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On A New Method Of Bias Reduction: Alternative To Approximately Unbiased Ratio Estimators

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Abstract

This paper proposes a new method of bias reduction from order n^{-1} to order n^{-2} resulting in a new approximately unbiased ratio estimator. The efficiency of this estimator for the ratio of population means of two characters is compared with the existing six other Ratio estimators under a linear regression model.
Keywords: *Approximately Unbiased, Mean Square error, Efficiency, Regression model, Ratio Estimator,Bias,Auxiliary Variable*

1.0 Introduction

Over the years survey samplers Pascual(1961),Beale(1962),Tin(1965),Sahoo(1983),Sahoo(1987),Adewara(2006),Oshungade(1986), among many others have been interested in methods of improving the precision of the estimates of population parameters both at the selection and estimation stages by making use of auxiliary information. Ratio estimators are often employed by these samplers to estimating the population mean of the characteristic of interest of the population ratio.

Let y and x be real variates taking y_i and x_i $(1 \leq i \leq N)$ for i^{th} unit of a population of Size N with means μ_y and μ_x respectively. Suppose that a simple random sample of size n units is drawn without replacement from the population. A commonly employed estimator in this context is traditional or classical ratio estimator $r = \frac{\bar{y}}{\bar{x}}$ where $\bar{y}$ and $\bar{x}$ denote the sample means of y and x values, respectively.

2.0 Approximately Unbiased Ratio Type Estimators

Generally, the classical ratio estimator*(r)* is biased; therefore, many authors Tin(1965),Sahoo(1983),Sahoo(1987),Cochran(1977) and others have attempted to reduce this biasness in a situation where freedom from bias is important. Many authors Tin(1965),Sahoo(1983),Sahoo(1987), had also, worked on adjusting the classical ratio estimator by the term that is capable of reducing it from order n^{-1} to order n^{-2} to attain small or moderate gain in efficiency. Several ratio type estimators which satisfy this criterion are called approximately or almost unbiased ratio type estimators.

The following approximately unbiased ratio estimators in the literature Beale(1962),Tin(1965),Sahoo(1983),Sahoo(1987), and others are considered for comparison.

Pascual (1961) came up with the approximately unbiased ratio estimator:

$$\bar{Y}_p = \bar{r} + \frac{(N-1)(\bar{y} - \bar{r}\bar{x})}{(n-1)N} \tag{2.1}$$

This estimator has been shown to be efficient as that of combined bias ratio estimator in stratified sampling. Beale (1962) proposed another approximately unbiased estimator of order $0(n^{-2})$ as

$$\hat{R}_B = \hat{R}(1+\theta\frac{S_{xy}}{\bar{x}\bar{y}})/(1+\theta\frac{S_X^2}{\bar{x}^2}) = \hat{R}(1+\theta C_{xy})/(1+\theta C_X^2) \tag{2.2}$$

Tin (1965) derived another approximately unbiased ratio estimator closely related to that of Beale (1962) which was called

Modified ratio estimator, defined as

$$\hat{R}_T = \hat{R}\left[1+\theta\left(\frac{S_{xy}}{\bar{x}\bar{y}} - \frac{S_x^2}{\bar{x}^2}\right)\right] = \hat{R}(1+\theta(C_{XY} - C_X^2)) \tag{2.3}$$

, where

$$S_{xy} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}), \tag{2.4}$$

$$S_x^2/\bar{x}^2 = C_x^2 \tag{2.5}$$

$$S_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \tag{2.6}$$

$$S_{xy}/(\bar{y}\,\bar{x}) = C_{xy} \tag{2.7}$$

And $\theta = \frac{1}{n} - \frac{1}{N}$ (2.8)

$\hat{R}_B$ and $\hat{R}_T$ have the same variance to order $0(n^{-2})$.

