Dynamics of a prey-predator model involving a prey refuge and disease in the predator

M.V. Ramana Murthy and Dahlia Khaled Bahlool*
Departments of Mathematics, College of Science, Osmania University, Hyderabad, India

Abstract

In this paper, a mathematical model consisting of a prey-predator involving a prey refuge and infectious disease in the predator has been proposed and analyzed. Two types of functional responses are used to describe the feeding of the predator on the available prey. The existence, uniqueness and boundedness of the solution of the system are discussed. The dynamical behavior of the system has been investigated locally as well as globally using suitable Lyapunov function. The persistence conditions of the system are established. Local bifurcation near the equilibrium points has been investigated. The Hopf bifurcation conditions around the positive equilibrium point are derived. Finally, numerical simulations are carried out to specify the control parameters and confirm the obtained results

Keywords: Prey-Predator, Disease, Refuge, Stability, Bifurcation.

1. Introduction

It is well known that the term ecology deals with the study of the interactions of organisms (individual living creature, either unicellular or multi-cellular) with their physical environment and with one another. The interaction between predator and their prey is one of the important subjects in both ecology and applied mathematics due to its wide existence in our life and importance. Although such problems may appear to be simple mathematically, they are very challenging and complicated [1]. Since the pioneering work of Lotka (1925) and Volterra (1926) on predator-prey system, the field of mathematical ecology has reached at the top level in theoretical biology and many research works have been down in literature, see for example [2-4] and the references therein. Moreover, later on Kaewmanee and Tang [5] studied the dynamics of an age-structured prey-predator system incorporating cannibalism. However, Tian and Xu [6] proposed and analyzed a predator-prey system with Holling type II functional response and stage structure in predator, and many other research works involving different factors have been proposed and studied.

On the other hand, epidemiology is the study of the spread of diseases in species. It is well known that after the pioneering work of Kermack–Mckendrick [7], which is based on classical susceptible, infected, recovered model, the field of epidemiology has come into sight and received a lot of attention from the researchers see for example [8-9] and the references therein. Moreover, later on Naji and Shafeeq [10], studied the effect of treatment and immigrants on the dynamics of SIS epidemic model. Zhou and Yao [11], investigated a host-vector epidemic model with stage-structure for the vector. Further investigations have been published in the field of epidemiology.

In fact recently there is increasing interest from the researchers by combining these two fields in one field that called eco-epidemiology, which has become a major field of study, see for example [12–19] and the references therein. Recently, Pal et al [20], presented an investigation deals with a predator–prey model with SI-type of disease that spreads among the predator species only. They assumed that the disease transfers by contact according to mass action law and the predator has another sources of food.

Finally according to nature, the prey population prefers staying in a safe area where the predation is prohibitive. This is known as prey refuge, which reduces the chance of extinction in prey species due to predation and damp the oscillation in the system. Therefore the existence of refugees will increase the size of prey population and then stabilizing the real world system. The effect of prey refuge on the behavior of prey-predator systems has been studied by many researchers in literature see for example [21-22] and the references therein. Recently, Pal and Samanta [1], proposed and analyzed a prey-predator model incorporating a prey refuge with disease in the prey-population.

In this paper however, we proposed and studied a mathematical model for prey-predator system involving prey refuge and SIS-type of disease in predator. It is assumed that the disease transfer through contact between the susceptible and infected individuals as well as through an external resources.

2. Model Formulation

In this section, a prey-predator system incorporating a prey refuge and disease in a predator is formulated mathematically. The following hypotheses are adopted in constricting the mathematical model that describe the dynamics of the above real world eco-epidemiological system.

1. In the absence of predator, the prey population that denoted by $X(t)$ grows logistically with carrying capacity $K > 0$ and intrinsic growth rate $r > 0$.
2. In the presence of infections disease, the predator population is divided into two compartments, namely susceptible predator that denoted by $S(t)$ and infected predator, which denoted by $I(t)$. Therefore at time *t* the total predator population is $P(t) = S(t) + I(t)$.
3. The disease doesn't transfer to prey population instead of that it spreads among the predator species by contact between the susceptible predator individual and infected predator individual with contact infection rate $c_1 > 0$, in addition to an external resources (air, food, etc) with external infection rate $c_2 > 0$. The infected predator doesn't become immune and may be recovers and becomes susceptible again with recovering rate $c_3 > 0$.
4. It is assuming that there is a refuge protecting nX of the prey species where $n \in (0,1)$ denotes the refuge rate constant. Thus there is $(1-n)X$ of prey species available to the predator.
5. Since the susceptible predator is more efficient compared with the infected predator, it is assuming that the susceptible predator consumes the prey according to Lotka-Voltter type of functional response with attack rate $a_1 > 0$. However the infected predator consumes the prey according to Holling type-II functional response with maximum attack rate $a_2 > 0$ and half saturation constant $b > 0$. The prey converts into predator with conversion rate $0 < e < 1$.
6. In the absence of prey the susceptible predator decays with natural death rate $d_1 > 0$, however the infected predator decays with death rate (natural death + disease death) $d_2 > 0$

According to the above hypotheses the dynamics of the above described eco-epidemiological real world system can be represented by the following set of differential equations.

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right) - a_1(1-n)XS - \frac{a_2(1-n)XI}{b + (1-n)X}$$

$$\frac{dS}{dt} = ea_1(1-n)XS - c_1SI - c_2S - d_1S + c_3I \qquad (1)$$

$$\frac{dI}{dt}=\frac{ea_2(1-n)XI}{b+(1-n)X}+c_1SI+c_2S-d_2I-c_3I$$

with initial data $X(0)\geq 0$, $S(0)\geq 0$ and $I(0)\geq 0$.

Clearly the interaction functions in system (1) are continuously differential functions on the domain $R_+^3=\{(X,S,I)\in R_+^3 : X(t)\geq 0, S(t)\geq 0, I(t)\geq 0\}$. Hence they are Lipschitzian. Therefore the solution of system (1) exits and is unique. Moreover all solutions of system (1) initiated in R_+^3 are uniformly bounded as shown in the following theorem.

Theorem (1): All the solutions of system (1), which initiate in R_+^3, are uniformly bounded.

Proof: Let $(X(t),S(t),I(t))$ is any solution of system (1) initiated in R_+^3, and let W is a function defined by $W(t)=X(t)+S(t)+I(t)$. Then by differentiate this function with respect to time gives

$$\frac{dW}{dt}=\frac{dX}{dt}+\frac{dS}{dt}+\frac{dI}{dt}\leq (r+1)X\left[1-\frac{X}{\delta_1}\right]-\mu W$$

here $\delta_1=\frac{K(r+1)}{r}$ and $\mu=\min\{1,d_1,d_2\}$. Now since we have

$$Sup.(r+1)X\left[1-\frac{X}{\delta_1}\right]\leq\frac{(r+1)\delta_1}{4}=\frac{K(r+1)^2}{4r}=\delta_2$$

Thus

$$\frac{dW}{dt}+\mu W\leq\delta_2$$

So by using Grownwall lemma we obtain that

$$0<W<\frac{\delta_2}{\mu}\left(1-e^{-\mu t}\right)+e^{-\mu t}W(0)$$

where $W(0)=W(X(0),S(0),I(0))$. Therefore for $t\to\infty$ we obtain

$$0<W(X(t),S(t),I(t))\leq\frac{\delta_2}{\mu}$$

Thus all the solutions of system (1) are uniformly bounded and enter the region

$$\Gamma_1=\left\{(X,S,I)\in R_+^3 : W\leq\frac{\delta_2}{\mu}+\varepsilon,\varepsilon>0\right\}$$ ■

3. Stability and persistence

In this section, the stability analysis of all possible equilibrium points of system (1) is investigated. The persistence conditions of system (1) are established. Now straightforward computation shows that the system (1) has at most three nonnegative equilibrium points. The existence conditions for each of them can be summarized as follows:

1. The vanishing equilibrium point, say $E_0=(0,0,0)$ always exists.

2. The predator free equilibrium point, namely $E_1 = (\overline{X}, 0, 0) = (K, 0, 0)$ always exists.

3. The predator can't survive in the absence of the prey, therefore there are no equilibrium points on the $S-$axis, $I-$axis or in the interior of $SI-$plane. Moreover, since the external infection rate c_2 and the recovering rate c_3 are assumed be positive, hence there are no equilibrium points in the $XS-$plane and $XI-$plane.

