Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.5, No.11, 2015

A zero-sum nonlinear quadratic differential games with closed-loop feedback solution by homotopy analysis method

Majid Darehmiraki

Department of Mathematics, Khatam Alanbia University of Technology, Behbahan, khouzestan, Iran

Abstract

In this paper, we consider zero-sum nonlinear quadratic differential games which the coefficients of the quadratic form are quadratic matrix, function of the state variable. Dynamic constraints are represented bilinear differential systems of the form $\dot{x} = A(x)x + B(x)u, x(0) = x_0$. The homotopy analysis method (HAM) approach is applied in obtaining the solution of the dependent state matrix algebraic Riccati equation. Finally we present certain significant case.

Keywords: Nonlinear quadratic differential game, Riccati equation, Homotopy analysis method

1. Introduction

Differential games have been extensively studied during the recent decades to analyze economic problems in areas such as industrial organization, resource and environmental economics or macroeconomic policy. The solution concept that is most often used is the open-loop Nash equilibrium (OLNE), where controls only depend on time (and the initial state of the system). As it is well known, the OLNE is weakly time-consistent but not strongly time-consistent (Bas-ar, 1989): it does not possess the Markov perfect property and is not robust against unexpected changes in the state of the system. Therefore, the feedback Nash equilibrium (FBNE) is a more satisfactory solution concept. It is derived in a dynamic programming framework, so that controls depend on time and state, and the solution is Markov perfect by construction. However, solutions are usually very difficult to derive.
Homotopy analysis method (HAM) initially proposed by Liao in [1, 2] is a powerful method to obtain series solution of various nonlinear problems. In recent years, this method has been successfully employed to solve many types of nonlinear problems in science and engineering such as the viscous flows of non-Newtonian fluids [3--13], the KdV-type equations [14--18], nonlinear heat transfer [19--21], nonlinear water waves [22], groundwater flows [23], Burgers–Huxley equation [24], time-dependent Emden–Fowlertype equations [25], differential-difference equation [26], Laplace equation with Dirichlet and Neumann boundary conditions [27], MHD Falkner–Skan flow [28], the Sharma–Tasso–Olver equation [29], the Kawahara equation [30], for multiple solutions of nonlinear boundary value problems (BVPs) [31--36] and Abbasbandy et al. [34] applied HAM to predict the multiplicity of the solutions of
nonlinear BVPs and shows that convergence-control parameter h plays basic role in prediction of multiplicity of solutions of nonlinear problems. In [37] a new technique of HAM form introducing a change in the using of HAM in solving high-order nonlinear initial value problems. HAM enjoys great freedom in choosing initial approximations and auxiliary linear operators. The HAM can guarantee the convergence of the series solutions by auxiliary parameters especially the so-called convergence-controller parameter h.
The State-Dependent algebraic Riccati Equation (SDARE) strategy is well-known and has become very popular within the control community over the last decade, providing a very effective algorithm for synthesizing nonlinear feedback controls by allowing nonlinearities in the system states while additionally offering great design flexibility through state-dependent weighting matrices. This method, first proposed by Pearson [38] and later expanded by Wernli & Cook [39], was independently studied by Mracek & Cloutier [40] and alluded to by Friedland [41].

The contribution of our paper is to apply the HAM for solving the SDARE. The application of HJB equation to the zero-sum nonlinear quadratic differential games results in a SDARE. As we will point out in Section 2, we can achieve the feedback optimal control law, by using SDARE.
The paper has been organized as follows. Section 2, describes the solution guidelines for linear optimal control system (1). Section 3, presentation Steady-state Riccati equation. In Section 4, HAM is applied for solving optimal control problem. Finally, conclusions are given in the last section.

2. Nonlinear zero-sum quadratic differential games

In this section, we consider a special class of the zero-sum differential games where the system is nonlinear and the cost functions are quadratic functions of the state vector and controls. In proposed differential games the coefficients of the quadratic form and state equation are function of the state variable.

