The general Common positive solutions for adjointable Bounded operator equations

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Abstract

The aim of this work, given the general common positive solutions of some types of adjointable bounded operator equations define on complex Hilbert space, also introduce the general common positive solutions for system of operator equations upon some necessary and sufficient conditions for existence these solutions via g-inverse operator.

Keywords: key words, positive solution, common positive solution.

1. Introduction

The subject of positive solution studied in initial on matrix equations after that this concept introduce on some types of bounded operator equations such as, Jacobe C, Andre C and Arie L in 1993, studies the matrix equation $X + A^* X^{-1} A = Q$ and found positive define solution, where Q is positive define and finally given the number of solutions[1],in 2005 Xiam Zhang introduce the necessary and sufficient conditions to get the nonnegative define solution of operator equations $AXA^* = BB^*$ and $CXC^* = DD^*$ [4], Qingxiang Xu, in 2008 given the general common Hermition and positive solutions for bounded linear operator equations AX = C and XB = D, also given the represented of this solution[5], in this work, we introduce the general common positive solutions for the adjointable operator equations $AXA^* = B^*B$ and $CXC^* = D^*D,$ (1) appeared in [4], by using g-inverse operator, also the general common positive solutions for system of adjointable operator equations $BXB^* = D^*D,\ AXA^* = C^*C,\ AXB^* = E^*E,\ BXA^* = F^*F.$ (2), it generalization to the adjointable operator equations (1).

At first we must introduce some basic concept of operator such as, adjoint of operator was define by: let $A: H \to K$ be a bounded operator from a Hilbert spaces H onto K then adjoint operator denoted by A^* and $A^*: K \to H$ such that $< Ax, y > = < x, A^* y >$, where for all $x \in H$ and $y \in K$ [2] also, the generalize inverse of operator $A: H \to K$ is denoted by $A^- \in B(K, H)$ satisfying the condition $AA^- A = A$ [3] and an operator $A: H \to K$ is said to be positive operator if $< Ax, x > \geq 0$, for all $x \in X$ and denoted by $A \geq 0,$ [2].

Proposition (1) :-

If $A: H \longrightarrow H$ positive operator, it follows $A \geq 0$ then, $x^* A x \geq 0$ for any $x \in H.$ [5]

1- general Common positive Solutions for some types of adjointable Operator Equations.

In this section, we introduce the common positive solution of adjointable bounded operator equations, $AXA^* = B^*B,\ CXC^* = D^*D$ (1) via g-inverse operator these equations appeared in [4],

Theorem (1.1):

Let $A, C \in B(H, K),\ B, D \in B(K, H)$ and A, C, M have g-inverse operators, then the adjointable operator equations (1) have a common positive solution if and only if $AA^- B^* B = B^* B,$ $D^* D - S - MM^- S(M^-)^* M^* - = 0,$ $(B^* B, D^* D \geq 0),$ where, $S = CA^- B^* B(A^-)^* C^*$ and $M = C(I - A^- A).$

Proof:-

We claim $X_0 = A^- B^* B(A^-)^* + (I - A^- A) M^- CA^- B^* BC^* (A^-)^* (M^-)^* (I - A^- A)^*$ be a common solution to the adjointable operator equations (1), since $B^* B, D^* D \geq 0$ then by using proposition (1), we can have X_0 is positive and substitute X_0 in the left side of adjointable operator equations (1) we get;

$$AX_0A^* = AA^-B^*B(A^-)^*A^* + A(I - A^-A)M^-CA^-B^*BC^*(A^-)^*(M^-)^*(I - A^-A)^*A^*$$
$$= AA^-B^*B(A^-)^*A^*$$
$$= AA^-B^*B$$

By the condition $AA^-B^*B = B^*B,$ so $AX_0A^* = B^*B$

And, $CX_0C^* = CA^-B^*B(A^-)^*C^* + C(I - A^-A)M^-CA^-B^*BC^*(A^-)^*(M^-)^*(I - A^-A)^*C^*$
$= S + MM^-S(M^-)^*M^*$

And By the condition $D^*D - S - MM^-S(M^-)^*M^* = 0,$ so $CXC^* = D^*D.$

Therefore; $X_0 = A^-B^*B(A^-)^* + (I - A^-A)M^-CA^-B^*BC^*(A^-)^*(M^-)^*(I - A^-A)^*$ is a positive common solution of adjointable operator equations (1).