4. The positive equilibrium point that denoted by $E_2 = (X^*, S^*, I^*)$, exists uniquely in the interior of R_+^3 provided that the following algebraic system has a unique positive solution.

$$\begin{aligned} & r\left(1-\frac{X}{K}\right) - a_1(1-n)S - \frac{a_2(1-n)I}{b+(1-n)X} = 0 \\ & ea_1(1-n)XS - c_1SI - (c_2+d_1)S + c_3I = 0 \\ & \frac{ea_2(1-n)XI}{b+(1-n)X} + c_1SI + c_2S - (d_2+c_3)I = 0 \end{aligned} \tag{2}$$

From the second equation we get

$$I = \left[\frac{ea_1mx - (c_2+d_1)}{c_1S - c_3}\right]S \tag{3a}$$

here $m = 1-n$ for simplicity and $c_1S - c_3 \neq 0$.
Now by substituting Eq.(3a) in the first and third equations gives the following two isoclines respectively.

$$\begin{aligned} f(X,S) = {} & rc_3mX^3 - rc_1msX^2 + \left[rc_1(Km-b) - a_1Km^2(c_3+a_2e)\right]XS \\ & + rc_3[b-Km]X + K\left[rbc_1 + m\left(a_1bc_3 + a_2(c_2+d_1)\right)\right]S \\ & - a_1bc_1KmS^2 - a_1c_1Km^2S^2X - rKbc_3 = 0 \end{aligned} \tag{3b}$$

$$\begin{aligned} g(X,S) = {} & ea_1m^2[ea_2 - (d_2+c_3)]SX^2 + ea_1c_1mS^2X^2 \\ & + m[(d_2+c_3)(c_2+d_1-eba_1) - ea_2(c_2+d_1) - c_2c_3]SX \\ & + c_1m[ea_1b - d_1]S^2X - bc_1d_1S^2 + b[d_2(c_2+d_1) + c_3d_1]S = 0 \end{aligned} \tag{3c}$$

Note that, in order to obtain a unique positive equilibrium point for system (1) in the interior of R_+^3, it is sufficient to show that the two isoclines (3b) and (3c) have a unique positive intersection point, namely (X^*, S^*).

It is clear from Eq. (3b) that, as $X \to 0$ we obtain

$$\sigma_1S^2 + \sigma_2S + \sigma_3 = 0 \tag{3d}$$

where $\sigma_1 = -a_1bc_1Km < 0$; $\sigma_2 = K\left[rbc_1 + m\left(a_1bc_3 + a_2(c_2+d_1)\right)\right] > 0$; $\sigma_3 = -rKbc_3 < 0$.
Straightforward computation gives that Eq. (3d) has two positive roots given by

$$S_1 = -\frac{\sigma_2}{2\sigma_1} + \frac{1}{2\sigma_1}\sqrt{\sigma_2^2 - 4\sigma_1\sigma_3}$$
$$S_2 = -\frac{\sigma_2}{2\sigma_1} - \frac{1}{2\sigma_1}\sqrt{\sigma_2^2 - 4\sigma_1\sigma_3}$$

Provided that the following condition holds

$$\sigma_2^2 > 4\sigma_1\sigma_3 \tag{3e}$$

Moreover Eq. (3d) has a unique maximum positive value given by

$$-\frac{\sigma_2^2}{4\sigma_1} + \sigma_3 = \gamma_1 \tag{3f}$$

Similarly, as $X \to 0$, Eq. (3c) gives that

$$\sigma_4 S^2 + \sigma_5 S = 0 \tag{3g}$$

where $\sigma_4 = -bc_1 d_1 < 0;\ \sigma_5 = b[(c_2 + d_1)d_2 + d_1 c_3] > 0$. Clearly Eq. (3g) has a zero root and a positive root given by:

$$S_3 = -\frac{\sigma_5}{\sigma_4}$$

Further Eq. (3g) has a unique maximum positive value given by

$$-\frac{\sigma_5^2}{4\sigma_4} = \gamma_2 \tag{3h}$$

Consequently the two isoclines (3b) and (3c) has a unique positive intersection point, denoted by (X^*, S^*) provided that condition (3e) holds along with the following two conditions

$$S_2 < S_3 < S_1 \tag{3i}$$

$$\gamma_2 < \gamma_1 \tag{3j}$$

Thus by substituting the value of (X^*, S^*) in Eq. (3a) gives the value of I^*, which is positive provided that one set of the following sets of conditions hold

$$X^* > \frac{c_2 + d_1}{ea_1 m} \quad \text{and} \quad S^* > \frac{c_3}{c_1} \tag{4a}$$

or

$$X^* < \frac{c_2 + d_1}{ea_1 m} \quad \text{and} \quad S^* < \frac{c_3}{c_1} \tag{4b}$$

In the following the local stability analysis of the above equilibrium points is carried out. The Jacobian matrix of system (1) at the point (X, S, I) can be written as

$$J(X, S, I) = (u_{ij})_{3\times 3} \tag{5}$$

here

$$u_{11} = X\left[-\frac{r}{K} + \frac{a_2(1-n)^2 I}{(b+(1-n)X)^2}\right] + r\left(1 - \frac{X}{K}\right) - a_1(1-n)S - \frac{a_2(1-n)I}{b+(1-n)X}$$

$$u_{12} = -a_1(1-n)X$$

$$u_{13} = -\frac{a_2(1-n)X}{b+(1-n)X}$$

$$u_{21} = ea_1(1-n)S$$
$$u_{22} = ea_1(1-n)X - c_1 I - (c_2 + d_1)$$
$$u_{23} = -c_1 S + c_3$$
$$u_{31} = \frac{ea_2 b(1-n)I}{(b+(1-n)X)^2}$$
$$u_{32} = c_1 I + c_2$$
$$u_{33} = \frac{ea_2(1-n)X}{b+(1-n)X} + c_1 S - (d_2 + c_3)$$

Therefore the Jacobian matrix at E_0 is

$$J(E_0) = \begin{pmatrix} r & 0 & 0 \\ 0 & -(c_2+d_1) & c_3 \\ 0 & c_2 & -(c_3+d_2) \end{pmatrix} \quad (6a)$$

Clearly the characteristic equation of $J(E_0)$ can be written as:

$$(r-\lambda_0)\left[\lambda_0^2 - T_0\lambda_0 + D_0\right] = 0 \quad (6b)$$

where $T_0 = -(c_2+d_1) - (c_3+d_2) < 0;\ D_0 = (c_2+d_1)d_2 + c_3 d_1 > 0$.

Accordingly, the eigenvalues of $J(E_0)$ in a the $X-$direction, $S-$direction and $I-$ direction can be written respectively

$$\begin{aligned} \lambda_{0X} &= r > 0 \\ \lambda_{0S} &= \frac{T_0}{2} + \frac{1}{2}\sqrt{T_0{}^2 - 4D_0} \\ \lambda_{0I} &= \frac{T_0}{2} - \frac{1}{2}\sqrt{T_0{}^2 - 4D_0} \end{aligned} \quad (6c)$$

Since $\lambda_{0X} = r > 0$ with other two eigenvalues have negative real parts, then E_0 is a saddle point.