For the *i*th player, i=1,2 the problem is to choose a control strategy $u_i = \psi_i(x,t)$ to minimize

$$J_i = \frac{1}{2}\int_0^{\infty}\left[x^T Q_i(x) x^T + \sum_{j=1}^{2} u_j{}^T S_{ij}(x) u_j\right] dt \qquad (1)$$

For which the state variable $x \in R^n$ and the control variables $u_j \in R^m, j = 1,2$ satisfy the system [10]

$$\dot{x} = A(x)x + \sum_{j=1}^{2} B_j(x) u_j\,, \qquad x(0) = x_0 \in R^n \qquad (2)$$

Where $Q_i(x) \in R^{n\times n}, S_{ij}(x) \in R^{m\times m}, i,j = 1,2$ are quadratic symmetrical matrices and $Q_2 = -Q_1$, $S_{12} = -S_{22}$, $S_{21} = -S_{11}$ for all $x \in R^n$. The matrices $A(x) \in R^{n\times n}, B_j(x) \in R^{n\times m}, j = 1,2$ are continuous function together with their derivatives.

The Hamiltonian for the *i*th player is

$$H_i(x,u,\lambda) = \frac{1}{2}x^T Q_i(x) x + \frac{1}{2}\sum_{j=1}^{2} u_j^T S_{ij}(x) u_j + \lambda_j^T (A(x)x + \sum_{j=1}^{2} B_j(x) u_j) \qquad (3)$$

With the necessary extreme conditions

$$H_{i_u} = 0 \qquad (a)$$

$$\dot{\lambda}_\iota = -H_{\iota_x} \qquad (b) \qquad (4)$$

$$\dot{x} = A(x)x + \sum_{j=1}^{N} B_j(x) u_j \qquad (c)$$

Using the value-function approach, one see that the game is normal and from the (4a) the optimal control for the *i*th player is

$$u_i^* = -S_{ii}^{-1}(x) B_i^T \lambda_i \qquad (5)$$

The Hamilton-Jacobi associated equation for the first player will be written by using (5)

$$\frac{\partial V_1}{\partial t}+\frac{1}{2}x^T Q_1(x)x+\frac{1}{2}u_1^T S_{11}u_1-\frac{1}{2}u_2^T S_{22}u_2+\left(\frac{\partial V_1}{\partial x}\right)^T A(x)x+\left(\frac{\partial V_1}{\partial x}\right)^T B_1(x)u_1+\left(\frac{\partial V_1}{\partial x}\right)^T B_2(x)u_2=0 \quad (6)$$

In the case of infinite time one will selects the solution of equation (6) as function of the state variable $V=V(x)$. We consider the solution of (6) of the form

$$\frac{\partial V_i}{\partial x}=P_i(x)x \qquad i=1,2 \qquad (7)$$

That $P_i(x), i=1,2$ is a symmetric positive semidefinite matrix. By

$$x^T P_i(x)A(x)x=x^T\frac{P_i(x)A(x)+A^T(x)P_i(x)}{2}x \quad i=1,2 \qquad (8)$$

And (7) and by the choice $P_1=-P_2=P$, the Hamilton-Jacobi equation (6) becomes

$$A^T(x)P(x)+P(x)A(x)-P(x)(B_2S_{22}^{-1}B_2^T-B_1S_{11}^{-1}B_1^T)P(x)+Q_1(x)=0 \qquad (9)$$

Above equation is a Riccati algebraic matrix equation of dependent state. The solution of (9) is the symmetric matrix $P(x)\geq 0$. Thus, the nonlinear feedback control could be writhed as

$$u_1^*=-S_{11}^{-1}(x)B_1^T Px, \qquad u_2^*=S_{22}^{-1}(x)B_2^T Px \qquad (10)$$

According to Hamilton-Jacobi equation and choice $P_1=-P_2=P$, we obtain

$$\lambda_1=-\lambda_2=P(x)x \qquad (11)$$

From (4b) it follows

$$\dot{\lambda}_1=-Q_1x-\frac{1}{2}x^T Q_{1_x}x-\frac{1}{2}u_1{}^T S_{11_x}u_1+\frac{1}{2}u_2{}^T S_{22_x}u_2-\left(x^T A_x^T+A^T+u_1^T B_{1_x}^T+u_2^T B_{2_x}^T\right)\lambda_1 \quad (12)$$