Conversely; let X_0 is a positive common solution to the adjointable operator equations (1), then it is satisfy this equation, $AX_0A^* = B^*B$ since $X_0 \geq 0$ and by proposition (1) we get $B^*B \geq 0.$

$$AX_0A^* = B^*B$$
$$AA^-B^*B(A^-)^*A^* + A(I - A^-A)M^-CA^-B^*BC^*(A^-)^*(M^-)^*(I - A^-A)^*A^* = B^*B$$
$$AA^-B^*B(A^-)^*A^* = B^*B, \text{ so } AA^-B^*B = B^*B$$

Also, $CXC^* = D^*D,$ since $X_0 \geq 0$ and by proposition (1) we get; $D^*D \geq 0.$

$$CXC^* = D^*D$$
$$CA^-B^*B(A^-)^*C^* + C(I - A^-A)M^-CA^-B^*BC^*(A^-)^*(M^-)^*(I - A^-A)^*C^* = D^*D$$
$$S + MM^-(S - D^*D(C^-)^*C^*) = D^*D.$$

Thus we can have $D^*D - S - MM^-S(M^-)^*M^* = 0.$

Now we show the general common positive solution of adjointable operator equations (1), with same condition appeared in above theorem.

Theorem (1.2):-

Let $A, C \in B(H, K),\ B, D \in B(K, H)$ and A, C, M have g-inverse operators, then the adjointable operator equations (1), have a general common positive solution if and only if, $AA^-B^*B = B^*B,$ $D^*D - S - MM^-S(M^-)^*M^* = 0$ and $B^*B, D^*D, Z \geq 0,$ where, $S = CA^-B^*B(A^-)^*C^*,$ $M = C(I - A^-A)$ and $(I - U^-U) = (I - A^-A)(I - M^-M).$

Proof:-

We claim $X = X_0 + (I - A^-A)(I - M^-M)Z(I - M^-M)^*(I - A^-A)^*$ be a general common solution to the adjointable operator equations (1), since $B^*B, D^*D, Z \geq 0$ then by using proposition (1), we can have X is positive and substitute X in the left side of adjointable operator equations (1) we get;

$$AXA^* = A(X_0 + (I - A^-A)(I - M^-M)Z(I - M^-M)^*(I - A^-A)^*)A^*$$
$$= AX_0A^* + A((I - A^-A)(I - M^-M)Z(I - A^-A)^*(I - M^-M)^*)A^*$$
$$= AX_0A^*, \text{ thus from theorem (1.1) we can have; } AXA^* = B^*B.$$

And, $CXC^* = CX_0C^* + C(I - A^-A)(I - M^-M)Z(I - M^-M)^*(I - A^-A)^*C^*$

$$= CX_0C^* + M(I - M^-M)Z(I - M^-M)^*M^*$$
$$= CX_0C^*, \text{ thus from theorem (1.1) we can have; } CXC^* = D^*D.$$

Therefore; $X = X_0 + (I - A^-A)(I - M^-M)Z(I - M^-M)^*(I - A^-A)^*$ is a general positive common solution of adjointable operator equations (1).

Conversely; let X is a positive common solution to the adjointable operator equations (1.6), then it is satisfy this equations, $AXA^* = B^*B,$ since $X \geq 0$ and by proposition (1) we get; $B^*B \geq 0.$

$$AXA^* = B^*B,$$

$A(X_0+(I-U^-U)Z(I-U^-U)^*)A^*=B^*B$

$AX_0A^*=B^*B,$ thus from theorem (1.1) we can have; $AA^-B^*B=B^*B.$

Also, $CXC^*=D^*D,$ since $X\geq 0$ and by proposition (1) we get $D^*D\geq 0.$

$CXC^*=D^*D$

$C(X_0+(I-U^-U)Z(I-U^-U)^*)C^*=D^*D$

$CX_0C^*=D^*D,$ Thus from theorem (1.1) we can have; $D^*D-S-MM^-S(M^-)^*M^*=0,$ and since X is positive common solution to (1), then from [5] one can have Z is positive.

2- General Common positive Solutions for system of adjointable Operator Equations.

Theorem (2.1):-

Let $A,B\in B(H,K),$ $D,C,E,F\in B(K,H)$ and A,B,N,M have g-inverse operators, then the system of adjointable operator equations (2) have a positive common solution if $D^*D-MM^-(S+Y)(M^-)^*M^*=0,$ $AA^-E^*E(B^-)^*B^*=E^*E,$ $C^*C-T-(N^-N)^*TN^-N=0,$ $BA^-E^*E(B^-)^*A^*=F^*F,$ $A^*(A^-)^*B^*=B^*$ and $A^-E^*E(B^-)^*,S,T,Y\geq 0,$ where, $W=BA^-E^*E(B^-)^*B^*,$ $S=BE^*E(B^-)^*B^*,$ $T=AA^-E^*E(B^-)^*A^*,$ $Y=B^*A^-E^*E(B^-)^*A^*,$ $M=B(I-A^-A)$ and $N=(I-B^-B)^*A^*.$