Sahoo (1983) proposed another approximately unbiased ratio estimator termed almost unbiased ratio estimator, defined as

$$\hat{R}_S = \hat{R}/[1+\theta(C_x^2 - C_{xy})] \tag{2.9}$$

It is said to be more efficient than $\hat{R}_B$ and $\hat{R}_T$.

Sahoo (1987) further derived a class of almost unbiased ratio estimators, among which we have the following as its members:

$$\hat{R}_{s1} = \hat{R}\ (1+\theta C_{xy})(1-\theta C_x^2) \quad (2.10)$$

$$\hat{R}_{s2} = \hat{R}\left(\frac{1-\theta C_x^2}{1-\theta C_{xy}}\right) \quad (2.11)$$

$$\hat{R}_{s3} = \frac{\hat{R}}{[(1-\theta C_{xy})(1+\theta C_x^2)]} \quad (2.12)$$

3.0 The Proposed Method Of Bias Reduction: Alternative To Approximately Unbiased Ratio (AAUR) Estimators

The proposal of this estimator was based on the following standard regularity conditions:

Let $v = (C_{xy}, C^2_x)$ assume values in a bounded, closed convex subset, S, of two dimensional real spaces containing the point $V = (C_{xy}, C^2_x)$.

Let $f(v)$ be a function of v (which in particular may be a polynomial in (C_{xy}, C^2_x) satisfying the following conditions:

(a) The function $f(v)$ is continuous and bounded in S

(b) The first and second order partial derivatives of $f(v)$ exist and are continuous and bounded in S.

(c) After expansion under the given conditions, we get

$$f(v) = 1+ (C_{xy}-C^2_x) + 0(v-^2)$$

Then, we have the following theorem:

Theorem: If $f(v)$ is differentiable in C_{xy} and C^2_x and fulfils the above regularity conditions then $\hat{R}^* = \hat{R} f(v)$ is an asymptotically unbiased ratio estimator.

The unbiased ratio estimator proposed is $\hat{R}_A = \hat{R}\,[1-\theta(C_{xy} + C^2_x)]$ (3.1)

where S_{xy}, $S^2x/\bar{x}^2$, S^2_x, $S_{xy}/\bar{y}\,\bar{x}$ and θ are as defined in equations (2.4) to (2.8)

Conventionally, $R_n = \frac{\bar{y}_n}{\bar{x}_n}$ is taken as a biased estimate of $R_N = \frac{\bar{y}_N}{\bar{x}_N}$, since both $\bar{y}_n$ and $\bar{x}_n$ are unbiased estimates of $\bar{y}_N$ and $\bar{x}_N$ respectively.

Proof:

$$\text{Let } R_n = \frac{\bar{y}_n}{\bar{x}_n} \quad (3.2)$$

$$\Rightarrow \bar{y}_n = R_n\,\bar{x}_n \quad (.3.3)$$

$$R_N = \frac{\bar{y}_N}{\bar{x}_N} = \frac{E(\bar{y}_n)}{E(\bar{x}_n)} \quad (3.4)$$

Substituting (3.3) in (3.4), we have

$$R_N = \frac{E(R_n\bar{x}_n)}{E(\bar{x}_n)} \quad (3.5)$$

Now

Bias in $R_n = E(R_n) - R_N$ (3.6)

$$B(R_n) = E(R_n) - \frac{E(R_n\bar{x}_n)}{E(\bar{x}_n)} \quad (3.7)$$

That is

$B(R_n) = \{E(R_n)\, E(\bar{x}_n) - R_n\, \bar{x}_n\}/E(\bar{x}_n)$ (3.8)

since the negative covariance between R_n and $\bar{x}_n = E(R_n)\, E(\bar{x}_n) - R_n\, \bar{x}_n$ and $E(\bar{x}_n) = \bar{x}_N$, then

$$B(R_n) = -\frac{Cov(R_n, \bar{x}_n)}{\bar{x}_N} \quad (3.9)$$

Obtaining an upper bound to (3.8), we have

$$|Bias\ \text{in}\ R_n| \leq \frac{\sigma_{R_n}\sigma_{\overline{X_n}}}{\overline{x_n}} = \sigma_{R_n}\sqrt{\frac{N-n}{Nn}}\, C_X \quad (3.10)$$

Where $C_x = \frac{S_x}{\bar{x}_N}$ is the coefficient of variation of x and σ_{R_n} and $\sigma_{\bar{x}_n}$ are the standard errors of R_n and x_n respectively.

