

On Quasi-similarity of a product of Quasi-similar operators in Hilbert Spaces.

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Abstract

In this paper we investigate the conditions under which two operators A and B are quasi-similar implies AB and BA are also quasi-similar. It is herein shown that if A and B satisfy the Putnam-Fuglede property and are quasi-similar with the intertwining quasiaffinities self adjoint, then AB and BA are also quasi-similar.

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1. INTRODUCTION

Let H be a complex Hilbert space and $B(H)$ denote the Banach algebra of all bounded linear operators on H. An operator $X \in B(H)$ is said to be a quasiaffinity if X is both one-to-one and has a dense range. Then two operators A and B are said to be similar if there exists an invertible operator S such that $AS = SB$, while A and B are said to be quasisimilar if there exists quasiaffinities X and Y such that $AX = XB$ and $BY = YA$.

It is well known in operator theory that two similar operators have the same spectrum. However, in the case of quasisimilarity, it is not necessarily true. This depends on the classes of operators. The concept of quasisimilarity and equality of spectra has been considered by a number of authors among them S. Clary[1] who showed that quasisimilar hyponormal operators have equal spectra. J.M. Khalagai and B.Nyamai[5] showed that if A and B are quasisimilar with A dominant and $B*$ M-hyponormal then A and B have equal spectra. B.P.Duggal[2] showed that if Ai : i = 1, 2 are quasisimilar p-hyponormal operators such that Ui is unitary in the polar decomposition $Ai = Ui|Ai|$ then $A1$ and $A2$ have equal spectra and also equal essential spectra. J.P.Williams[6] and [7] showed that there are several cases which imply that two operators A and B have equal essential spectra under quasisimilarity. For example, if A and B are both hyponormal or are both quasinormal or are both partial isometries.

In this paper, we first investigate the conditions under which quasisimilarity of operators A and B implies quasisimilarity of AB and BA. We then deduce a number of results on equality of spectra and essential spectra of the operators AB and BA

2. NOTATIONS, DEFINITIONS AND TERMINOLOGIES

Given a complex Hilbert space H and operators $A, B \in B(H)$. We define the commutator:

$C(A, B) : B(H) \longrightarrow B(H)$ by $C(A, B)X = AX - XB$ for some $X \in B(H)$

Let C denote the class of contractions with C_o completely non-unitary parts. Let C_o denote the class of contractions $A \in C$ which satisfy the property: $(P2)$ if the restriction of A to an invariant subspace M is normal then M reduces A. Also, let C_1 be the class of contractions $A \in C_o$ with defect operator $D_A = (1 - A*A)^{1/2}$ being of the Hilbert-Schmidt class C_2 and which are such that neither the pure part of A has empty point spectrum or the eigenvalues of A are all simple.

The classical Putnam-Fuglede commutativity theorem states that if A and B are normal operators then $C(A, B)X = 0$ for some operator $X \in B(H)$ implies $C(A^*, B^*)X = 0$. This result has been extended to other classes of operators. Among such results is the fact that if $A \in C_0$ and $B^* \in C_1$ then $C(A, B)X = 0$ for some operator X then $C(A^*, B^*)X = 0$ see[3]

The spectrum of $A_\lambda \in B(H)$ will be denoted by $\sigma(A)$. Thus:

$\sigma(A) = \{\lambda \in \mathrm{C} : A - \lambda I$ is not invertible$\}$ where C is the complex number field.

The point spectrum of A will be denoted by $\sigma_P(A)$. Thus: $\sigma_P(A) = \{\lambda \in \mathrm{C} : A - \lambda I$ is not one-to-one$\}$

The essential spectrum of A will be denoted by $\sigma_e(A)$. Thus: $\sigma_e(A) = \{\lambda \in \mathrm{C} : A - \lambda I$ is not Fredholm$\}$

An operator A is called:

- Self-adjoint if $A = A^*$
- Normal if $[A^*, A] = 0$
- Hyponormal if $A^*A \geq AA^*$
- Dominant if to each $\lambda \in \mathrm{C}$, $\exists M_\lambda > 0$ such that $||(A - \lambda)^* x|| \leq M_\lambda ||(A - \lambda)x||$
- Quasinormal if $[A^*A, A] = 0$
- p-hyponormal if $(A^*A)^p \geq (AA^*)^p$ for $0 < p < 1$
- Partial isometry if $A = AA^*A$
- Isometry if $A^*A = I$
- Unitary if $A^*A = AA^* = I$
- Fredholm if its range denoted by ran$A$ is dense and both the null space, Ker$A$ and Ker A^* are finite dimensional.

2.1 CLASS INCLUSIONS

The following class inclusions hold and are proper:

Unitary $\subseteq$ *Isometry* $\subseteq$ *Partial Isometry* $\subseteq$ *Contraction*

Normal $\subseteq$ *Quasinormal* $\subseteq$ *Subnormal* $\subseteq$ *Hyponormal* $\subseteq$ *M − hyponormal* $\subseteq$ *Dominant*

3. RESULTS

3.1 THEOREM 1

Let $A, B \in B(H)$ be quasisimilar operators with the intertwining quasiaffinities self adjoint. Then AB and BA are also quasisimilar provided that A and B satisfy the Putnam-Fuglede property.

