A study of Some Geometric Properties of Meromorphic Multivalent Functions Defined by Ruscheyeweh Derivative Operator

A.H.Majeed and Kassim A. Jassim
Department of Mathematics, College of Science ,University of Baghdad, Baghdad , Iraq

Abstract

In the present paper, we introduced a new class of meromorphic multivalent functions $W_p^{\lambda}(\beta)$ defined by Ruscheyeweh derivative operator. We obtained some properties, like , coefficient inequalities , distortion and growth theorems ,convex set , Convolution property and radii of starlikeness and convexity.

Keywords: Multivalent functions , Ruscheyeweh derivative operator , distortion and growth ,convex set, Convolution property and radii of starlikeness and convexity.

1.Introduction

Let $\sum_p$ be the class of functions f of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p}\, z^{k-p}, \quad (p \in \mathbb{N}, k \geq p), \tag{1.1}$$

which are meromophic multivalent in the punctured unit disk $\mathbb{U}^* = \{z \in \mathbb{C}; 0 < |z| < 1\}$. Let W_p be the subclass of $\sum_p$ of functions of the form

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p}\, z^{k-p}, \quad (a_{k-p} \geq 0, p \in \mathbb{N}, k \geq p), \tag{1.2}$$

The convolution (Hadamard product) of two functions of $f(z)$ and $g(z)$ [6] , is given by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p}\, z^{k-p}, \tag{1.3}$$

where $g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p}\, z^{k-p}$ and $f(z)$ is given by (1.2) .We must recall a Ruscheweyeh derivative operator [1] , as follows:-

$$D_*^{\lambda,p} f(z) = \frac{z^{-p}}{(1-z)^{\lambda+p}} * f(z), \quad (\lambda > -p, f \in W_p) \tag{1.4}$$

The equation (1.4) can be written as follows:-

$$D_*^{\lambda,p} f(z) = z^{-p} + \sum_{k=1}^{\infty} \binom{\lambda+k}{k} a_{k-p}\, z^{k-p} \quad ,(\lambda > -p, f \in W_p) \tag{1.5}$$

Now, we denote a new class $W_p^{\lambda}(\beta)$ of functions f of the form (1.2) which satisfies the condition

$$\left| \frac{z^{p+2}(D_*^{\lambda,p} f(z))'' - p^2 - p}{z^{p+1}(D_*^{\lambda,p} f(z))' + D_*^{\lambda,p} f(z) + p} \right| < \beta \tag{1.6}$$

where $\beta (0 \leq \beta < 1)$.For [6] , f be p-valently meromorphic starlike of order $\delta (0 \leq \delta < p)$ if

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta, f(z) \neq 0 \text{ for } z \in \mathbb{U}^* \tag{1.7}$$

Also, f is p-valently meromorphic convex of order $\delta(0 \leq \delta < p)$ if

$$-Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \delta, z \in \mathbb{U}^*. \tag{1.8}$$

Many authors were studied and discussed another classes consisting of meromorphic univalent and meromorphic multivalent functions , like, Atshan [1,2,3,4] , N. E. Cho and I. H. Kim [5], Frasin and M. Darus [7] , J. L. Liu and H. M. Srivastava [8,9], Miller [10] , Mogra [11] and Raina and H. M. Srivastava [12].

The geometric properties of the class $W_p^{\lambda}(\beta)$ are discussed as following:-

2. Coefficient inequality

Theorem(2.1): Let $f \in W_p$. Then $f \in W_p^{\lambda}(\beta)$ if and only if

$$\sum_{k=1}^{\infty} \binom{\lambda+k}{k} [(k-p)(k-p-1) + \beta(k-p) + \beta] a_{k-p} \leq \beta, \tag{2.1}$$

where $0 \leq \beta < 1$, $\lambda > -p$, $k \geq p$ and $p \in \mathbb{N}$.