The Jacobian matrix of system (1) at the predator free equilibrium point E_1 can be written as:

$$J(E_1) = \begin{pmatrix} -r & -a_1(1-n)K & -\frac{a_2(1-n)K}{b+(1-n)K} \\ 0 & ea_1(1-n)K - (c_2+d_1) & c_3 \\ 0 & c_2 & \frac{ea_2(1-n)K}{b+(1-n)K} - (d_2+c_3) \end{pmatrix} = (b_{ij}) \quad (7a)$$

Hence the characteristic equation of $J(E_1)$ is:

$$(-r-\lambda_1)\left[\lambda_1^2 - T_1\lambda_1 + D_1\right] = 0 \quad (7b)$$

here

$$T_1 = ea_1(1-n)K - (c_2+d_1) + \frac{ea_2(1-n)K}{b+(1-n)K} - (c_3+d_2)$$
$$D_1 = \frac{e\,(1-n)K}{b+(1-n)K}\left[a_2\left(ea_1(1-n)K - (c_2+d_1)\right) - a_1(c_3+d_2)(b+(1-n)K)\right] + (c_2+d_1)d_2 + c_3 d_1$$

Therefore, the eigenvalues of $J(E_1)$ in the X, S, and $I-$directions respectively are

$$\lambda_{1X} = -r < 0$$
$$\lambda_{1S} = \frac{T_1}{2} + \frac{1}{2}\sqrt{T_1^{\,2} - 4D_1} \qquad (7c)$$
$$\lambda_{1I} = \frac{T_1}{2} - \frac{1}{2}\sqrt{T_1^{\,2} - 4D_1}$$

Straightforward computations show that, λ_{1S} and λ_{1I} have negative real parts if the following conditions are satisfied.

$$ea_1(1-n)K < (c_2 + d_1) \qquad (8a)$$

$$\frac{ea_2(1-n)K}{b+(1-n)K} < (c_3 + d_2) \qquad (8b)$$

$$\frac{e(1-n)K}{b+(1-n)K}\left[a_2\left(c_2 + d_1 - ea_1(1-n)K\right) + a_1\left(c_3 + d_2\right)\left(b+(1-n)K\right)\right] < \left(c_2 + d_1\right)d_2 + c_3 d_1 \qquad (8c)$$

Accordingly, the predator free equilibrium point E_1 is locally asymptotically stable under the above conditions.

Theorem (2): Suppose that the positive equilibrium point E_2 exists, then it is locally asymptotically stable in R_+^3 if the following conditions hold:

$$\frac{a_2(1-n)^2 I^*}{R^{*2}} < \left(\frac{r}{K}\right)\frac{c_1 I^* + c_2}{ea_1 b + c_1 I^* + c_2} \qquad (9a)$$

$$ea_1(1-n)X^* < c_1 I^* + c_2 + d_1 \qquad (9b)$$

$$\frac{ea_2(1-n)X^*}{R^*} + c_1 S^* < c_3 + d_2 \qquad (9c)$$

$$c_3 < c_1 S^* \qquad (9d)$$

$$(c_1 I^* + c_2)S^* R^* < b(c_1 S^* - c_3)I^* \qquad (9e)$$

here $R^* = b + (1-n)X^*$.

Proof: According to Eq. (5) the Jacobian matrix at E_2 is given by

$$J(E_2) = \left(a_{ij}\right)_{3\times 3} \qquad (10a)$$

where

$$a_{11} = X^*\left[-\frac{r}{K} + \frac{a_2(1-n)^2 I^*}{R^{*^2}}\right]; a_{12} = -a_1(1-n)X^* < 0; a_{13} = -\frac{a_2(1-n)X^*}{R^*} < 0$$

$$a_{21} = ea_1(1-n)S^* > 0; a_{22} = ea_1(1-n)X^* - c_1 I^* - \left(c_2 + d_1\right); \; a_{23} = -c_1 S^* + c_3$$

$$a_{31} = \frac{ea_2 b(1-n)I^*}{R^{*2}} > 0; a_{32} = c_1 I^* + c_2 > 0; a_{33} = \frac{ea_2(1-n)X^*}{R^*} + c_1 S^* - \left(d_2 + c_3\right)$$

Therefore the characteristic equation of $J(E_2)$ can be written as:

$$\lambda_2^3 + A_1\lambda_2^2 + A_2\lambda_2 + A_3 = 0 \qquad (10b)$$

where

$$A_1 = -(a_{11} + a_{22} + a_{33})$$
$$A_2 = a_{11}a_{22} - a_{12}a_{21} + a_{22}a_{33} - a_{23}a_{32} + a_{11}a_{33} - a_{13}a_{31}$$
$$A_3 = -a_{33}(a_{11}a_{22} - a_{12}a_{21}) - a_{23}(a_{12}a_{31} - a_{11}a_{32}) - a_{13}(a_{21}a_{32} - a_{22}a_{31})$$
$$\Delta = A_1A_2 - A_3 = -(a_{11} + a_{22})(a_{11}a_{22} - a_{12}a_{21}) - (a_{11} + a_{33})(a_{11}a_{33} - a_{13}a_{31})$$
$$-(a_{22} + a_{33})(a_{22}a_{33} - a_{23}a_{32}) - 2a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

According to Routh-Hurwitz criterion Eq. (10b) has roots (eigenvalues) with negative real parts if and only if $A_1 > 0$, $A_3 > 0$ and $\Delta > 0$.

Straightforward computation shows that all the requirements of Routh-Hurwitz criterion are satisfied provided that the condition (9a)-(9e) hold. Hence E_2 is locally asymptotically stable.

Now, the global dynamics of system (1) is investigated using suitable Lyapunov functions as shown in the following theorem.

Theorem (3): Assume that the predator free equilibrium point E_1 is locally asymptotically stable, then it is a globally asymptotically stable in R_+^3 provided that

$$K < \min\left\{\frac{d_1}{ea_1(1-n)}, \frac{d_2 b}{ea_2(1-n)}\right\} \qquad (11)$$

Proof: Consider the positive definite function

$$L_1 = e\left(X - K - K\ln\left(\frac{X}{K}\right)\right) + S + I$$

It is clear that $L_1 : R_+^3 \to R$ and $L_1(K,0,0) = 0$ with $L_1(X,S,I) > 0$; $\forall (X,S,I) \neq (K,0,0)$

in R_+^3. Since the derivative of L_1 with respect to time can be written as

$$\frac{dL_1}{dt} < -e\frac{r}{K}(X-K)^2 - [d_1 - ea_1(1-n)K]S - \left[d_2 - e\frac{a_2(1-n)K}{b}\right]I$$

Therefore, by using condition (11) we get $\frac{dL_1}{dt}$ is negative define in R_+^3. Thus according to Lyapunov second theorem E_1 is globally asymptotically stable. ■

Theorem (4): Assume that the positive equilibrium point $E_2 = (X^*, S^*, I^*)$ is locally asymptotically stable, then it is globally asymptotically stable in R_+^3 provided that

$$\frac{a_2(1-n)^2 I^*}{bR^*} < \frac{r}{K} \qquad (12a)$$

$$\left[\left(\frac{R^*}{b} - 1\right)c_1 + \frac{c_3}{S} + \frac{R^*}{b}\frac{c_2}{I}\right]^2 < 4\left(\frac{c_3 I^*}{SS^*}\right)\left(\frac{c_2 R^* S^*}{bI\,I^*}\right) \qquad (12b)$$

Proof: Consider the following positive definite

$$L_2 = \left(X - X^* - X^* \ln \frac{X}{X^*} \right) + \frac{1}{e}\left(S - S^* - S^* \ln \frac{S}{S^*} \right) + \frac{R^*}{eb}\left(I - I^* - I^* \ln \frac{I}{I^*} \right)$$