Proof:-

We claim

$X_0=A^-A(A^-E^*E(B^-)^*(A^-A)^*+(I-A^-A)M^-S(M^-)^*(I-A^-A)^*+(I-B^-B)(N^-)^*TN^-(I-B^-B)^*$
$+(I-A^-A)M^-Y(M^-)^*(I-A^-A)^*.$ be a common solution to the system of adjointable operator equations (2), since $A^-E^*E(B^-)^*,S,T,Y\geq 0,$ then by using proposition (1), we can have X_0 is positive, to show that, we substitute in the system of adjointable operator equations(2), thus;

$$\begin{aligned}BX_0B^*&=BA^-A(A^-E^*E(B^-)^*(A^-A)^*B^*+B(I-A^-A)M^-S(M^-)^*(I-A^-A)^*B^*\\&\quad+B(I-B^-B)(N^-)^*TN^-(I-B^-B)^*B^*+B(I-A^-A)M^-Y(M^-)^*(I-A^-A)^*B^*\\&=B^*A^-E^*E(B^-)^*B^*+MM^-S(M^-)^*M^*+MM^-Y(M^-)^*M^*\\&=S+MM^-(S+Y)(M^-)^*M^*\end{aligned}$$

And, by using the condition, $D^*D-MM^-(S+Y)(M^-)^*M^*=0$ we get; $BXB^*=D^*D.$

Also,
$$\begin{aligned}AX_0A^*&=AA^-A(A^-E^*E(B^-)^*(A^-A)^*A^*+A(I-A^-A)M^-S(M^-)^*(I-A^-A)^*A^*\\&\quad+A(I-B^-B)(N^-)^*TN^-(I-B^-B)^*A^*+A(I-A^-A)M^-Y(M^-)^*(I-A^-A)^*A^*\\&=AA^-E^*E(B^-)^*A^*+N^*(N^-)^*TN^-N\\&=T+N^*(N^-)^*TN^-N\end{aligned}$$

Since $C^*C-T-(N^-N)^*TN^-N=0$ we get; $AXA^*=C^*C$

And,
$$\begin{aligned}AX_0B^*&=AA^-A(A^-E^*E(B^-)^*(A^-A)^*B^*+A(I-A^-A)M^-S(M^-)^*(I-A^-A)^*B^*\\&\quad+A(I-B^-B)(N^-)^*TN^-(I-B^-B)^*B^*+A(I-A^-A)M^-Y(M^-)^*(I-A^-A)^*B^*\\&=AA^-E^*E(B^-)^*A^*(A^-)^*B^*+A(I-A^-A)M^-S(M^-)^*M^*+N^*(N^-)^*TN^-(I-B^-B)^*B^*\\&\quad+A(I-A^-A)M^-Y(M^-)^*M^*\\&=AA^-E^*E(B^-)^*A^*(A^-)^*B^*\end{aligned}$$

By the condition, $A^*(A^-)^*B^*=B^*$ we get; $AX_0B^*=AA^-E^*E(B^-)^*B^*$

and by condition, $AA^-E^*E(B^-)^*B^*=E^*E$ we get; $AXB^*=E^*E$

Finally,
$$\begin{aligned}BX_0A^*&=BA^-A(A^-E^*E(B^-)^*(A^-A)^*A^*+B(I-A^-A)M^-S(M^-)^*(I-A^-A)^*A^*\\&\quad+B(I-B^-B)(N^-)^*TN^-(I-B^-B)^*A^*+B(I-A^-A)M^-Y(M^-)^*(I-A^-A)^*A^*\\&=BA^-E^*E(B^-)^*A^*+MM^-S(M^-)^*(I-A^-A)^*A^*+B(I-B^-B)(N^-)^*TN^-N\\&\quad+MM^-Y(M^-)^*(I-A^-A)^*A^*\\&=BA^-E^*E(B^-)^*A^*\end{aligned}$$

And By using the condition $BA^-E^*E(B^-)^*A^*=F^*F$ one can have $BXA^*=F^*F$

Therefore;

$X_0 = A^-A(A^-E^*E(B^-)^*(A^-A)^* + (I - A^-A)M^-S(M^-)^*(I - A^-A)^* + (I - B^-B)(N^-)^*TN^-(I - B^-B)^*$
$+ (I - A^-A)M^-Y(M^-)^*(I - A^-A)^*$. is a positive common solution of the system of adjointable operator equations (2).

Now, we show the general common positive solution of the system of adjointable operator equations (2), with same condition appeared in above theorem.

Theorem (2.2):-

Let $A, B \in B(H, K)$, $D, C, E, F \in B(K, H)$ and A, B, N, M have g-inverse operators, then the system of adjointable operator equations (2), have a general positive common solution if $AA^-E^*E(B^-)^*B^* = E^*E$, $D^*D - MM^-(S+Y)(M^-)^*M^* = 0$, $A^*(A^-)^*B^* = B^*$, $C^*C - T - (N^-N)^*TN^-N = 0$, $BA^-E^*E(B^-)^*A^* = F^*F$ and $A^-E^*E(B^-)^*, S, T, Y, Z \geq 0$, where, $W = BA^-E^*E(B^-)^*B^*$, $S = BE^*E(B^-)^*B^*$, $T = AA^-E^*E(B^-)^*A^*$, $M = B(I - A^-A)$, $Y = B^*A^-E^*E(B^-)^*A^*$ and $N = (I - B^-B)^*A^*$.