The result is sharp for function

$$f(z) = z^{-p} + \frac{\beta}{\binom{\lambda+k}{k}[(k-p)(k-p-1)+\beta(k-p)+\beta]} z^{k-p}, \quad (a_{k-p} \geq 0, p \in \mathbb{N}, k \geq p)$$

Proof: Assume that inequality (2.1) is hold. Then from (1.6), we have

$$\left|z^{p+2}(D_*^{\lambda,p}f(z))'' - p^2 - p\right| - \beta \left|z^{p+1}\left(D_*^{\lambda,p}f(z)\right)' + D_*^{\lambda,p}f(z) + p\right|$$

$$= \left|z^{p+2}\left[p(p+1)z^{-p-2} + \sum_{k=1}^{\infty}\binom{\lambda+k}{k}a_{k-p}(k-p)(k-p-1)z^{k-p-2}\right] - p^2 - p\right| - \beta \times$$

$$\left|z^{p+1}\left[-pz^{-p-1} + \sum_{k=1}^{\infty}\binom{\lambda+k}{k}a_{k-p}(k-p)z^{k-p-1}\right] + z^{-p} + \sum_{k=1}^{\infty}\binom{\lambda+k}{k}a_{k-p}z^{k-p} + p\right|$$

$$\left|p(p+1) + \sum_{k=1}^{\infty}\binom{\lambda+k}{k}a_{k-p}(k-p)(k-p-1)z^k - p(p-1)\right| - \beta\left|-p + \sum_{k=1}^{\infty}\binom{\lambda+k}{k}a_{k-p}(k-p)z^k + \right.$$

$$\left. z^{-p} + \sum_{k=1}^{\infty}\binom{\lambda+k}{k}a_{k-p}z^{k-p} + p\right|$$. Then

$$\left|\sum_{k=1}^{\infty}\binom{\lambda+k}{k}a_{k-p}(k-p)(k-p-1)z^k\right| - \beta\left|\sum_{k=1}^{\infty}\binom{\lambda+k}{k}a_{k-p}(k-p)z^k + z^{-p} + \right.$$

$$\left.\sum_{k=1}^{\infty}\binom{\lambda+k}{k}a_{k-p}z^{k-p}\right|.$$

$$\leq \sum_{k=1}^{\infty}\binom{\lambda+k}{k}a_{k-p}(k-p)(k-p-1) + \beta\sum_{k=1}^{\infty}\binom{\lambda+k}{k}a_{k-p}(k-p) - \beta + \beta\sum_{k=1}^{\infty}\binom{\lambda+k}{k}a_{k-p}$$

$$\leq \sum_{k=1}^{\infty}\binom{\lambda+k}{k}[(k-p)(k-p-1)+\beta(k-p)+\beta]a_{k-p} - \beta.$$ So,

$\sum_{k=1}^{\infty}\binom{\lambda+k}{k}[(k-p)(k-p-1)+\beta(k-p)+\beta]a_{k-p} - \beta \leq 0$, by hypothesis. Hence, by the of maximum modules principle , we get $f \in W_p^{\lambda}(\beta)$.

Conversely, assume $f \in W_p^{\lambda}(\beta)$ is hold. From (1.6), we have

$$\left|\frac{z^{p+2}(D_*^{\lambda,p}f(z))'' - p^2 - p}{z^{p+1}(D_*^{\lambda,p}f(z))' + D_*^{\lambda,p}f(z) + p}\right| < \beta.$$

Therefore,

$$Re\left\{\frac{z^{p+2}(D_*^{\lambda,p}f(z))'' - p^2 - p}{z^{p+1}(D_*^{\lambda,p}f(z))' + D_*^{\lambda,p}f(z) + p}\right\} < \beta.$$

Now, choosing values of z on the real axis as z→ $1-$, then we get the inequality (2.1).Sharpness of our result follows by setting

$$f(z) = z^{-p} + \frac{\beta}{\binom{\lambda+k}{k}[(k-p)(k-p-1)+\beta(k-p)+\beta]} z^{k-p}, \quad (a_{k-p} \geq 0, p \in \mathbb{N}, k \geq p)$$

Corollary (2.1): Let $f \in W_p^{\lambda}(\beta)$. Then

$$a_{k-p} \leq \frac{\beta}{\binom{\lambda+k}{k}[(k-p)(k-p-1)+\beta(k-p)+\beta]}, (a_{k-p} \geq 0, p \in \mathbb{N}, k \geq p), \quad (2.2)$$

3. Distortion and growth property

Theorem(3.1): Let the function $f \in W_p^{\lambda}(\beta)$. Then

$$|f(z)| \leq |z|^{-1} + \frac{1}{\binom{\lambda+1}{1}}$$

and

$$|f(z)| \geq |z|^{-1} - \frac{1}{\binom{\lambda+1}{1}} \quad (3.1)$$

Proof: Let $f(z)$ be a function in $W_p^{\lambda}(\beta)$ of the form (1.2).Hence

$|f(z)| = |z^{-p} + \sum_{k=1}^{\infty}a_{k-p}z^{k-p}| \leq |z^{-p}| + |\sum_{k=1}^{\infty}a_{k-p}z^{k-p}| \leq |z|^{-p} + |z|^{1-p}\sum_{k=1}^{\infty}a_{k-p}$.