Clearly $L_2 : R_+^3 \to R$ and $L_2(X^*, S^*, I^*) = 0$ with $L_2(X,S,I) > 0$; $\forall (X,S,I) \neq (X^*, S^*, I^*)$ in R_+^3. Now, by differentiating L_2 with respect to time and then simplifying the resulting terms we obtain that:

$$\frac{dL_2}{dt} = \left(\frac{X - X^*}{X} \right)\frac{dX}{dt} + \frac{1}{e}\left(\frac{S - S^*}{S} \right)\frac{dS}{dt} + \frac{R^*}{eb}\left(\frac{I - I^*}{I} \right)\frac{dI}{dt}$$

$$= -\left[\frac{r}{K} - \frac{a_2(1-n)^2 I^*}{RR^*} \right](X - X^*)^2 - \frac{1}{e}\frac{c_3 I^*}{SS^*}(S - S^*)^2$$

$$+ \frac{1}{e}\left[\left(\frac{R^*}{b} - 1 \right)c_1 + \frac{c_3}{S} + \frac{R^*}{b}\frac{c_2}{I} \right](S - S^*)(I - I^*) - \frac{1}{e}\frac{c_2 R^* S^*}{bI I^*}(I - I^*)^2$$

Consequently by using conditions (12a)-(12b) we obtain

$$\frac{dL_2}{dt} \le -\left[\frac{r}{K} - \frac{a_2(1-n)^2 I^*}{bR^*} \right](X - X^*)^2 - \frac{1}{e}\left[\sqrt{\frac{c_3 I^*}{SS^*}}(S - S^*) - \sqrt{\frac{c_2 R^* S^*}{bI I^*}}(I - I^*) \right]^2$$

which is negative definite function. Therefore according to Lyapunov second theorem E_2 is a globally asymptotically stable. ■

It is well known that, the persistence of an ecological system means the coexistence of all the species for all positive time. Since the coexistence of all the species for all the positive time is satisfied mathematically if the solution of the system doesn't has omega limit set on the boundary planes, therefore the conditions of the persistence of the system (1) are established in the following theorem.

Theorem (5): Suppose that

$$\frac{e(1-n)K}{b + (1-n)K}\left[a_2(c_2 + d_1 - ea_1(1-n)K) + a_1(c_3 + d_2)(b + (1-n)K) \right] > (c_2 + d_1)d_2 + c_3 d_1 \quad (13)$$

Then system (1) is uniformly persistence.

Proof. Let p be any point in the positive octant and let $o(p)$ be the orbit through it. Let $\Omega(p)$ denotes the omega limit set of the orbit through the point p. Clearly $\Omega(p)$ is bounded due to the boundedness of the system (1). We claim that $E_0 \notin \Omega(p)$. If $E_0 \in \Omega(p)$ then according to the Butler-McGehe lemma [23], there is a point $q \in \Omega(p) \cap W^s(E_0)$, where $W^s(E_0)$ represents the stable manifold of E_0. Now since $o(q)$ lies in $\Omega(p)$ and $W^s(E_0)$ is the $SI-$plane, then the orbit through q, which denoted by $o(q)$, is unbounded orbit which leads to contradiction.

Now we claim that $E_1 \notin \Omega(p)$, otherwise $E_1 \in \Omega(p)$. Since E_1 is saddle point due to condition (13) with stable manifold represented by $X-$ axis, hence again by Butler-McGehe lemma there is a point $q \in \Omega(p) \cap W^s(E_1)$, where $W^s(E_1)$ is the stable manifold of E_1. Moreover since $o(q)$ lies in $\Omega(p)$ and

$W^s(E_1)$ is the $X-$axis then the orbit through q that denoted by $o(q)$ is unbounded orbit which leads to contradiction too.

Therefore $\Omega(p)$ doesn't intersect any of boundary planes of the R^3_+, then system (1) is persistent. In addition to that since system (1) is bounded system then according to theorem of Butler et al [24], system (1) becomes uniformly persistent. ■

4. Local bifurcation

It is clear that system (1) consisting of three first order nonlinear differential equations, which depends on parameters that characterize properties of the real worlds system being modeled. The bifurcation theory is concerned with changes in the qualitative behavior of the solution of system (1) as a control parameters varied. Therefore it is important to determine the set of parameters that control the system's behavior. In this section, the occurrence of local bifurcation near the equilibrium points of system (1) is investigated, with the help of Sotomoyor's theorem [25], in case of varying one parameter keeping the others fixed. It is well known that the existence of a non hyperbolic equilibrium point is a necessary but not sufficient condition for occurrence of bifurcation. Therefore the parameters that change the equilibrium points E_1 and E_2 from hyperbolic to non hyperbolic equilibrium points are considered as a candidate bifurcation parameters of system (1) as shown in the next theorems.

Consider first the general Jacobian matrix of system (1) that is given in Eq. (5), then it is easy to verify that

$$D^2F(V,V) = (u_{ij})_{3\times 1} \tag{14}$$

here

$$u_{11} = -2\frac{r}{K}v_1^{\,2} + 2\frac{a_2 b(1-n)^2 I}{(b+(1-n)X)^3}v_1^{\,2} - 2a_1(1-n)v_1v_2 - 2\frac{a_2 b(1-n)}{(b+(1-n)X)^2}v_1v_3$$

$$u_{21} = 2ea_1(1-n)v_1v_2 - 2c_1v_1v_3$$

$$u_{31} = -2\frac{ea_2 b(1-n)^2 I}{(b+(1-n)X)^3}v_1^{\,2} + 2\frac{ea_2 b(1-n)}{(b+(1-n)X)^2}v_1v_3 + 2c_1v_2v_3$$

with $V=(v_1,v_2,v_3)^T$ is any vector in R^3_+ and $F(X,S,I)=(f_1,f_2,f_3)^T$; $f_i\,(i=1,2,3)$ are given in the right hand sides of system (1).

Now, since D_0 is always positive in Eq.(6a), therefore the determinate of $J(E_0)$ can't be zero an hence E_0 is a hyperbolic equilibrium point which indicates to non existence of local bifurcation near E_0.

Theorem (6): Suppose that conditions (8a)-(8b) hold together with the following condition

$$\hat{D}_1\hat{D}_2(c_3+d_2) + \hat{D}_3c_2 > \hat{D}_2c_2d_2 + \hat{D}_1\hat{D}_3 \tag{15}$$

where $\hat{D}_1 = ea_1(1-n)K,\ \hat{D}_2 = b+(1-n)K,\ \hat{D}_3 = ea_2(1-n)K$. Then as the susceptible predator natural death rate parameter passes through the value $d_1 = d_1^*$, system (1) near the predator free equilibrium point E_1

undergoes transcritical bifurcation but neither saddle node nor pitchfork bifurcation can occur.

$$d_1^* = \frac{(\hat{D}_1(c_3+d_2) - c_2 d_2)\hat{D}_2 + \hat{D}_3(c_2 - \hat{D}_1)}{\hat{D}_2(c_3+d_2) - \hat{D}_3} \quad (16)$$

Proof: It is easy to verify that $D_1 = 0$, in Eq.(7b) at the value of d_1^*, which is positive under the conditions (8a), (8b) and (15). Therefore the Jacobian matrix of system (1) at E_1 and d_1^* becomes

$$\hat{J}(E_1, d_1^*) = (\hat{b}_{ij})_{3\times 3}$$

where $\hat{b}_{ij} = b_{ij}$ in Eq. (7a); for all $i, j = 1,2,3$ with $\hat{b}_{22} = -\frac{c_2 c_3 \hat{D}_2}{(c_3 + d_2)\hat{D}_2 + \hat{D}_1} < 0$.

Clearly $\det.[\hat{J}(E_1, d_1^*)] = 0$, and hence $\hat{J}(E_1, d_1^*)$ has zero eigenvalue, say $\hat{\lambda} = 0$. This means that E_1 becomes a non hyperbolic point at d_1^*.