Derivating of expression (11) and using the dynamic constraints (2) and optimal control value (10) it results

$$\dot{P}x+\frac{1}{2}x^T Q_x x+\frac{1}{2}u_1{}^T S_{11_x}u_1-\frac{1}{2}u_2{}^T S_{22_x}u_2+x^T A_x^T Px-x^T P(x)\left(B_2S_{22}^{-1}B_{2_x}^T-B_1S_{11}^{-1}B_{1_x}^T\right)P(x)x$$
$$+x^T[A^T(x)P(x)+P(x)A(x)-P(x)(B_2S_{22}^{-1}B_2^T-B_1S_{11}^{-1}B_1^T)P(x)+Q_1(x)]x=0 \quad (13)$$

Using (9) and substituting the controls $u_1, u_2$with its optimal control value (10), equation (13) is reduced to

$$\dot{P}x+\frac{1}{2}x^T Q_x x-\frac{1}{2}\left(P(x)(B_2S_{22}^{-1}B_2^T-B_1S_{11}^{-1}B_1^T)P(x)\right)+x^T A_x^T Px$$
$$-x^T P(x)\left(B_2S_{22}^{-1}B_{2_x}^T-B_1S_{11}^{-1}B_{1_x}^T\right)P(x)x=0 \qquad (14)$$

The Riccati differential equation of dependent states (14) represents the optimality criterium.

3. Special class of differential game

We consider the class of differential game that for each player

$$J_i = \int_0^\infty L_1(x, u_1, u_2)\, dt \qquad i = 1,2 \tag{15}$$

Satisfying the constraints

$$\dot{x} = f(x, u_1, u_2) \tag{16}$$

Where $L: R^{n\times m} \to R,\ x\epsilon R^n, u_1, u_2 \epsilon R^m$.

Where control functions u_1, u_2 are continuous on $[0, \infty)$, differential equation (16) has a unique solution on $[0, \infty)$ and one determinate the control function $u_i, i = 1,2$ which minimize (15) and implies that $x(t) \to 0$ as $t \to \infty$.

Theorem 1. Assume that there exist continuous differentiable functions $V_i(x): R^n \to R, i = 1,2$ positevly defined satisfying for each $x\epsilon R^n$, Bellman state equation:

$$min_{u\epsilon\Omega}\{L_i(x, u_1, u_2) + V_{1_x}(x) f(x, u_1, u_2)\} = 0 \tag{17}$$

Let $u_i{}^*, i = 1,2$ be the optimal control defined by:

$$u^*(x) = argmin_{u\epsilon\Omega}\{L_i(x, u_1, u_2) + V_{1_x}(x) f(x, u_1, u_2)\} = 0 \tag{18}$$

Such that the solution of the differential system (16) corresponding to $u_i{}^*, i = 1,2$ approaches zero as $\to \infty$. In these conditions it follows:

$$\min J_i = V_i(x_0), \qquad i = 1,2$$

Proof. Along the trajectory of (16) we have:

$$V_i(x(t)) - V_i(x(0)) - \int_0^t \frac{dV_i}{d\tau} d\tau = 0 \qquad i = 1,2 \tag{19}$$

This can be written as

$$V_i(x(t)) - V_i(x(0)) - \int_0^t V_{i_x}(x) f(x, u_1, u_2) d\tau = 0 \qquad i = 1,2 \tag{20}$$

Using the identity

$$\int_0^t min_{u\epsilon\Omega}\{L_i(x, u_1, u_2) + V_{i_x}(x) f(x, u_1, u_2)\} d\tau = 0 \qquad i = 1,2 \tag{21}$$

The expression of the functional J_i will become

$$J_i = \int_0^t L_i(x, u_1, u_2) d\tau + V_i(x(0)) - V_i(x(t)) + \int_0^t \{V_{i_x}(x) f(x, u_1, u_2) - min_{u\epsilon\Omega}\{L_i(x, u_1, u_2) + V_{i_x}(x) f(x, u_1, u_2)\}\} d\tau = 0 \qquad i = 1,2 \tag{22}$$

According to hypothesis, we will consider only the control that for which $x(t) \to 0\ as\ t \to \infty$. Since $V(x)$ is positively defined thus $V(x(t)) \to 0\ as\ t \to \infty$.