By theorem (2.1) , we get

$$\sum_{k=1}^{\infty}a_{k-p} \leq \frac{\beta}{\binom{\lambda+1}{1}[(1-p)(-p)+\beta(1-p)+\beta]}.$$

Therefore,

$$|f(z)| \leq |z|^{-p} + \frac{\beta}{\binom{\lambda+1}{1}[(1-p)(-p)+\beta(1-p)+\beta]}|z|^{1-p}.$$

Similarly , we get

$$|f(z)| \geq |z|^{-p} - \frac{\beta}{\binom{\lambda+1}{1}[(1-p)(-p)+\beta(1-p)+\beta]}|z|^{1-p}.$$

Since $k \geq p$, k=1 and $p \in N$, then p=1.Hence,

$$|f(z)| \leq |z|^{-1} + \frac{1}{\binom{\lambda+1}{1}}$$

Similarly , we get

$$|f(z)| \geq |z|^{-1} - \frac{1}{\binom{\lambda+1}{1}}.$$

Corollary (3.1) : Let the function $f \in W_p^{\lambda}(\beta)$. Then

$$|f'(z)| \leq |z|^{-2}$$

and

$$|f'(z)| \geq |z|^{-2} \tag{3.2}$$

4. Convex set

Theorem(4.1): The class $W_p^{\lambda}(\beta)$ is convex set.

Proof. Let functions f and g be in the class $W_p^{\lambda}(\beta)$. Then for every $0 \leq m \leq 1$, we must show that

$$(1-m)f(z) + mg(z) \in W_p^{\lambda}(\beta). \tag{4.1}$$

Therefore, we have

$(1-m)f(z) + mg(z) = z^{-p} + \sum_{k=1}^{\infty}[(1-m)a_{k-p} + mb_{k-p}]\, z^{k-p}$ .Therefore ,by theorem(2.1)

$\sum_{k=1}^{\infty}\binom{\lambda+k}{k}[(k-p)(k-p-1) + \beta(k-p) + \beta][(1-m)a_{k-p} + mb_{k-p}]$

$= (1-m)\sum_{k=1}^{\infty}\binom{\lambda+k}{k}[(k-p)(k-p-1) + \beta(k-p) + \beta]\, a_{k-p} + m\sum_{k=1}^{\infty}\binom{\lambda+k}{k}[(k-p)(k-p-1) + \beta(k-p) + \beta]\, b_{k-p}$

$\leq (1-m)\beta + m\beta = \beta.$

5. Arithmetic mean

Theorem (5.1): Let $f_1(z), f_1(z), \ldots, f_r(z)$ defined by

$$f_i(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p,i}\, z^{k-p} \tag{5.1}$$

where ($a_{k-p} \geq 0$, i=1,2,…,r , $k \geq p$) be in the class $W_p^{\lambda}(\beta)$. Then arithmetic mean of $f_i(z)$ $(i = 1,2,\ldots,r)$ defined by

$$h(z) = \frac{1}{r}\sum_{i=1}^{r} f_i(z) \tag{5.2}$$

is also in the class $W_p^{\lambda}(\beta)$.

Proof. By equations (5.1) and (5.2) , we can write $h(z) = \frac{1}{r}\sum_{i=1}^{r}(z^{-p} + \sum_{k=1}^{\infty} a_{k-p,i}\, z^{k-p})$ $= z^{-p} + \sum_{k=1}^{\infty}(\frac{1}{r}\sum_{i=1}^{r} a_{k-p,i})z^{k-p}$. Since $f_i \in W_p^{\lambda}(\beta)$ for every (i=1,2,…,r) then by using Theorem (2.1), we get $\sum_{k=1}^{\infty}\binom{\lambda+k}{k}[(k-p)(k-p-1) + \beta(k-p) + \beta](\frac{1}{r}\sum_{i=1}^{r} a_{k-p,i})$

$= \frac{1}{r}\sum_{i=1}^{r}(\sum_{k=1}^{\infty}\binom{\lambda+k}{k}[(k-p)(k-p-1) + \beta(k-p) + \beta]a_{k-p,i}) \leq \frac{1}{r}\sum_{i=1}^{r}\beta = \beta.$