From (3.10) above, we can see that if n is sufficiently large, the bias in the ratio estimate R_n is negligible as compared to its standard deviation.

Let $y_i = \bar{y}_N + \Delta_i$ (3.11)

So that

$$\overline{y_n} = \bar{y}_N + \overline{\Delta}_n \quad (3.12)$$

Where

$$E(\overline{\Delta_n}) = 0 \text{ and } E(\overline{\Delta^2}_n) = (1\text{-f})\frac{S^2 y}{n} \quad (3.13)$$

Similarly,

Let $x_i = \overline{x_N} + \overline{\Delta_i^*}$ (3.14)

So that $\overline{x_n} = \overline{x_N} + \overline{\Delta}_{ni}{}^*$ (3.15)

where

$E(\overline{\Delta_n}{}^*) = 0$ and $E(\overline{\Delta_n}{}^{*2}) = (1-f)\dfrac{S^2{}_x}{n}$ (3.16)

To obtain the expected value of R_n, it is convenient to express it in terms of Δ_n and $\Delta_n{}^*$ we have

$$R_n = \frac{\overline{y_N}(1+\dfrac{\overline{\Delta_n}}{\overline{y_N}})}{\overline{x_N}(1+\dfrac{\overline{\Delta^*_n}}{\overline{x_N}})} \quad (3.17)$$

It is assumed that $\overline{x_n} \neq 0$, $\overline{x_N} \neq 0$, $\overline{y_N} \neq 0$ and $\left|\dfrac{\overline{\Delta_n}}{\overline{x_N}}\right| < 1$ so that we may expand $\left(1+\dfrac{\overline{\Delta_n{}^*}}{\overline{x_N}}\right)^{-1}$

Using Taylor's series expansion in powers of $\overline{\Delta}_n{}^*$, and expanding and multiplying out, we have

$$R_n = R_N\left\{1+\frac{\overline{\Delta_n}}{\overline{y_N}} - \frac{\overline{\Delta^*{}_n}}{\overline{x_N}} + \frac{\overline{\Delta^{*2}{}_n}}{\overline{x^2{}_N}} - \frac{\overline{\Delta_n}}{\overline{y_N}}\frac{\overline{\Delta^*{}_n}}{\overline{x_N}} + \frac{\overline{\Delta_n}}{\overline{y_N}}\frac{\overline{\Delta^{*2}}}{\overline{x_N}^2} - \frac{\overline{\Delta^{*3}{}_n}}{\overline{x^3{}_N}} + \frac{\overline{\Delta_n{}^{*4}}}{\overline{x^4{}_N}} - \frac{\overline{\Delta_n}}{\overline{y_N}}\frac{\overline{\Delta^{*3}{}_n}}{\overline{x_N}^3} + \ldots\right\}$$

(3.18)

Now taking the expectation of (3.18) term by term we obtain,

$$E(R_n) = R_N + R_N.E\left\{\frac{\overline{\Delta^{*2}{}_n}}{\overline{x^2{}_N}} - \frac{\overline{\Delta_n}}{\overline{y_N}}\frac{\overline{\Delta^*{}_n}}{\overline{x_N}} + \frac{\overline{\Delta_n}}{\overline{y_N}}\frac{\overline{\Delta^{*2}}}{\overline{x_N}^2} - \frac{\overline{\Delta^{*3}{}_n}}{\overline{x^3{}_N}} + \frac{\overline{\Delta^{*4}}}{\overline{x^4{}_N}} - \frac{\overline{\Delta_n}}{\overline{y_N}}\frac{\overline{\Delta^{*3}{}_n}}{\overline{x^3{}_N}} + \ldots\right\}\ldots$$