Let $\hat{V} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)^T$ be the eigenvector of $\hat{J}(E_1, d_1^*)$ that corresponding $\hat{\lambda} = 0$, then $\hat{J}\hat{V} = 0$, which gives:

$$\hat{V} = (\hat{\alpha}_1 \hat{v}_3, \hat{\alpha}_2 \hat{v}_3, \hat{v}_3) \quad (17a)$$

where $\hat{v}_3$ be any nonzero real number and $\hat{\alpha}_1 = \frac{\hat{b}_{12}\hat{b}_{23} - \hat{b}_{13}\hat{b}_{22}}{\hat{b}_{11}\hat{b}_{22}} < 0$ under condition(8a) with $\hat{\alpha}_2 = -\frac{\hat{b}_{23}}{\hat{b}_{22}} > 0$ under condition(8a).

Let $\hat{\psi} = (\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3)^T$ be the eigenvector of $[\hat{J}(E_1, d_1^*)]^T$ that corresponding $\hat{\lambda} = 0$, then $\hat{J}^T \hat{\psi} = 0$ that gives that

$$\hat{\psi} = (0, \hat{\alpha}_3 \hat{\varphi}_3, \hat{\varphi}_3)^T \quad (17b)$$

where $\hat{\varphi}_3$ be any nonzero real number and $\hat{\alpha}_3 = -\frac{\hat{b}_{32}}{\hat{b}_{22}} > 0$ under condition (8a).

Now, by computing the derivative of $F(X,S,I)$ with respect to d_1, we obtain

$$\frac{\partial F(\Lambda, d_1)}{\partial d_1} = F_{d_1}(\Lambda, d_1) = \begin{pmatrix} 0 \\ -S \\ 0 \end{pmatrix} \quad , \quad \Lambda = (X,S,I) \quad (17c)$$

Thus $F_{d_1}(E_1, d_1^*) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Therefore $\hat{\psi}^T F(E_1, d_1^*) = 0$ and hence according to Sotomoyor's theorem saddle node bifurcation can't occur.

Further by determining the Jacobian to $F_{d_1}(\Lambda, d_1)$ given by (17c) we get

$$DF_{d_1}(\Lambda, d_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence $\hat{\psi}^T[DF(E_1, d_1^*)\hat{V}] = -\hat{\alpha}_2\hat{\alpha}_3\hat{v}_3\hat{\varphi}_3 \neq 0$

Now by substituting E_1 and d_1^* in Eq (14), then after doing some algebraic calculations we get that:

$$\hat{\psi}^T[D^2F(E_1, d_1^*)(\hat{V}, \hat{V})] = 2\hat{v}_3{}^2\hat{\varphi}_3\left[e(1-n)\hat{\alpha}_1\left(a_1\hat{\alpha}_2\hat{\alpha}_3 + \frac{a_2 b}{\hat{D}_2^2}\right) + c_1\hat{\alpha}_2(1-\hat{\alpha}_3)\right] \neq 0$$

Therefore system (1) undergoes transcritical bifurcation near E_1 at $d_1 = d_1^*$, but no pitchfork bifurcation can occur. ■

Theorem (7): Suppose that conditions (9b)-(9d) hold together with

$$\left(\frac{r}{K}\right)\frac{c_1 I^* + c_2}{e a_1 b + c_1 I^* + c_2} < \frac{a_2(1-n)^2 I^*}{R^{*2}} < \frac{r}{K} \tag{18}$$

Then as the infected predator death rate parameter passes through the value $d_2 = d_2^*$, system (1) near the positive equilibrium point E_2 undergoes a saddle bifurcation but neither transritical nor pitchfork bifurcation can occur.

$$d_2^* = \frac{a_{23}^*(a_{12}^* a_{31}^* - a_{11}^* a_{32}^*) + a_{13}^*(a_{21}^* a_{32}^* - a_{22}^* a_{31}^*)}{a_{11}^* a_{22}^* - a_{12}^* a_{21}^*} + \left(\frac{e a_2(1-n)X^*}{R^*} + c_1 S^* - c_3\right) \tag{19}$$

Proof: It is easy to verify that $A_3 = 0$, in Eq. (10b) at the value of d_2^*, which is positive under the conditions (9b)-(9a) and (18).

Therefore the Jacobian matrix of system (1) at E_2 and d_2^* becomes

$$J^*(E_2, d_2^*) = (a_{ij}^*)_{3\times3}$$

where $a_{ij}^* = a_{ij}$ in Eq. (10a) with $a_{33}^* = a_{33}(d_2^*) < 0$.

Since $\det.\lfloor J^*(E_2, d_2^*)\rfloor = -A_3 = 0$. Then $J^*(E_2, d_2^*)$ has a zero eigenvalue, namely $\lambda^* = 0$ and hence E_2 is a non hyperbolic equilibrium point at $d_2 = d_2^*$.

Let $V^* = (v_1^*, v_2^*, v_3^*)^T$ be the eigenvector of $J^*(E_2, d_2^*)$ that corresponding $\lambda^* = 0$. So $J^*V^* = 0$ gives that

$$V^* = (\alpha_2^* v_3^*, \alpha_1^* v_3^*, v_3^*) \tag{20a}$$

where $\alpha_1^* = \frac{a_{13}^* a_{21}^* - a_{11}^* a_{23}^*}{a_{11}^* a_{22}^* - a_{12}^* a_{21}^*} < 0$ under given conditions (9b), (9d) and (18), while conditions (9c), (9d) and (18) guarantee that $\alpha_2^* = -\frac{a_{12}^*(a_{13}^* a_{21}^* - a_{11}^* a_{23}^*) + a_{13}^*(a_{11}^* a_{22}^* - a_{12}^* a_{21}^*)}{a_{11}^*(a_{11}^* a_{22}^* - a_{12}^* a_{21}^*)} < 0$.

Let $\psi^* = (\varphi_1^*, \varphi_2^*, \varphi_3^*)^T$ representing the eigenvector of J^{*T} that corresponding the eigenvalue $\lambda^* = 0$, then, $J^{*T} \psi^* = 0$ gives that

$$\psi^* = (\sigma_2^* \varphi_3^*, \sigma_1^* \varphi_3^*, \varphi_3^*)^T \tag{20b}$$

where φ_3^* be any nonzero real number and $\sigma_1^* = -\frac{a_{11}^* a_{32}^* - a_{12}^* a_{31}^*}{a_{11}^* a_{22}^* - a_{12}^* a_{21}^*} < 0$ under the condition (18), while the conditions (9b) and (18) guarantee that $\sigma_2^* = -\frac{a_{21}^*(a_{12}^* a_{31}^* - a_{11}^* a_{32}^*) + a_{31}^*(a_{11}^* a_{22}^* - a_{12}^* a_{21}^*)}{a_{11}^*(a_{11}^* a_{22}^* - a_{12}^* a_{21}^*)} > 0$.

Now, since

$$\frac{\partial F(\Lambda, d_2)}{\partial d_2} = F_{d_2}(\Lambda, d_2) = \begin{pmatrix} 0 \\ 0 \\ -I \end{pmatrix} \Rightarrow F_{d_2}(E_2, d_2^*) = \begin{pmatrix} 0 \\ 0 \\ -I^* \end{pmatrix}$$

Moreover, Since

$$\hat{\psi}^T F_{d_2}(E_2, d_2^*) = -I^* \varphi_3^* \neq 0$$

Hence according to Sotomoyor's theorem, system (1) satisfied the first condition of saddle node bifurcation near E_2 when $d_2 = d_2^*$.