6.Convolution Property

Theorem(6.1): Let $f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p}\, z^{k-p}$ and $g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p}\, z^{k-p}$ are in the class $W_p^{\lambda}(\beta)$, then the hadamard product $f * g$ is in the class $W_p^{\lambda}(\gamma)$, where

$$\gamma = \frac{\beta^2(k-p)(k-p-1)}{\binom{\lambda+k}{k}[(k-p)(k-p-1) + \beta(k-p) + \beta]^2 - \beta^2(k-p) - \beta^2} \tag{6.1}$$

Proof: For functions f and g in the class $W_p^{\lambda}(\beta)$, we get

$$\sum_{k=1}^{\infty}\binom{\lambda+k}{k}\frac{[(k-p)(k-p-1)+\beta(k-p)+\beta]}{\beta}a_{k-p}\leq 1 \text{ and } \sum_{k=1}^{\infty}\binom{\lambda+k}{k}\frac{[(k-p)(k-p-1)+\beta(k-p)+\beta]}{\beta}b_{k-p}\leq 1$$

by using theorem (2.1) .Then we must find a smallest γ such that

$$\sum_{k=1}^{\infty}\binom{\lambda+k}{k}\frac{[(k-p)(k-p-1)+\gamma(k-p)+\gamma]}{\gamma}a_{k-p}b_{k-p}\leq 1.$$

By Cauchy – Schwartz inequality, we get

$$\sum_{k=1}^{\infty}\binom{\lambda+k}{k}\frac{[(k-p)(k-p-1)+\beta(k-p)+\beta]}{\beta}\sqrt{a_{k-p}b_{k-p}}\leq 1\,. \tag{6.2}$$

To prove our theorem, we have to show that

$$\binom{\lambda+k}{k}\frac{[(k-p)(k-p-1)+\gamma(k-p)+\gamma]}{\gamma}a_{k-p}b_{k-p}\leq\binom{\lambda+k}{k}\frac{[(k-p)(k-p-1)+\beta(k-p)+\beta]}{\beta}\sqrt{a_{k-p}b_{k-p}}$$

So, this inequality to have be shown

$$\sqrt{a_{k-p}b_{k-p}}\leq\frac{\gamma[(k-p)(k-p-1)+\beta(k-p)+\beta]}{\beta[(k-p)(k-p-1)+\gamma(k-p)+\gamma]}.$$

From (6.2) , we get

$$\sqrt{a_{k-p}b_{k-p}}\leq\frac{\beta}{\binom{\lambda+k}{k}[(k-p)(k-p-1)+\beta(k-p)+\beta]}.$$

It is sufficient to show

$$\frac{\beta}{\binom{\lambda+k}{k}[(k-p)(k-p-1)+\beta(k-p)+\beta]}\leq\frac{\gamma[(k-p)(k-p-1)+\beta(k-p)+\beta]}{\beta[(k-p)(k-p-1)+\gamma(k-p)+\gamma]} \tag{6.3}$$

Therefore, from (6.3) we get

$$\gamma\geq\frac{\beta^2(k-p)(k-p-1)}{\binom{\lambda+k}{k}[(k-p)(k-p-1)+\beta(k-p)+\beta]^2-\beta^2(k-p)-\beta^2}=\varphi(k) \tag{6.4}$$

Since $\varphi(k)$ is decreasing function of k (k≥ 1),setting k=3 in (6.4) , we get

$$\gamma\geq\varphi(3)=\frac{\beta^2(3-p)(2-p)}{\binom{\lambda+3}{3}[(3-p)(2-p)+\beta(3-p)+\beta]^2-\beta^2(3-p)-\beta^2}$$

So, the proof is done.