(3.19)

neglecting terms in $\overline{\Delta_n}$ and $\overline{\Delta_n{}^*}$ higher than the second term i.e. $\overline{\Delta_n}\,\overline{\Delta_n{}^{*2}}$, $\overline{\Delta^{*3}{}_n}$, $\overline{\Delta_n}\,\overline{\Delta_n{}^{*3}}$

This approximation gives

$$E(R_n) = R_N + R_N\left[\frac{E(\overline{\Delta^{*2}{}_n})}{\overline{x^2{}_N}} - E\frac{(\overline{\Delta_n}\,\overline{\Delta_n{}^*})}{\overline{y_N}\,\overline{x_N}}\right] \quad (3.20)$$

$$= R_N\left\{1+\frac{N-n}{Nn}\left(\frac{S^2{}_x}{\overline{x_N}^2} - \rho\frac{S_y}{\overline{y_N}}\frac{S_x}{\overline{x_N}}\right)\right\}$$

$$= R_N\left\{1+\frac{N-n}{Nn}(C_x{}^2 - \rho CxCy)\right\} \quad (3.21)$$

Where $C_x = \dfrac{S_x}{\overline{x_N}}$ and $C_y = \dfrac{S_y}{\overline{y_N}}$

$\therefore$ E [R_n] = R_N [1 + θ(C^2_x - C_{xy})] (3.22)

Now, following the approach of Tin (1965) in equation (2.3), by adjusting equation (3.22) for bias and by subtracting an estimate of the bias from E [R_n] ,

we obtain the estimator:

$\hat{R}_A$ = E [R_n] -2R_N θC^2_x (3.23)

$\hat{R}_A$ = $\hat{R}$ [1 + θ(C^2_x - C_{xy})] -2R_N θC^2_x

$\therefore$ $\hat{R}_A$ = $\hat{R}$ [1- θ (C_{XY} +C^2_X)] (3.24)

4.0 Derivations Of The Mean Square Error For The Alternative Unbiased Ratio Type Estimator Under The Regression Model:

$y_i = \beta x_i + e_i$, , i = 1,2,…,N. (4.1)

where β is unknown real constant and e_i's are random variables with the following conditional expectations:

E (e_i/x_i) = 0 (4.2)

E (e^2_i /x_i) = δx^t_i , (4.3)

With

δ > 0,

0 ≤ **t** ≤ **2**. (4.4)

and E ($e_i e_j$ / $x_i x_j$)= 0 , i ≠ j.

It is also assumed that x has gamma distribution with parameter k as often encountered in real life survey situations.

Now, assuming (4.1), we have

μ_x = k; μ_y = βk ; R =β; $\bar{y} = \beta \bar{x} + \bar{e}$ and $S_{xy} = \beta s^2_x + s_{ex}$ (4.5)

Where

$$\bar{e} = \frac{\sum e_i}{n} \; ; \quad \text{and} \quad s_{ex} = \frac{\sum e_i (x_i - x)}{n-1} \qquad (4.6)$$

Let E (./x) and E_x (.) denote respectively the expectation operators for given x and with respect to the distribution of x, so that E (.) = E_x (E (. /x)) denoting the mean square error for R_i as MSE (R_i) under this model , we have

MSE (R_i) = E (R_i- β)^2 = E_x[E (R_i- β)^2 /X_i] i = 1,2, …,8. (4.7)