It follows that for $t \to \infty$, the equation (27) will be written

$$J_i = \int_0^t V_i(x(0)) + \int_0^t \{L_i(x, u_1, u_2) + V_{i_x}(x) f(x, u_1, u_2) - min_{u\epsilon\Omega}\{L_i(x, u_1, u_2) + V_{i_x}(x) f(x, u_1, u_2)\}\} d\tau = 0 \qquad i = 1,2 \tag{22}$$

Consequently

$$min\, J_i = V_i\big(x(0)\big) \tag{23}$$

This ends the proof of the theorem. □

Consider the scalar function $V: R^n \to R$, given by:

$$V(x) = x^T P(x) x \tag{24}$$

Where $P(x)$ is a symmetric matrix and $P(x) \geq 0$.

We determine $P(x)$. Then we construct the optimal controls and we solve the above differential game.

4. HAM to solve dependent state algebraic Riccati equation

To solve dependent state Riccati algebraic equation (9) by means of the HAM, let us define:

$$G(P) = A^T(x)P(x) + P(x)A(x) - P(x)(B_2 S_{22}^{-1} B_2^T - B_1 S_{11}^{-1} B_1^T)P(x) + Q_1(x) = 0 \tag{25}$$

We construct 0th-order deformation equation

$$(1 - q)\big(G[\varphi(q)] - G(P_0)\big) = qhG[\varphi(q)] \tag{26}$$

Since $h \neq 0$, the above equation at $q = 1$ becomes $hG[\varphi(q)] = 0$, which is equivalent to the original equation $G(p) = 0$, provided $P = \varphi(1)$. Taking the 1st -order homotopy-derivative on both sides of (26), we have the corresponding 1st –order deformation equation

$$Y_1 G'(P_0) - hG(P_0) = 0 \tag{27}$$

Whose solution is

$$P_1 = h\frac{G(P_0)}{G'(P_0)} \tag{28}$$

Taking the 2nd –order homotopy-derivative on both sides of (26) gives the 2nd –order deformation equation:

$$P_2 G'(P_0) - (1 + h)P_1 G'(P_0) + \frac{1}{2}P_1^2 G''(P_0) = 0 \tag{29}$$

P_2 is obtained as follows:

$$P_2 = (1+h)P_1 - \frac{Y_1^2 G''(P_0)}{2G'(P_0)} = h(1+h)\frac{G(P_0)}{G'(P_0)} - \frac{h^2}{2}\frac{G^2(P_0)G''(P_0)}{[G'(P_0)]^3} \quad (30)$$

In this way, one obtains P_k one by one in the order $k = 1, 2, 3, \dots$. Here, we emphasize that all of these high order deformation are linear, and therefore are easy to solve. Then, we have the 1st-order homotopy-series approximation

$$P \cong P_0 + P_1 = P_0 + h\frac{G(P_0)}{G'(P_0)} \quad (31)$$

And the 2nd-order homotopy-series approximation

$$P \cong P_0 + P_1 + P_2 = P_0 + (2h + h^2)\frac{G(P_0)}{G'(P_0)} - \frac{h^2}{2}\frac{G^2(P_0)G''(P_0)}{[G'(P_0)]^3} \quad (32)$$

Obviously, (31) when $h = -1$, is exactly the same as the famous Newton's iteration formula, and thus (32) when $h = -1$, can be regarded as the 2nd-order Newton's iteration formula. In fact, one can give a family of Newton's iteration formula in a similar way.