Theorem(6.2): Let the functions f_j (j=1,2) defined by (2.1) be in the class $W_p^{\lambda}(\beta)$ .Then the function h defined by

$$h(z)=z^{-p}+\sum_{k=1}^{\infty}((a_{k-p,1})^2+(a_{k-p,2})^2)z^{k-p}\,, \tag{6.5}$$

belong to the class $W_p^{\lambda}(\epsilon)$, where

$$\epsilon=\frac{2(k-p)(k-p-1)}{\binom{\lambda+k}{k}\left(\frac{[(k-p)(k-p-1)+\beta(k-p)+\beta]}{\beta}\right)^2-(k-p)-1}$$

Proof: We must find the smallest ϵ such that

$$\sum_{k=1}^{\infty}\binom{\lambda+k}{k}\frac{[(k-p)(k-p-1)+\epsilon(k-p)+\epsilon]}{\epsilon}((a_{k-p,1})^2+(a_{k-p,2})^2)\leq 1.$$

Since $f_j\in W_p^{\lambda}(\beta)$ $(j=1,2)$, we get

$$\sum_{k=1}^{\infty}\left(\binom{\lambda+k}{k}\frac{[(k-p)(k-p-1)+\beta(k-p)+\beta]}{\beta}\right)^2(a_{k-p,1})^2$$

$$\leq\left(\sum_{k=1}^{\infty}\binom{\lambda+k}{k}\frac{[(k-p)(k-p-1)+\beta(k-p)+\beta]}{\beta}(a_{k-p,1})\right)^2\leq 1\,, \tag{6.6}$$

and

$$\sum_{k=1}^{\infty}\left(\binom{\lambda+k}{k}\frac{[(k-p)(k-p-1)+\beta(k-p)+\beta]}{\beta}\right)^2(a_{k-p,2})^2$$
$$\leq\left(\sum_{k=1}^{\infty}\binom{\lambda+k}{k}\frac{[(k-p)(k-p-1)+\beta(k-p)+\beta]}{\beta}(a_{k-p,2})\right)^2\leq 1. \qquad (6.7)$$

Combining the inequalities (6.6) and (6.7), gives

$$\sum_{k=1}^{\infty}\frac{1}{2}\left(\binom{\lambda+k}{k}\frac{[(k-p)(k-p-1)+\beta(k-p)+\beta]}{\beta}\right)^2\left((a_{k-p,1})^2+(a_{k-p,2})^2\right)\leq 1\,. \qquad (6.8)$$

But $h\in W_p^{\lambda}(\epsilon)$ if and only if

$$\sum_{k=1}^{\infty}\binom{\lambda+k}{k}\frac{[(k-p)(k-p-1)+\epsilon(k-p)+\epsilon]}{\epsilon}\left((a_{k-p,1})^2+(a_{k-p,2})^2\right)\leq 1\,. \qquad (6.9)$$

The inequality (6.9) will be satisfied if

$$\binom{\lambda+k}{k}\frac{[(k-p)(k-p-1)+\epsilon(k-p)+\epsilon]}{\epsilon}\leq\frac{1}{2}\left(\binom{\lambda+k}{k}\frac{[(k-p)(k-p-1)+\beta(k-p)+\beta]}{\beta}\right)^2 \qquad (6.10)$$

so that,

$$\epsilon\geq\frac{2(k-p)(k-p-1)}{\binom{\lambda+k}{k}\left(\frac{[(k-p)(k-p-1)+\beta(k-p)+\beta]}{\beta}\right)^2-(k-p)-1}=M(k) \qquad (6.11)$$

Since $M(k)$ is decreasing function of k (k≥ 1),setting k=3 in (6.10) , we get

$$\epsilon\geq M(3)=\frac{2(3-p)(2-p)}{\binom{\lambda+3}{3}\left(\frac{[(3-p)(2-p)+\beta(3-p)+\beta]}{\beta}\right)^2-(3-p)-1}$$

So, the proof is done.

7. Radii of starlikeness and convexity

The following results giving the radii of starlikeness and convexity of the functions $f(z)\in W_p^{\lambda}(\beta)$.

Theorem(7.1): If $f\in W_p^{\lambda}(\beta)$, then f is meromorphic multivalent starlike function of order $\rho(0\leq\rho<1)$ in the disk |z|<r_1, where

$$r_1(\lambda,k,p)=inf\left(\frac{\binom{\lambda+k}{k}[(k-p)(k-p-1)+\beta(k-p)+\beta](p-\rho)}{\beta(k-p-\rho+2)}\right)^{\frac{1}{k}} \qquad (7.1)$$

Proof: It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)}+1\right|\leq 1-\rho\;,\qquad (0\leq\rho<1),$$

for |z|<$r_1(\lambda,k,p)$.

