beta x); \forall\ \alpha, \beta \in F, x \in X$

$1.x = x, 1 \in F, x \in X.$

1.1.2 Normed Linear Space.

A normed linear space X is a linear space on which is defined a norm (i. e a satisfactory notion of distance from an arbitrary element to the origin), that is a function.

$\|.\|: x \rightarrow \mathbb{R}$

Which assigns to each element $x \in X$ a real number $|x|$ such that the following axioms hold:

(i) $\|x\|\geq 0$ and $\|x\| = \Leftrightarrow x=0$

(ii) $\| x + y \leq \| x \| + \| y \|$

(iii) $\| \alpha x \| = |X| \| x \|$

(iv)

1.1.3. Algebra

An Algebra is a linear space A with a ring characterization;

Such that the middle associative rule:

$\alpha (xy) = (\alpha x) y = x(\alpha y)$

is satisfied. The algebra A is real or complex according as the scalar involved is real or complex.

1.1.4 Banach Space

A Banach space is a normed linear space in which every Cauchy sequence is convergent.

1.1.5 Banach Algebra

A Banach Algebra is a Banach Space which is also an algebra with identify 1, and in which the multicative structure is related to the norm by

(i) $\| xy \| \leq \| x \| \| y \|$

(ii) $\| 1 \| = 1$

1.1.6. Involution

Let B be Banach-algebra. An **involution** on B is a mapping

$*: B \longrightarrow B$

Such that the following properties hold.

(i). $(x+y)^* = x^* + y^*\ \forall\ x, y \in B$.

(ii) $(\alpha x)^* = \alpha x^*$, $x \in B$, $\alpha \in F$, where F is the field of Scalars.

(iii) $(xy)^* = y^* x^*$

(iv) $x^{**} = x$

i.e. involution on B is a conjugate linear mapping such that

$(xy)^* = y^*x^*$ and $x^{**} = x\ \forall\ x, y \in B$

1.1.7 $\mathbb{C}$*-Algebra.

A $\mathbb{C}$*-Algebra A is a Banach-algebra A having an involution, *(= a conjugate linear mapping of A out itself) such that

(i). $x^{**}=x$ (ii) $(xy)^* = y^*x^*$

And in addition,

$\|x^*x\| = \| x \|^2\ \forall\ x, y \in B$

1.1.8. $\mathbb{R}$ The set $\mathbb{R}$ is regarded as part of $\mathbb{C}$ and defined by

$\mathbb{R} = [z: Im(z)=0] = [z: z = z\ )]$

1.1.9. B(H) is the set of all continuous bounded linear operators defined on the Hilbert Space, H.

1.1.10 $\mathbb{C}$(X) is the Banach Space of bounded continuous scalar valued functions on a topological space X with norm, $\| f \| = \text{sap} \mid f(x) \mid, x \in X$

Concrete examples of $\mathbb{C}$* -Algebras

In this section, we give examples of concrete $\mathbb{C}$*-Algebras by showing that the axioms of a $\mathbb{C}$*-Algebra are satisfied.

2.1. $\mathbb{R}$, the set of real numbers.

Since it is basic fact of analysis that $\mathbb{R}$ is a Banach Space and indeed, a real Banana algebra, it remains to show that $\mathbb{R}$ is conjugate linear satisfying

$x^{**} = x\ \forall\ x \in \mathbb{R}$; $(xy)^* = y^* x^*$

and $\| x^*x \| = \| x \|^2$ if we take the complex conjugation as the involution, it follows that

$(x + y) = \bar{x} + y = x + y\ \forall\ x, y \in \mathbb{R}$ and that

$(\alpha x\ ) = \alpha x = \alpha\ x\ \forall\ x \in, \alpha \in \mathbb{R}$. Showing conjugate linear

and that $\bar{x} = x$ and $\mid \bar{x} x \mid = \mid \bar{x} \|\mid x \mid = \mid x \| x \mid = \mid x \mid^2$; also $xy = \bar{x}\ y = y\ \bar{x} = yx$

2.2 $\mathbb{C}$, the set of Complex Number

As with $\mathbb{R}$, $\mathbb{C}$ is fundamentally a Banach algebra. To show that it is a $\mathbb{C}$*-Algebras, it remains to show that $\mathbb{C}$ is conjugate linear and that $\mathbb{C}$ satisfied

$x^{**} = x\ \forall\ x \in \mathbb{C}$, and $\| x^*x \| = \mid x \mid^2, \forall\ x \in \mathbb{C}$ and $(xy)^* = y^* x^*$

If we take the complex Conjugation as involution, then simple calculations show that

$Z_1 + z_2 = z_1 + z_2$ and that

$(\alpha z\ ) = \alpha z$. Showing conjugate linearity.