Finally, by substituting the value of E_2 and d_2^* in Eq. (14), and then doing some algebraic computations, it is observed that

$$\begin{aligned}\psi^{*T}[D^2 F(E_2, d_2^*)(V^*, V^*)] = -2 v_3^{*^2} \varphi_3^* \Bigg[& \frac{r}{K} \alpha_2^* \sigma_2^* - \frac{a_2 b (1-n)^2 I^*}{R^{*^3}} \alpha_2^{*2} \sigma_2^* \\ & + a_1 (1-n) \alpha_1^* \alpha_2^* \sigma_2^* + \frac{a_2 (1-n) b}{R^{*^2}} \alpha_2^* \sigma_2^* - e a_1 (1-n) \alpha_1^* \alpha_2^* \sigma_1^* + c_1 \alpha_1^* \sigma_1^* \\ & + \frac{e a_2 b (1-n)^2 I^*}{R^{*^3}} \alpha_2^{*2} - \frac{e a_2 b (1-n)}{R^{*^2}} \alpha_2^* - c_1 \alpha_1^* \Bigg] \neq 0\end{aligned}$$

Thus saddle node bifurcation occurs but neither transcitical nor pitchfork bifurcation can occur and hence the proof is complete. ■

5. Hopf-bifurcation

In this section, the possibility of occurrence of a simple Hopf bifurcation near the positive equilibrium point of system (1) is studied. It is well known that, system (1) undergoes a simple Hopf bifurcation near E_2 if and only if there exists a parameter say r, such that the following conditions hold.

1. The Jacobian matrix $J(E_2)$ of system (1) has a simple pair of complex conjugate eigenvalues $\lambda^*(r)=\xi_1(r)\pm i\xi_2(r)$, which become pure imaginary at $r=r^*$, while the third eigenvalue remain real and negative.
2. $\left.\frac{d\xi_1(r)}{dr}\right|_{r=r^*}\neq 0$, which is known as transversality condition

Consequently, in order to satisfying the above conditions we should have that

$$\Delta(r^*)=A_1A_2-A_3=0 \tag{21}$$

where A_1, A_2 and A_3 are given in Eq. (10b). Therefore the characteristic equation (10b) at the specific value of the bifurcation parameter that satisfies Eq. (21), say $r=r^*$, will be written as

$$P_3(\lambda)=(\lambda+A_1)(\lambda^2+A_2)=0 \tag{22}$$

which gives that

$$\lambda_1^*(r^*)=-A_1 \tag{23a}$$

$$\lambda_2^*(r^*)=i\sqrt{A_2}=i\xi_2(r^*) \tag{23b}$$

$$\lambda_3^*(r^*)=-i\sqrt{A_2}=-i\xi_2(r^*) \tag{23c}$$

Then the first condition of Hopf bifurcation accrued when $A_i>0; i=1,2,3$ and condition (21) are hold. Moreover to establish the conditions that guarantee the occurrence of the trasversality condition, the following is down.

It is well known that in the neighborhood of $r=r^*$ the complex eigenvalues can be written as $\lambda^*(r)=\xi_1(r)\pm i\xi_2(r)$. Substituting the value of $\lambda^*(r)$ in Eq. (22) and determine the derivative with respect to the bifurcation parameter r and then comparing the two sides of the resulting equation with equating their real and imaginary parts we obtain that

$$\begin{aligned} H_1(r)\xi_1'(r)-H_2(r)\xi_2'(r)&=-H_3(r)\\ H_2(r)\xi_1'(r)+H_1(r)\xi_2'(r)&=-H_4(r)\end{aligned} \tag{24}$$

here

$$H_1(r)=3\xi_1^2(r)-3\xi_2^2(r)+A_2(r)+2\xi_1(r)A_1(r)$$

$$H_2(r)=6\xi_1(r)\xi_2(r)+2\xi_2(r)A_1(r)$$

$$H_3(r)=\xi_1(r)A_2'(r)+\xi_1^2(r)A_1'(r)-\xi_2^2(r)A_1'(r)+(A_1(r)A_2(r))'$$

$$H_4(r)=2\xi_1(r)\xi_2(r)A_1'(r)+\xi_2(r)A_2'(r)$$

Thus by solving system (24), its obtain that

$$\xi_1'(r) = -\frac{H_1(r)H_3(r) + H_2(r)H_4(r)}{H_1^2(r) + H_2^2(r)}$$
$$\xi_2'(r) = \frac{-H_1(r)H_4(r) + H_2(r)H_3(r)}{H_1^2(r) + H_2^2(r)} \tag{25}$$

Clearly in order for the transversality condition holds, we should have

$$H_1(r^*)H_3(r^*) + H_2(r^*)H_4(r^*) \neq 0 \tag{26}$$

Consequently the Hopf bifurcation conditions of system (1) near E_2 are derived in the following theorem.

Theorem (8): Suppose that conditions (9a)-(9d) are satisfied together with

$$(c_1 I^* + c_2)S^* R^* > b(c_1 S^* - c_3)I^* \tag{27a}$$

$$M_3 < 0 \tag{27b}$$

where M_1, M_2 and M_3 are given in the proof. Then system (1) undergoes a Hopf bifurcation around the positive equilibrium point E_2 in the interior of R_+^3 when the intrinsic growth rate parameter r passes through the value.

$$r^* = \frac{M_2}{2M_1} + \frac{1}{2M_1}\sqrt{M_1^2 - 4M_1 M_3}$$

here M_1, M_2 and M_3 are given in proof.

Proof: Clearly according to the $J(E_2)$ conditions (9a)-(9d) guarantee that $A_i > 0,\ \forall i = 1,2,3.$ while condition (27a) guarantees that $a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} < 0$, otherwise $\Delta > 0$ always.

Now, straightforward computation shows that $\Delta = A_1 A_2 - A_3$ can be written as a function of parameter r in the form

$$\Delta(r) = M_1 r^2 - M_2 r + M_3 \tag{28}$$

where

$$M_1 = -\left(\frac{X^*}{K}\right)^2 (a_{22} + a_{33}) > 0$$

$$M_2 = \left(\frac{X^*}{K}\right)\left[-2(a_{22} + a_{33})\frac{a_2(1-n)^2}{R^*} X^* I^* + a_{12}a_{21} + a_{13}a_{31} - (a_{22} + a_{33})^2\right]$$

$$\begin{aligned} M_3 = &-(a_{22} + a_{33})\left[\frac{a_2(1-n)^2}{R^*} X^* I^*\right]^2 \\ &+ \left[a_{12}a_{21} + a_{13}a_{31} - (a_{22} + a_{33})^2\right]\left[\frac{a_2(1-n)^2}{R^*} X^* I^*\right] \\ &+ \big[a_{22}(a_{12}a_{21} + a_{23}a_{32}) - a_{22}a_{33}(a_{22} + a_{33}) \\ &\quad + a_{33}(a_{23}a_{32} + a_{13}a_{31}) + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}\big] \end{aligned}$$

Therefore due to condition (27b), $r = r^*$ represents the positive root of Eq. (28), that satisfies $\Delta(r^*) = A_1(r^*)A_2(r^*) - A_3(r^*) = 0$.

Consequently, the Jacobian matrix $J(E_2, r^*)$ has two pure imaginary eigenvalues with the third real and negatives given in Eq. (23a)-(23c).

Now in order to complete the proof we have to satisfy the transversality condition as given in Eq. (26). Thus by substituting the value of r^* in Eq. (26), we get after doing some calculations that

$$H_1(r^*) = -2A_2(r^*)$$

$$H_2(r^*) = -2(a_{11} + a_{22} + a_{33})\sqrt{A_2(r^*)}$$

$$H_3(r^*) = -\left(\frac{X^*}{K}\right)[a_{11}a_{22} - a_{12}a_{21} + a_{11}a_{33} - a_{13}a_{31}]$$

$$H_4(r^*) = -\left(\frac{X^*}{K}\right)(a_{22} + a_{33})\sqrt{A_2(r^*)}$$

Thus by substituting theses values in Eq. (26), its observed that

$$H_1(r^*)H_3(r^*) + H_2(r^*)H_4(r^*) = 2\left(\frac{X^*}{K}\right)A_2(r^*) \times \left(2a_{11}(a_{22} + a_{33}) + (a_{22} + a_{33})^2 - a_{12}a_{21} - a_{13}a_{31}\right) \neq 0$$

Therefore all the conditions of Hopf bifurcation near E_2 are satisfied when $r = r^*$, and hence the proof is complete. ■

6. Numerical simulation

In this section the dynamical behavior of system (1) is investigated numerically. Two objectives are assigned from such type of investigation, the first is verification of our obtained analytical results and the second is to specify the control set of parameters that characterize the behavior of the system. It is observed that for the following hypothetical biologically feasible set of parameters the solution of system (1) approaches asymptotically to the positive equilibrium point as shown in Fig. (1).

$$\begin{aligned} &r = 1, K = 100, a_1 = 1, n = 0.6, a_2 = 0.75, b = 4, e = 0.5, \\ &c_1 = 0.2, c_2 = 0.1, d_1 = 0.1, c_3 = 0.4, d_2 = 0.2 \end{aligned} \tag{29}$$

According to Fig. (1), system (1) has a globally asymptotically stable positive equilibrium point for the set of data (29), starting from three different initial points. This is verifying our obtained analytical result regarding to global stability and persistence of the system. Now in order to investigate the effect of varying each parameter on the system's behavior we start varying one parameter at a time from the set (29) and drawing the solution of system (1).