Proof

Assume $A, B \in B(H)$ are quasisimilar. Then there exist quasiaffinities X and Y such that

$AX = XB$ and $BY = YA$

Now by the Putnam-Fuglede property, we have

$A^*X = XB^*$ and $B^*Y = YA^*$

Taking adjoints we get

$BX = XA$ and $AY = YB$

Therefore $ABX = AXA = XBA$ and $BAY = BYB = YAB$

Hence AB and BA are also quasisimilar through the same intertwining quasiaffinities.

3.2 COROLLARY 1

Let $A, B \in B(H)$ be quasisimilar operators with the intertwining quasiaffinities self adjoint then AB and BA are quasisimilar under any one of the following conditions:

- A is dominant and B^* is M-hponormal.
- A is dominant and B^* is p-hyponormal.
- A and B^* are p-hyponormal
- $A \in C_o$ and $B^* \in C_1$

Proof

This is trivial since the classes of operators in this corollary satisfy the classical Putnam-Fuglede property.

3.3 COROLLARY 2

Let $A, B \in B(H)$ be quasisimilar operators with the intertwining quasiaffinities self adjoint. If AB and BA are also hyponormal then:

$$\sigma(AB) = \sigma(BA)$$
$$\sigma_e(AB) = \sigma_e(BA)$$

Proof

We note that for quasisimilar hyponormal operators A and B we have that $\sigma(A) = \sigma(B)$ $\sigma_e(AB) = \sigma_e(BA)$ see[7]

3.4 REMARK

We note that in the theorem above if the operator B is replaced with A^* then we can drop the condition of self-adjointness on the implementing quasiaffinities as is shown in theorem 2.

3.5 THEOREM 2

Let A and A^* satisfy the Putnam-Fuglede property and be quasisimilar. Then A^*A and AA^* are also quasisimilar. Consequently $\sigma(A^*A) = \sigma(AA^*)$ and $\sigma_e(AB) = \sigma_e(BA)$

Proof

Now A and A^* are quasisimilar implies there exist quasiaffinities X and Y such that:

$$AX = XA^* \text{ and } A^*Y = YA$$

Since A and A^* satisfy Putnam-Fuglede property we also have:

$$A^*X = XA \text{ and } AY = YA^*$$
$$\text{i.e. } A^*AX = A^*XA^* \text{ and } A^*XA = XAA^*$$

Also $AA^*Y = AYA = YA^*A$

Hence A^*A and AA^* are quasisimilar with the same implementing quasiaffinities. We also note that quasisimilar hyponormal operators have equal spectra and equal essential spectra. Now A^*A and AA^* being positive operators are a subclass of hyponormal operators. Hence, we have that $\sigma(A^*A) = \sigma(AA^*)$ and $\sigma_e(A^*A) = \sigma_e(AA^*)$

3.6 COROLLARY 3

If an M-hyponormal operator A is quasisimilar to its adjoint A^* then $\sigma(A^*A) = \sigma(AA^*)$ and $\sigma_e(A^*A) = \sigma_e(AA^*)$

Proof

We first note that for an M-hyponormal operator A we have that $AX = XA^*$ implies $A^*X = XA$ and $A^*Y = YA$ implies $AY = YA^*$ Hence, by theorem 2 we have the result.

3.7 COROLLARY 4

If a hyponormal operator A is quasisimilar to its adjoint A^* then we have $\sigma(A) = \sigma(A^*)$, $\sigma(A^*A) = \sigma(AA^*)$, $\sigma_e(A) = \sigma_e(A^*)$ and $\sigma_e(A^*A) = \sigma_e(AA^*)$

Proof

This follows from the fact that quasisimilar hyponormal operators have same spectra and same essential spectra together with the fact that they satisfy the Putnam-Fuglede property and are a subclass of hyponormal operators.

3.8 REMARK

Note that for an operator $B \in B(H)$, we say that B is consistent in invertibility (with respect to multiplication) or briefly that B is a *CI* operator if for any other operator $A \in B(H)$, AB and BA are invertible or non-invertible together. Thus B is *CI* if implies $\sigma(AB) = \sigma(BA)$ W. Gong and D. Han[4] proved among other results that an operator $B \in B(H)$ is *CI* operator iff $\sigma(B^*B) = \sigma(BB^*)$. In view of this result the following corollary is immediate.

3.9 COROLLARY 5

An M-hyponormal operator A which is quasisimilar to its adjoint A^* is a *CI* operator.

Proof

Since A is M-hyponormal and quasisimilar to its adjoint A^*, we by corollary 3 that; $\sigma(A^*A) = \sigma(AA^*)$. Hence, from the definition of a *CI* operator, it follows that A^* is *CI* operator

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