Therefore,

$$\left|\frac{zf'(z)}{f(z)}+1\right|=\left|\frac{zf'(z)+f(z)}{f(z)}\right|=\left|\frac{-pz^{-p}+\sum_{k=1}^{\infty}(k-p)a_{k-p}z^{k-p}+z^{-p}+\sum_{k=1}^{\infty}a_{k-p}z^{k-p}}{z^{-p}+\sum_{k=1}^{\infty}a_{k-p}z^{k-p}}\right|$$

$$=\left|\frac{(1-p)+\sum_{k=1}^{\infty}(k-p+1)a_{k-p}z^k}{1+\sum_{k=1}^{\infty}a_{k-p}z^k}\right|$$

$$\leq\frac{(p-1)+\sum_{k=1}^{\infty}(k-p+1)a_{k-p}|z^k|}{1-\sum_{k=1}^{\infty}a_{k-p}|z^k|}$$

The last expression must bounded by $1-\rho$ if

$$\frac{\sum_{k=1}^{\infty}(k-p-\rho+2)a_{k-p}\,|z^k|}{p-\rho}\leq 1.$$

The last inequality will be true if

$$\frac{(k-p-\rho+2)}{p-\rho}|z^k|\leq\frac{\binom{\lambda+k}{k}[(k-p)(k-p-1)+\ \beta(k-p)+\beta]}{\beta}$$

Hence,

$$|z|\leq\left(\frac{\binom{\lambda+k}{k}[(k-p)(k-p-1)+\ \beta(k-p)+\beta](p-\rho)}{\beta(k-p-\rho+2)}\right)^{\frac{1}{k}}$$

Putting $|z|=r_1$, we get the result.

Theorem(7.2): If $f\in W_p^{\lambda}(\beta)$, then f is meromorphic multivalent convex function of order $\rho(0\leq\rho<1)$ in the disk $|z|<r_2$, where

$$r_2(\lambda,k,p)=inf\left(\frac{\binom{\lambda+k}{k}[(k-p)(k-p-1)+\ \beta(k-p)+\beta]p(p-\rho)}{\beta(k-p)(k-p-\rho+2)}\right)^{\frac{1}{k}} \qquad (7.2)$$

Proof: It is sufficient to show that

$$\left|\frac{zf''(z)}{f'(z)}+2\right|\leq 1-\rho\ ,\qquad (0\leq\rho<1),$$

for $|z|<r_2(\lambda,k,p)$.

Therefore,

$$\left|\frac{zf''(z)}{f'(z)}+2\right|=\left|\frac{zf''(z)+2f'(z)}{f'(z)}\right|$$

$$=\left|\frac{p(p+1)z^{-p-1}+\sum_{k=1}^{\infty}(k-p)(k-p-1)a_{k-p}z^{k-p-1}-2pz^{-p-1}+2\sum_{k=1}^{\infty}(k-p)a_{k-p}z^{k-p-1}}{-pz^{-p-1}+\sum_{k=1}^{\infty}(k-p)a_{k-p}z^{k-p-1}}\right|$$

$$=\left|\frac{p(p-1)+\sum_{k=1}^{\infty}(k-p)(k-p+1)a_{k-p}z^k}{-p+\sum_{k=1}^{\infty}(k-p)a_{k-p}z^k}\right|$$

$$=\left|\frac{p(p-1)+\sum_{k=1}^{\infty}(k-p)(k-p+1)a_{k-p}z^k}{p-\sum_{k=1}^{\infty}(k-p)a_{k-p}z^k}\right|$$

$$\leq\frac{p(p-1)+\sum_{k=1}^{\infty}(k-p)(k-p+1)a_{k-p}|z|^k}{p-\sum_{k=1}^{\infty}(k-p)a_{k-p}|z|^k}$$

The last expression must bounded by $1-\rho$ if

$$\frac{\sum_{k=1}^{\infty}(k-p)(k-p-\rho+2)a_{k-p}\,|z^k|}{p(p-\rho)}\leq 1.$$

The last inequality will be true if

$$\frac{(k-p)(k-p-\rho+2)}{p(p-\rho}|z|^k\leq\frac{\binom{\lambda+k}{k}[(k-p)(k-p-1)+\ \beta(k-p)+\beta]}{\beta}$$

Hence,

$$|z| \leq \left(\frac{\binom{\lambda+k}{k}[(k-p)(k-p-1) + \beta(k-p) + \beta]p(p-\rho)}{\beta(k-p)(k-p-\rho+2)} \right)^{\frac{1}{k}}$$

Putting |z|=r_2, we get the result.

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