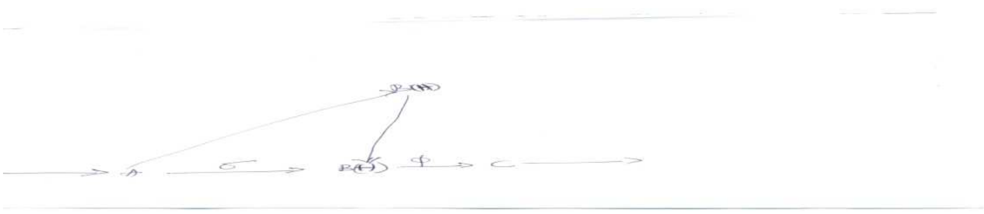
Now, $z = z$; so;$xy = yx = y\ \bar{x}$

2.3 B(H);

Operators on the Hilbert Space, H

We recall that B(H) is an algebra with respect to the usual operations of (a) addition:
$\forall T_1T_2 \in B(H), T_1 + T_2) (x) = T_1x + T_2x$. (b) Multiplication by Scalars :
$\forall\ \alpha \in \mathbb{C}, T \in B(H), (\alpha\ T)x = \alpha\ (Tx)$. Multiplication or iteration or transformations :
$\forall\ T_1, T_2 \in B\ (H), (T_1\ T_2)\ x = T(Tx)$.
B(H) becomes normed by the function $\|.\|: B(H) \longrightarrow \|B(H)\|$
Where $\forall\ T \in B(H)$,
$\|T\| = \sup\ \{\|\ T\ x\ \|: \|\ x\ \| \leq 1\} \leq \sup\ \{\|T\|\ \|x\| : \|\ x\ \| \leq 1\}$.
B(H) is complete in the norm since every canchy sequence {Tm} in B(H) is convergent: given
$\varepsilon > 0, \exists$ no $\in \mathbb{N} \ni$ m, n $\geq$no $\Rightarrow \|\ T_m\text{-}T_n\ \| \prec \varepsilon$.
$\|T_1 + T_2\| = \sup \|\ \{T_1 + T_2\}\ (\ x)\ \|: \|\ x\ \| \leq 1\} \leq \sup\ \{\|T\|\ \|x\| : \|\ x\ \| \leq 1\}. \leq \sup\ \{\|T_1(x): \|x\| \leq 1\} + \sup\ \{\|T_2\ (x) : \|x\| \leq 1$
That is,
$\|T_1 + T_2\| \ \leq\ \|T_1\| + \|T_2\|$.
It remains to show that B(H) is a $\mathbb{C}$*-Algebra (or involution algebra) with the involution defined by
$:B(H) \longrightarrow B(H)^*$
Satisfying the following properties:

Extension of $\mathbb{C}$*- Algebras.
Adelodun [1] described an extension of a $\mathbb{C}$*-Algebras A by C as a commutative diagram



Where B is a $\mathbb{C}$*-Algebra with identity σ is an injection, ker $\emptyset$ = A $\subset$ B(H) and B(H) is the algebra of bounded linear operators on (H).
It is the purpose of this paper to describe the extension just as short exact sequence.
Now, Im σ = ker $\emptyset$ (since A=ker σ). This suggests that
$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$
is a short exact sequence.
This suggest that an extension of A by C was has been given as a commutative diagram can also be given as short exact sequence
$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$
To say that the sequence is a short exact sequence is to say that the sequence is exact at B. This amounts to say that Im σ = ker $\emptyset$
(1). It follows that the composition mapping
(i). $\emptyset\ \ 0\ \ \sigma = 0$ (i.e. im $\sigma \subseteq$ ker $\emptyset$. Every b is of the form.
(ii) σ (a) = b (i.e. ker $\emptyset \subseteq$ Im σ for every a $\in$ A. Hence by using the antisymmetsic law on (i) and (ii), it follows that im σ = ker $\emptyset$.

Conclusion

The author gave the definitions of the technical words he used, including the definition of a $\mathbb{C}$*-Algebra. He gave concrete examples of $\mathbb{C}$*-Algebras so as to make the definition of $\mathbb{C}$*Algebra devoid of abstraction. Then he described two ways of representing an extension of a $\mathbb{C}$*-Algebra. Showing that the ways are essentially the same. To God be the Glory for all that we 've done.

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