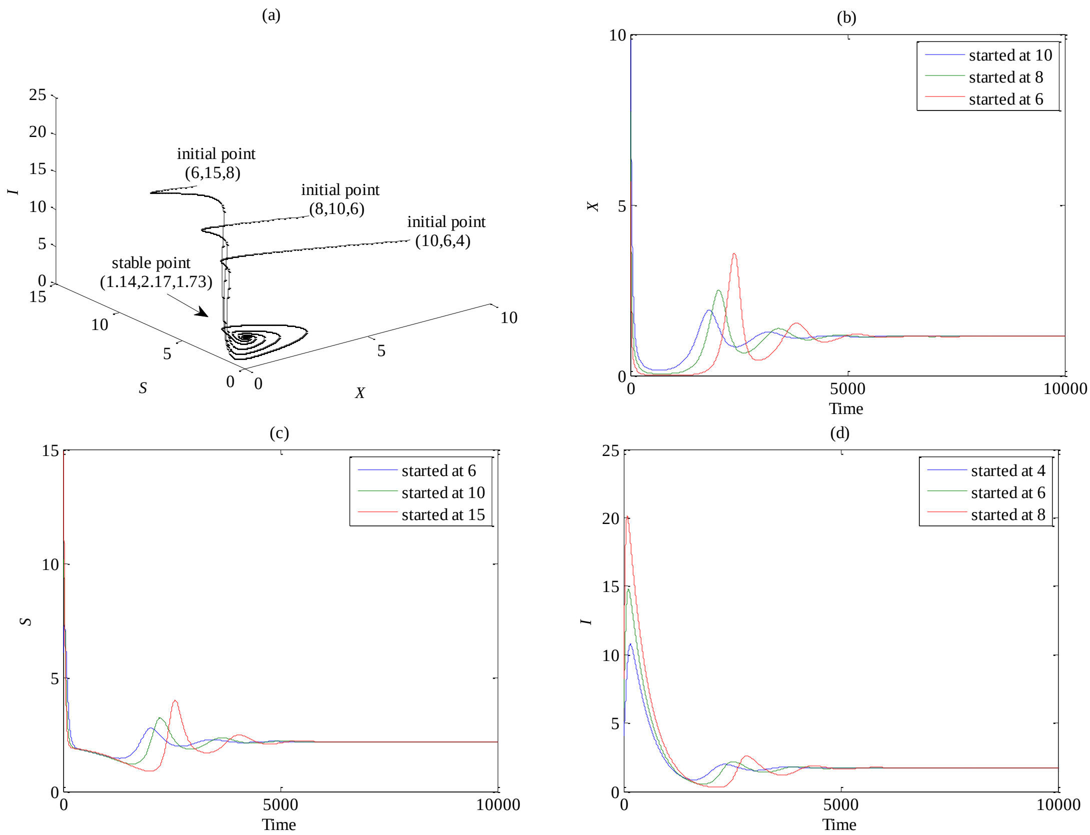

Fig. 1: The solution of system (1) approaches asymptotically to $E_2 = (1.14, 2.17, 1.73)$ for the data (29) starting from three different initial points. (a) 3D positive point attractor. (b) The trajectory of X as a function of time. (c) The trajectory of S as a function of time. (d) The trajectory of I as a function of time.

For varying the parameter r keeping the rest of parameters as in (29), its observed that, decreasing the parameter r has no effect on the dynamical behavior of the system while increasing the value of r in the range $r \geq 2.27$ destabilizes the positive equilibrium point and the solution of system (1) approaches asymptotically to periodic dynamics as shown in Fig. (2) for $r = 2.3$ and Fig. (3) for $r = 2.5$. Clearly the last two figures show the occurrence of Hopf bifurcation around the positive equilibrium point. In fact as the bifurcation parameter r increases the period size of limit cycle increases too.

Now increasing the parameter a_1 above the given value in (29) keeping the rest of other parameters fixed do not has effect on the solution of system (1) and the system still approaches to a positive equilibrium point. However decreasing the value of a_1 in the range $a_1 \leq 0.43$ destabilizes the positive equilibrium point and again Hobf bifurcation occurred as shown in Fig. (4).

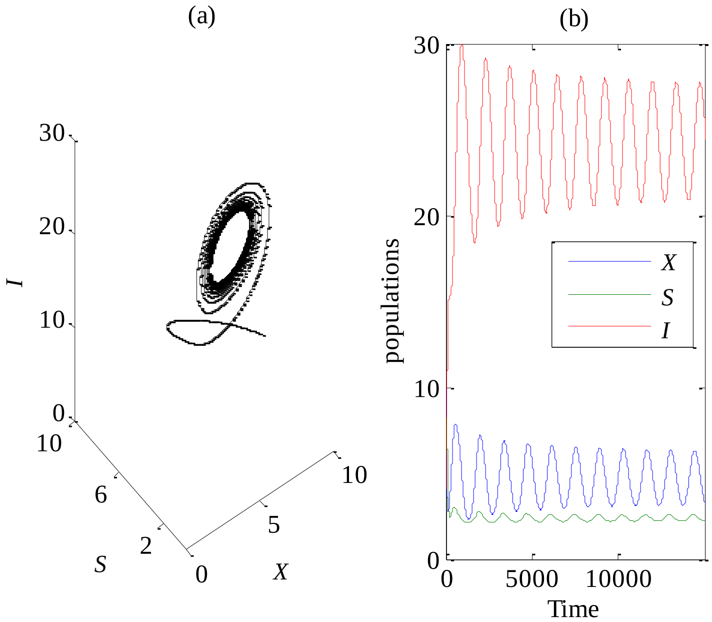

Fig. 2: The solution of system (1) for the data (29) with $r = 2.3$. (a) 3D periodic attractor. (b) Time series of the attractor in (a).

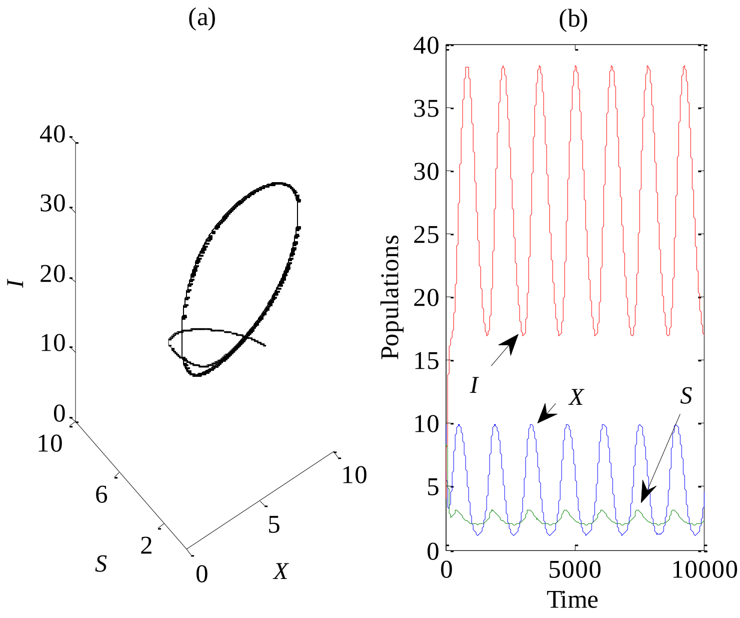

Fig. 3: The solution of system (1) for the data (29) with $r = 2.5$. (a) 3D periodic attractor. (b) Time series of the attractor in (a).