Let $Z_i = x^t_i$ so that we have

$$E(z) = \bar{Z} = \frac{\Gamma(k+t)}{\Gamma(k)} \qquad (4.8)$$

$$E(e_2 / x) = \frac{\delta}{n} \bar{Z} \qquad (4.9)$$

$$E(\bar{e}\, S_{ex} / x) = \frac{\delta}{n} S_{zx} \quad (4.10)$$

and

$$E(S^{2}_{ex} / x) = \frac{\delta}{n} q. \quad (4.11)$$

Where

$$\bar{Z} = \frac{\sum Z_i}{n} \; ; S_{zx} = \frac{\sum Z_i (x_i - \bar{x})}{n-1} \text{ and } q = \frac{n\sum Z_i (x_i - \bar{x})^2}{(n-1)^2}. \quad (4.12)$$

Considering (1.13) and the regression model

$$\bar{y} = \beta\, \bar{x} + \bar{e} \quad (4.13)$$

We have

$$\beta = \frac{(\bar{y} - \bar{e})}{\bar{x}} \quad (4.14)$$

and

$$S_{xy} = \beta\, s^{2}_{x} + s_{ex}$$

$$S_{xy} = [\frac{(\bar{y} - \bar{e})}{\bar{x}}]\, s^{2}_{x} + s_{ex} \quad (4.15)$$

Now, substituting directly into the regression model and expanding under the assumptions and approximations Tin (1965), we obtain,

$$MSE(R_i) = E(R_i - \beta)^2 = E_x[E(R_i - \beta)^2 / x] \quad (4.16)$$

For $R_i = \hat{R}_A$, we have,

$$(\hat{R}_A - \beta)^2 = \{R(1 - \theta(C_{xy} - C^2_X)) - \beta\}^2 = \left\{\frac{\bar{y}}{\bar{x}}(1 - \theta\frac{Sxy}{\bar{\bar{xy}}} - \theta\frac{S^2_X}{\bar{X}^2}) - [\frac{\bar{y}}{\bar{x}} - \frac{\bar{e}}{\bar{x}}]\right\}^2$$

$$= \left\{\frac{\bar{y}}{\bar{x}} - \theta\frac{Sxy}{\bar{x}^2} - \theta\bar{y}\frac{S^2_X}{\bar{x}^3} - \frac{\bar{y}}{\bar{x}} + \frac{\bar{e}}{\bar{x}}\right\}^2 = \left\{\theta\frac{Sxy}{\bar{x}^2} - \theta\bar{y}\frac{S^2_X}{\bar{x}^3} + \frac{\bar{e}}{\bar{x}}\right\}^2 \quad (4.17)$$

Substituting for S_{xy}, we obtain,

$$(\hat{R}_A - \beta)^2 = \left\{\frac{\bar{e}}{\bar{x}} - \frac{\theta}{\bar{x}^2}[(\frac{\bar{y}}{\bar{x}} - \frac{\bar{e}}{\bar{x}})s^2_x + s_{ex}] - \theta\bar{y}\frac{S^2_x}{\bar{x}^3}\right\}^2$$

$$=\left\{\frac{\bar{e}}{x}-\theta\frac{\bar{y}}{x^3}s_x^2+\theta\frac{\bar{e}}{x^3}s_x^2-\frac{\theta}{x^2}s_{ex}-\theta\bar{y}\frac{S_x^2}{x^3}\right\}^2 =$$

$$\left\{\frac{\bar{e}}{x}+\theta\frac{\bar{e}}{x^3}s_x^2-\frac{\theta}{x^2}s_{ex}-2\theta\bar{y}\frac{S_x^2}{x^3}\right\}^2$$

$$=\left\{\frac{\bar{e}}{x}[1+\theta\frac{s_x^2}{x^2}]-\frac{\theta}{x^2}[s_{ex}+2\bar{y}\frac{S_x^2}{x}]\right\}^2 \qquad (4.18)$$