5. Application

Example 1. Consider zero-sum nonlinear quadratic differential game as follows:

Minimizing the functional:

$$-J_2 = J_1 = \frac{1}{2}\int_0^\infty [x_1^4 + x_2^4 - 2u_1^2 - 2u_2^2]dt , \quad (33)$$

Subject to

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}\sqrt{2}\frac{x_2^2}{x_1^2} & 0\\ x_2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{x_2^2}{x_1}\\ 0\end{bmatrix}(u_1 + u_2), \qquad X(0) = \begin{bmatrix}x_1^0\\ x_2^0\end{bmatrix} \quad (34)$$

Defined in the domain

$$D = \{(x_1, x_2): x_1 > 0, x_2 \neq 0\} \quad (35)$$

Expressing (33) in form (1), it follows

$$Q(x) = \begin{bmatrix}x_1^2 & 0\\ 0 & x_2^2\end{bmatrix}, \quad S_{11} = -S_{22} = -2 \quad (36)$$

Example 2. Consider the system

$$\dot{x} = x(t)^2 + x(t)u_1(t) + \sqrt{2x(t)}u_2(t)$$

Objective function in this differential game as follows:

$$J_1 = -J_2 = \frac{1}{2}\int_0^\infty [-x(t)^4 - u_1^2 - u_2^2]\, dt$$

Example 3. This case corresponds to the zero-sum nonlinear quadratic differential game of a bilinear system of the form

$\dot{X} = B(X)X(u_1 + u_2)$

Where $X\epsilon R^n, u_1, u_2\epsilon R$ and $B(X)$ is a quadratic matrix of range $n \times n$.

Objective function in this differential game as follows:

$J_1 = -J_2 =$
$\frac{1}{2}\int_0^\infty X^T QXdt$

The matrix $Q(X)$ is positively defined, and the controls belongs to domain

$\Omega = \{u: |u| \leq 1\}$

The bellman equation () associated to the differential game is given by

$$min_{u_1}\left[\frac{1}{2}X^T QX + X^T P(X)B(X)X(u_1 + u_2)\right] = 0$$

The condition () is satisfied by the optimal control

$$u_1^* = -sign[X^T P(X)B(X)X]$$

Or

$$u_1^* = \begin{cases} 1 & real\ part\ of\ the\ eigenvalues\ of\ B(X) is\ negative \\ -1 & real\ part\ of\ the\ eigenvalues\ of\ B(X) is\ positive \end{cases}$$

And $u_2^* = -u_1^*$.

6. Conclusions

This paper studies the two-person, zero-sum linear quadratic differential games on a finite horizon. Some necessary and sufficient conditions for the existence of the value of the game are derived. Although we obtain the open loop − open loop saddle points whenever the value of the game exists, nothing is said about their synthesis as state feedback. In subsequent papers, we shall further investigate the relationship among open loop saddle points, closed loop saddle points, value of the game, and the Riccati differential equations. Another future research will discuss infinite horizontal differential game problems.

References

1. Liao SJ. The proposed homotopy analysis techniques for the solution of nonlinear problems. Ph.D. Thesis, Shanghai Jiao Tong University, Shanghai, 1992 (in English).
2. Liao SJ. Beyond Perturbation: Introduction to Homotopy Analysis Method. Chapman & Hall/CRC Press: Boca Raton, 2003.
13. Sajid M, Hayat T, Asghar S. Non-similar solution for the axisymmetric flow of a third-grade fluid over radially stretching sheet. Acta Mechanica 2007; 189:193--205.
14. Abbasbandy S. The application of the homotopy analysis method to solve a generalized Hirota–Satsuma coupled KdV equation. Physics LettersA 2007; 361:478--483.
15. Abbasbandy S. Soliton solutions for the 5th-order KdV equation with the homotopy analysis method. Nonlinear Dynamics 2008; 51:83--87.
19. Abbasbandy S. The application of the homotopy analysis method to nonlinear equations arising in heat transfer. Physics Letters A 2006; 360:109--113.