Proof:-

We claim $X = X_0 + (I - A^-A)(I - M^-M)Z(I - M^-M)^*(I - A^-A)^*$ be a common solution to the system of adjointable operator equations (2) since $A^-E^*E(B^-)^*, S, T, Y, Z \geq 0$, then by using proposition (1) we can have X is positive and substitute X in the left side of adjointable operator equations (2) we get;

$$BXB^* = B(X_0 + (I - A^-A)(I - M^-M)Z(I \_ NN^-)(I - B^-B)^*)B^*$$
$$= BX_0B^* + M(I - M^-M)Z(I \_ NN^-)(I - B^-B)^*B^*$$

$= BX_0B^*$, thus from theorem (2.1) we can have; $BXB^* = D^*D$.

Also, $AXA^* = AX_0A^* + A(I - A^-A)(I - M^-M)Z(I - M^-M)^*(I - A^-A)^*A^*$
$= AX_0A^* + A(I - A^-A)(I - M^-M)Z(I \_ M^-M)^*(I - A^-A)^*)A^*$

$= AX_0A^*$, thus from theorem (2.1) we can have; $AX_0A^* = C^*C$.

And, $AXB^* = AX_0B^* + A(I - A^-A)(I - M^-M)Z(I - M^-M)^*(I - A^-A)^*B^*$

$= AX_0B^* + A(I - A^-A)(I - M^-M)Z(I \_ M^-M)^*(I - A^-A)^*B^*$

$= AX_0B^*$, Thus from theorem (2.1) we can have; $AXB^* = E^*E$.

Finally, $BXA^* = BX_0A^* + B(I - A^-A)(I - M^-M)Z(I - M^-M)^*(I - A^-A)^*A^*$
$= BX_0A^* + M(I - M^-M)Z(I \_ M^-M)^*(I - A^-A)^*)A^*$

$= BX_0A^*$, thus from theorem (2.1) we can have; $AXA^* = C^*C$.

Therefore; $X = X_0 + (I - A^-A)(I - M^-M)Z(I - M^-M)^*(I - A^-A)^*$ is a general positive common solution of the system of adjointable operator equations (2).

Remark (2.3):-

The converse of theorems (2.1), (2.2) not necessary to be true. To illustrate this consider the following example.

Example (2.4):-

Let $A = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$, $D^*D = \begin{bmatrix} 24 & 24 \\ 24 & 24 \end{bmatrix}$, $C^*C = \begin{bmatrix} 24 & 72 \\ 72 & 216 \end{bmatrix}$, $E^*E = \begin{bmatrix} 24 & 24 \\ 72 & 72 \end{bmatrix}$, $F^*F = \begin{bmatrix} 24 & 72 \\ 24 & 72 \end{bmatrix}$, then $X = \begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}$ is a common positive solution of the system of adjointable operator equations (2).

$$BXB^* = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 24 & 24 \\ 24 & 24 \end{bmatrix}$$

$$AXA^* = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 24 & 72 \\ 72 & 216 \end{bmatrix}$$

$$AXB^* = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 24 & 24 \\ 72 & 72 \end{bmatrix}$$

$$BXA^* = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 24 & 72 \\ 24 & 72 \end{bmatrix}$$

But, $A^{-}E^{*}E(B^{-})^{*} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 24 & 24 \\ 72 & 72 \end{bmatrix}\begin{bmatrix} 0.25 & 0 \\ 0.25 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}$, it is not positive operator.

References

1- C.jacob, C.Andre & L.Arie, "Necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation $X + A^* X^{-1} A = Q$ ", linear algebra and its applications 186,pp.255-275,Avenue of the Americas, New York,1993.

2- Erwin K., " Introducation Funcational Analysis", Jon Wiley and Sons Inc., 1978.

3- John Z. Hearon," Generalized Inverses and Solutions of Linear Systems ", Journal of research of the Notional Bureau of Standards - B. Mathematical Sciences, Vol. 72B, No.4, (1968).

4- Xian Zhang "the general common Hermitian nonnegative definite solution to the matrix equations AXA*= BB* and CXC*= DD* with applications in statistics", Journal of Multivariate analysis 93 PP. 257–266, (2005).

5- Xu, Q, "Common hermitian and positive solutions to the adjointable operator equations $AX = C$, $XB = D$", linear algebra and its applications 429, pp.1-11, (2008).