Recall as N tends to infinity, θ becomes $\frac{1}{n}$, then

$$(\hat{R}_A-\beta)^2=\left\{\frac{\bar{e}}{x}[1+\frac{s_x^2}{nx^2}]-\frac{1}{nx^2}[s_{ex}+2\bar{y}\frac{S_x^2}{x}]\right\}^2=\left\{\frac{\bar{e}}{x}+\frac{\bar{e}}{nx^3}s_x^2-\frac{1}{nx^2}s_{ex}-2\bar{y}\frac{S_x^2}{nx^3}\right\}^2$$

$$=\frac{\bar{e^2}}{x^2}+\bar{e^2}\frac{s_x^2}{nx^4}-\bar{e}\frac{s_{ex}}{nx^3}+\bar{e^2}\frac{s_x^2}{nx^4}+\bar{e^2}\frac{(s_x^2)^2}{n^2x^6}-\bar{e}s_x^2\frac{s_{ex}}{n^2x^5}-\bar{e}\frac{s_{ex}}{nx^3}-\bar{e}s_x^2\frac{s_{ex}}{n^2x^5}+\frac{s_{ex}^2}{n^2x^4}$$ + terms involving y's.

$$=\frac{\bar{e^2}}{x^2}+2\,\bar{e^2}\frac{s_x^2}{nx^4}-2\,\bar{e}\frac{s_{ex}}{nx^3}+\bar{e^2}\frac{(s_x^2)^2}{n^2x^6}-2\,\bar{e}s_x^2\frac{s_{ex}}{n^2x^5}+\frac{s_{ex}^2}{n^2x^4}$$ + terms involving y's.

$$=\frac{\bar{e^2}}{x^2}-\frac{2}{n}\left[\bar{e}\frac{s_{ex}}{x^3}-\bar{e^2}\frac{s_x^2}{x^4}\right]+\frac{1}{n^2}\left[\frac{s_{ex}^2}{x^4}-2\,\bar{e}s_x^2\frac{s_{ex}}{x^5}+\bar{e^2}\frac{(s_x^2)^2}{x^6}\right]$$ + terms involving y's (4.19)

Substituting the (4.9), (4.10) and (4.11) in (4.19) we obtain,

$$E[((\hat{R}_A-\beta)^2/X]=\frac{\delta\bar{z}}{nx^2}-\frac{2\delta}{n^2}\left[\frac{s_{zx}}{x^3}-\bar{z}\frac{s_x^2}{x^4}\right]+\frac{\delta}{n^3}\left[\frac{q}{x^4}-\frac{2s_{zx}}{x^5}s_x^2+\bar{z}\frac{(s_x^2)^2}{x^6}\right] \qquad (4.20)$$

Applying the well known Taylor's series expansion method, we have,

$$E_X(\frac{\bar{z}}{x^2})=\frac{Z}{k^2}\left(1+\frac{3}{nk}-\frac{2t}{nk}+\frac{7}{n^2k^2}-\frac{9t}{n^2k^2}+\frac{3t^2}{n^2k^2}\right)+o(n^{-3}) \qquad (4.21)$$

$$E_X(\frac{s_{zx}}{x^3}-\bar{z}\frac{s_x^2}{x^4})=\frac{Z}{k^2}\left(-\frac{1}{k}+\frac{t}{k}-\frac{2}{nk^2}+\frac{6t}{nk^2}-\frac{4t^2}{nk^2}\right)+o(n^{-2}) \qquad (4.22)$$

$$E_x(\frac{q}{x^4})=\frac{Z}{k^4}\left(k+t+t^2\right)+o(n^{-1}) \qquad (4.23)$$

$$E_X(\bar{z}\frac{(s_x^2)^2}{x^6})=\frac{Z}{k^4}+o(n^{-1}) \qquad (4.24)$$

and

$$E_x\left(\frac{s_{zx}}{\bar{x}^5}s_x^2\right)=\frac{Z}{k^4}t+o(n^{-1}) \quad (4.25)$$