20. Abbasbandy S. Homotopy analysis method for heat radiation equations. International Communications in Heat and Mass Transfer 2007; 34:380--387.
22. Tao L, Song H, Chakrabarti S. Nonlinear progressive waves in water of finite depth—an analytic approximation. Coastal Engineering 2007; 54:825--834.
23. Song H, Tao L. Homotopy analysis of 1D unsteady, nonlinear groundwater flow through porous media. Journal of Coastal Research 2007; 50:292--295.
24. Molabahrami A, Khani F. The homotopy analysis method to solve the Burgers–Huxley equation. Nonlinear Analysis: Real World Applications 2009; 10:589--600.
25. Bataineh AS, Noorani MSM, Hashim I. Solutions of time-dependent Emden–Fowler type equations by homotopy analysis method. Physics Letters A 2007; 371:72--82.
26. Wang Z, Zou L, Zhang H. Applying homotopy analysis method for solving differential-difference equation. Physics Letters A 2007; 369:77--84.
27. Inc M. On exact solution of Laplace equation with Dirichlet and Neumann boundary conditions by the homotopy analysis method. Physics Letters A 2007; 365:412--415.
28. Abbasbandy S, Hayat T. Solution of the MHD Falkner–Skan flow by homotopy analysis method. Communications in Nonlinear Science and Numerical Simulation 2009; 14:3591--3598.
29. Abbasbandy S, Ashtiani M, Babolian E. Analytic solution of the Sharma–Tasso–Olver equation by Homotopy analysis method. Zeitschrift für Naturforschung 2010; 65a:285--290.
30. Abbasbandy S. Homotopy analysis method for the Kawahara equation. Nonlinear Analysis: Real World Applications 2010; 11:307--312.
31. Li S, Liao SJ. An analytic approach to solve multiple solutions of a strongly nonlinear problem. Applied Mathematics and Computation 2005; 169:854--865.
32. Liao SJ. A new branch of solutions of boundary-layer flows over a permeable stretching plate. International Journal of Non-Linear Mechanics 2007; 42:819--830.
33. Abbasbandy S, Magyari E, Shivanian E. The homotopy analysis method for multiple solutions of nonlinear boundary value problems. Communications in Nonlinear Science and Numerical Simulation 2009; 14:3530--3536.
34. Abbasbandy S, Shivanian E. Prediction of multiplicity of solutions of nonlinear boundary value problems: novel application of homotopy analysis method. Communications in Nonlinear Science and Numerical Simulation 2010; 15:3830--3846.
35. Abbasbandy S, Shirzadi A. Homotopy analysis method for multiple solutions of the fractional Sturm–Liouville problems. Numerical Algorithm 2010; 54:521--532.
37. Hassan HN, El-Tawil MA. An efficient analytic approach for solving two-point nonlinear boundary value problems by homotopy analysis method. Mathematical Methods in the Applied Sciences 2011; DOI: 10.1002/mma.1416
37. Hassan HN, El-Tawil MA. A new technique of using homotopy analysis method for solving high-order nonlinear differential equations. Mathematical Methods in the Applied Sciences 2010; DOI: 10.1002/mma.1400.
38. Pearson, J.D. (1962). Approximation methods in optimal control. Journal of Electronics and Control, 13, 453-469.
39. Wernli, A. and G. Cook (1975). Suboptimal control for the nonlinear quadratic regulator problem. Automatica, 11, 75-84.
40. Mracek, C.P. and J.R. Cloutier (1998). Control designs for the nonlinear benchmark problem via the state-dependent Riccati equation method. International Journal of Robust and Nonlinear Control, 8, 401-433.
41. Friedland, B. (1996). Advanced Control System Design, 110-112. Prentice-Hall, Englewood Cliffs NJ.
42. Ceimen, T. (2008). State-Dependent Riccati Equation (SDRE) Control: A Survey, Proceedings of the 17th World Congress, The International Federation of Automatic Control, Seoul, Korea.
43. Popescu M, Dumitrache A. Stabilization of feedback control and stabilizability optimal solution for nonlinear quadratic problems, Commun Nonlinear Sci Numer Simulat 16 (2011) 2319-2327.
44. Zahedi M.S, Nik H.S. On homotopy analysis method applied to linear optimal control problems, Applied Mathematical Modelling, 37(23) (2013) 9617-9629.