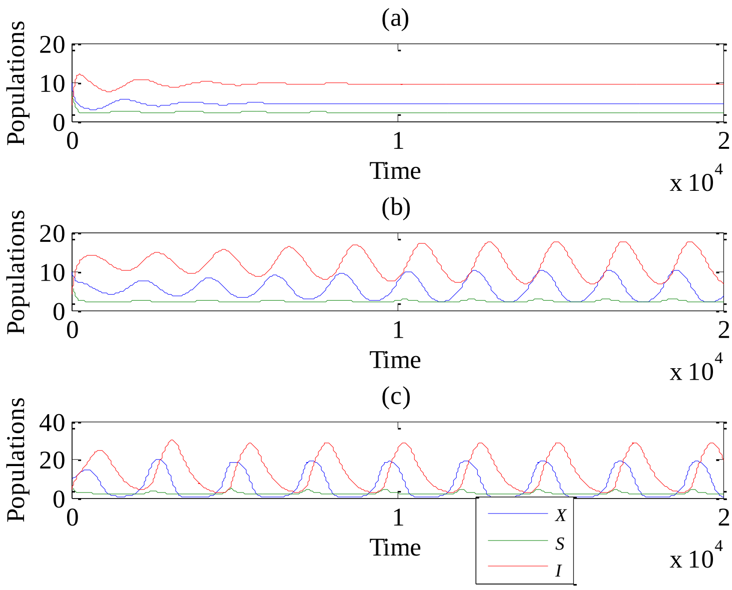

Fig. 4: The solution of system (1) for the data (29) with $a_1 = 0.5, 0.4, 0.3$ respectively. (a) The trajectory of system (1) approaches to positive point when $a_1 = 0.5$. (b) The trajectory of system (1) approaches to periodic attractor when $a_1 = 0.4$. (c) The trajectory of system (1) approaches to periodic attractor when $a_1 = 0.3$.

Note that, although the parameter a_1 do not used as a bifurcation parameter analytically, system (1) still sensitive to the changing in this parameter. This indicates to important of simulation study to specify the control set of parameters. In addition to that the forms of bifurcation parameters given in Eq. (16) and Eq. (19) depend on different parameters of the system so any change in those parameters my causes changing in the bifurcation parameter it self and then leads to changing in dynamical behavior of the system (1).

Further numerical analysis shows that, increasing the parameter n or decreasing the parameter e causes destabilizing of the positive equilibrium point and the solution of system (1) approaches asymptotically to the predator free equilibrium point as shown in Fig. (5). Clearly the Fig. (5), indicates to occurrence of bifurcation in system (1) as changing in the parameters n and e respectively. Moreover, it is observed that the parameters a_2 and c_1 have the same effect on the dynamical behavior of system (1) as that of the parameter r. While the parameter b has similar effect, as that of a_1, on the dynamics of system (1).

Finally for the same set of data (29), it is observed that changing the other parameters, one at a time, do not affect the behavior of the system. However these parameters may have clear effect on the dynamics of system (1) for other set of hypothetical data.

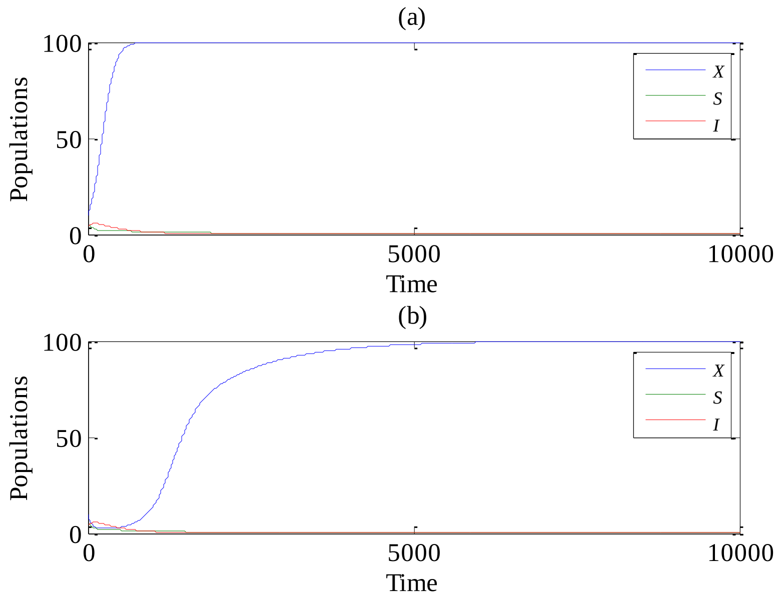

Fig. 5: The solution of system (1) for the data (29) with $n = 0.999$ and $e = 0.001$ respectively. (a) The trajectory of system (1) approaches $E_1 = (100,0,0)$ when $n = 0.999$. (b) The trajectory of system (1) approaches to $E_1 = (100,0,0)$ when $e = 0.001$.

6. Discussion and conclusion

In this paper, an eco-epidemiological model consisting of prey-predator system with infectious disease in predator has been proposed and analyzed. The effect of existence of prey's refuge on the system is included in the model. Two types of functional responses are used. The existence, uniqueness and boundedness of the solution of the system are studied. The local stability and global stability of the all possible equilibrium points are investigated. The persistence conditions of the system are derived analytically and shown numerically. Local bifurcation near the equilibrium points and the Hopf bifurcation around the positive equilibrium point are investigated. Finally, numerical simulations are carried out to specify the control parameters and confirm our obtained analytical results using biologically feasible set of parameters given in (29). Finally in the following we summarize the obtained numerical results.

1. The system (1) has only to types of dynamics approaches to an equilibrium point or approaches to a periodic dynamics.
2. Decreasing the intrinsic growth rate parameter of the prey don't change the dynamical behavior of system (1) and the solution still approaches asymptotically to the positive equilibrium point, however increasing this parameter causes destabilizing of positive equilibrium point and the solution approaches asymptotically to a stable limit cycle in the interior of positive octant, which indicate to occurrence of Hopf bifurcation. Consequently, system (1) still persists for all values of intrinsic growth. Similar behavior has been observed in case of varying the maximum attack rate of the infected predator and contact infection rate as that of intrinsic growth rate.
3. Increasing the attack rate of the susceptible predator don't change the dynamical behavior of system (1) and the solution still approaches asymptotically to the positive equilibrium point, however decreasing this parameter causes destabilizing of positive equilibrium point and the solution approaches asymptotically to a stable limit cycle in the interior of positive octant, which indicate to occurrence of

Hopf bifurcation too. Again, system (1) still persists for all values of attack rate of the susceptible predator. Similar behavior has been observed in case of varying the half saturation constant as that of intrinsic growth rate.

4. Decreasing the value of prey's refuge rate or increasing the value of conversion rate don't have any effect on the dynamical behavior of the system (1) and the solution still approaches to the positive equilibrium point. However, increasing the value of prey's refuge rate or decreasing the value of conversion rate lead to destabilizing the positive equilibrium point and the solution approach to the predator free equilibrium point, which means that system (1) losses its persistence . This is indicates to occurrence of local bifurcation in the system (1).
5. For the set of hypothetical data (29) it is observed that changing other parameters don't have affect on the dynamical behavior of the system (1) and the system still persists at the positive equilibrium point.

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