Now, the Mean Squared Error is given by:

$$E[(\hat{R}_A-\beta)^2/X] = \frac{\delta\bar{z}}{n\bar{x}^2} - \frac{2\delta}{n^2}\left[\frac{s_{zx}}{\bar{x}^3}-\bar{z}\frac{s_x^2}{\bar{x}^4}\right] + \frac{\delta}{n^3}\left[\frac{q}{\bar{x}^4}-\frac{2s_{zx}}{\bar{x}^5}s_x^2+\bar{z}\frac{(s_x^2)^2}{\bar{x}^6}\right]$$

$$=\frac{\delta Z}{nk^2}\left(1+\frac{3}{nk}-\frac{2t}{nk}+\frac{7}{n^2k^2}-\frac{9t}{n^2k^2}+\frac{3t^2}{n^2k^2}\right)-\frac{2\delta Z}{n^2k^2}\left(-\frac{1}{k}+\frac{t}{k}-\frac{2}{nk^2}+\frac{6t}{nk^2}-\frac{4t^2}{nk^2}\right)$$

$$+\frac{\delta Z}{n^3k^4}\left(k+t+t^2\right)\frac{2\delta Zt}{n^3k^4}+\frac{\delta Z}{n^3k^4} = \frac{\delta Z}{n^3k^4}\left[n^2k^2+5nk-4nkt+12-22t+12t^2+k\right] \quad (4.26)$$

Thus,

$$\text{MSE}(\hat{R}_A)=\frac{\delta Z}{n^3k^4}\left\{nk(nk+1)+4nk(1-t)+12(1+t^2)-22t+k\right\} \quad (4.27)$$

Similarly, the mean square errors of all the estimators are shown below:

$$\text{MSE}(\hat{R}_B)=\frac{\delta Z}{n^3k^4}\left\{nk(nk+1)+k+4(1+t^2)+2\right\} \quad (4.28)$$

$$\text{MSE}(\hat{R}_T)=\frac{\delta Z}{n^3k^4}\left\{nk(nk+1)+k+4(1+t^2)+2t\right\} \quad (4.29)$$

$$\text{MSE}(\hat{R}_S)=\frac{\delta Z}{n^3k^4}\left\{nk(nk+1)+k+4(1+t^2)+2t\right\} \quad (4.30)$$

$$\text{MSE}(\hat{R}_{S1})=\frac{\delta Z}{n^3k^4}\left\{nk(nk+1)+k+4(1+t^2)\right\} \quad (4.31)$$

$$\text{MSE}(\hat{R}_{S2})=\frac{\delta Z}{n^3k^4}\left\{nk(nk+1)+k+4(1+t^2)+2(2t-1)\right\} \quad (4.32)$$

$$\text{MSE}(\hat{R}_{S3})=\frac{\delta Z}{n^3k^4}\left\{nk(nk+1)+k+4(1+t^2)+2(2t+1)\right\} \quad (4.33)$$

5.0 Condition Under Which $\hat{R}_A$ Is Better Than The Existing Estimators

We intend to establish the conditions under which our proposed estimator is better than the existing estimators Beale(1962),Tin(1965),Sahoo(1983),Sahoo(1987). In comparison of two estimators e_1 and e_2, when $V(e_1) < V(e_2)$ or MSE (e_1) < MSE (e_2), then, e_1 is better than e_2. Hence, in order to establish the conditions when $\hat{R}_A$ is better than other existing estimators Pascual(1961),Beale(1962),Tin(1965),Sahoo(1983),Sahoo(1987), we compared the mean square errors of our alternative estimator with the existing ones.

Past researchers Dalabehera and Sahoo(1994), Dalabehera and Sahoo(1995) and Sahoo et al(2006) have shown that $\hat{R}_{S1}$ is better than others; therefore we compared the mean square errors of $\hat{R}_A$ and $\hat{R}_{s1}$.

In our comparison it was discovered that MSE ($\hat{R}_A$) < MSE ($\hat{R}_{s1}$) whenever, $1 < t \leq 2.0$ and $n\,k > 8$

6.0 Empirical Investigations

In this section, an empirical study is carried out using a Monte Carlo technique to compare the performance of our alternative unbiased ratio estimator along side with the classical one. We shall be considering the usual model for ratio estimator $y_i = \beta x_i + e_i$, i = 1,2,3,…,N.

This work shall be viewed from the angle of real life situation which is always encountered in sampling practice by assuming that the auxiliary variable x follows a gamma (i.e. skewed population).

The approximately unbiased ratio estimators shall be compared under the following assumed model:

the regression of y on x is linear i.e, $y = \alpha + \beta x + e$ with x having a gamma distribution with parameter(2, 1) ,that is, $x_i \sim G(2 , 1)$and e having a gamma distribution with parameter (0.25, 1), that is, $e_i \sim G(0.25x_i , 1.0)$. Under varying values of intercept (α) = 0, 0.5, 1.2.

We considered the simulation of Variance, Bias, mean square error (MSE) and efficiency of the proposed alternative approximately unbiased ratio estimator ($\hat{R}_A$) along side with that **of** ($\hat{R}_{s1}$) which has been established to be the best in previous studies [9-11]

We shall consider the values t = 1, 1.5. 2.0 For n = 20, 40, 100, 200 and k = 2 under the regression model.

(i) For t = 1, we have,

$$y_i = 0.25x_i + e_i ; \quad e_i \sim G(0.25x_i , 1.0); \quad x_i \sim G(2 , 1).$$

(ii) For t = 1.5, we have,

$$y_i = 0.25x_i + e_i\sqrt{x_i} ; e_i \sim G(0.25\sqrt{x_i} , 1.0); \quad x_i \sim G(2 , 1).$$

(iii) For t = 2.0, we have,

$$y_i = 0.25x_i + e_i x_i; \quad e_i \sim G(0.25 , 1.0); \quad x_i \sim G(2 , 1).$$

Table 4.1: Simulations for t = 1

EST	N	Variance	Bias	MSE
R_{S1}	20	0.02229141	0.4957136	0.26802338225
R_A	20	0.02035064	0.4512071	0.2239384870904
R_{S1}	40	0.01163586	0.4911259	0.25284050965081
R_A	40	0.1121222	0.4682516	0.33138176090256
R_{S1}	100	0.00477786	0.5061191	0.2609344033848
R_A	100	0.00471523	0.4985879	0.25330512402641
R_{S1}	200	0.001758906	0.5378979	0.2910930568244
R_A	200	0.001743158	0.534885	0.287845121225

Table 4.2: Simulations for t = 1.5

EST	N	Variance	Bias	MSE
R_{S1}	20	0.05887116	0.5420939	0.35273695641721
R_A	20	0.04969045	0.4892565	0.289070690225
R_{S1}	40	0.01163586	0.4911259	0.25284050965081
R_A	40	0.01121222	0.4682516	0.2304717809026
R_{S1}	100	0.00477786	0.5061191	0.26093432938481
R_A	100	0.00471523	0.4985879	0.2533051240264
R_{S1}	200	0.003271526	0.80595569	0.6528361002433761
R_A	200	0.003247954	0.8021401	0.646676694028

Table 4.3: Simulations for t = 2.0

EST	N	Variance	Bias	MSE
R_{S1}	20	0.01470531	0.3578919	0.14279192208561
R_A	20	0.0151919	0.3350565	0.1274547581923
R_{S1}	40	0.00531964	0.3857655	0.15413466099025
R_A	40	0.005348987	0.3753843	0.1462623596865
R_{S1}	100	0.002499927	0.3738429	0.14225844088041
R_A	100	0.002493612	0.3699303	0.1393420388581
R_{S1}	200	0.0006446923	0.364413	0.133441526869
R_A	200	0.0006427695	0.3626546	0.1321611284012

Conclusion: The simulations results in this study have confirmed that $\hat{R}_A$ estimator is better than the existing ones in terms of the bias and MSE whenever $1 \leq t \leq 2.0$ and n k > 8.

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