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Collocation Orthonormal Berntein Polynomials method for Solving Integral Equations.

Suha. N. Shihab; Asmaa. A. A.; Mayada. N.Mohammed Ali

University of Technology Applied Science Department

Abstract:

In this paper, we use a combination of Orthonormal Bernstein functions on the interval $[0,1]$ for degree $m = 5$,and 6 to produce anew approach implementing Bernstein Operational matrix of derivative as a method for the numerical solution of linear Fredholm integral equations of the second kind and Volterra integral equations. The method converges rapidly to the exact solution and gives very accurate results even by low value of m. Illustrative examples are included to demonstrate the validity and efficiency of the technique and convergence of method to the exact solution.

Keywords: Bernstein polynomials, Operational Matrix of Derivative, Linear Fredholm Integral Equations of the Second Kind and Volterra Integral Equations.

1. Introduction:

In the Survey of solutions of integral equations, a large number of analytical but afew approximate methods for solving numerically various classes of integra equations [1]. Orthogonal functions and polynomial series have received considerable attention in dealing with various problems of dynamical Systems. The main Characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem [2,14]. While in recent years interest in the solution of integral and differential equations, such as Fredholm, Volterra, and integro-differential equations [3]. The general form of Fredholm, Volterra integral equations respectively are

1- Freholm integral equation: (FIE)

$$u(x) = g(x) + \int_0^1 k(x,t)u(t)dt \qquad x \in [0,1] \qquad (1)$$

2- Volterra integral equation: (VIE)

$$u(x) = f(x) + \int_0^x k(x,t)u(t)dt \qquad x \in [0,1] \qquad (2)$$

Integral equations are widely used for solving many problems in mathematical physics and engineering. In recent years, many different basic functions have been used to estimate the solution of integral equations, such as Block-Pulse functions [4,5], Hybrid Legendre and Block-Pulse functions. Bernstein polynomials play a promineut role in various areas of mathematics. These polynomials, have been frequently used in the solution of integral equations,differential equations and approximation theory,see [6]. Recently the various operational matrices of the polynomials have been developed to cover the numerical solution of differential, integral and integro-differential equations. In [7] the operational matrices of Bernstein polynomials are introduced. Doha [8] has drived the shifted Jacobi operational matrix of fractional derivatives which is applied together with the Spectral Tau method for the numerical solution of dynamical systems.Yousefi et al.in [9], [10] and [11] have presented Legendre wavelets and Berntein operational matrices and used them to solve miscellaneous systems. Lakestani et al. [12] constracted the operational matrix of fractional derivatives using B-Spline functions. Another motivation is concerned with the direct solution techniques for solving the Fredholm and Volterra integral equations respectively on the interval [0,1] using the method based on the derivatives of orthonormal (B-polynomials) sense for m=5 and 6. Finally, the accuracy of the proposed algorithm is demonstrated by test problems.

2-Bernstein Polynomials (B-Polynomials):

The Bernstein polynomias (B-Polynomials) [13], are some usfel polynomials defined on [0,1]. The Bernstein Polynamials of degree m form a basis for the power polynomials of degree m. we can mentioned, B-Polynomials are aset of Polynomials

$$B_{k,m}(x) = \binom{m}{k} x^k (1-x)^{m-k} \qquad , 0 \leq k \leq m, \qquad (3)$$

Note that each of these m+1 Polynomials having degree m is normalization, i.e $\sum_{k=0}^{m} B_{k,m}(x) = 1$, has one root, each of multiplicity k and m-k, at x=0 and x=1 respectively,also $B_{k,m}(x)$ in which $k \notin \{0, m\}$has asingle unique local maximum of $k_m^{k-m}(m-k)^{m-k}\binom{m}{k}$, it can provide flexibility applicable to impose boundary conditions at the end points of the interval. First derivatives of the generalized Bernstein basis polynomials.

$$\frac{d}{dx} B_{k,m}(x) = m\left[B_{k-1,m-1}(x) + B_{k,m-1}(x)\right] \qquad (4)$$

In this paper, we use $\Psi_m(x)$ notation to show

$$\Psi_m(x) = [B_{0m}(x),\ B_{1m}(x), \dots, B_{mm}(x)]^T \qquad (5)$$

where we can have

$$\Psi_m(x) = A_m(x)\Delta_m(x) \qquad (6)$$

that A is the matrix and $(k+1)^{th}$ row of A is

$A_{k+1} = [0,0, \dots^{k\ times}, 0, s\_(0,k,m), s\_(1,k,m), \dots, s\_(m,k,m)\ ]$

$=\left[0,0, \dots^{k\ times}, 0, (-1)^0 \binom{m}{k}\binom{m-k}{0}, (-1)^1 \binom{m}{k}\binom{m-k}{1}, \dots, (-1)^{m-k} \binom{m}{k}\binom{m-k}{m-k}\right]$ (7)

where

$$S_{i,k,m} = (-1)^i \binom{m}{k}\binom{m-k}{i}$$

and $\Delta_n(x) = \begin{bmatrix} x^0 \\ x^1 \\ \vdots \\ x^m \end{bmatrix}$ (8)

using MATHEMATICA code, the first six (B-Polynomials) of degree five over the interval [0,1], are given

$B_{05}(x) = (1-x)^5$

$B_{15}(x) = 5x(1-x)^4$

$B_{25}(x) = 10x^2(1-x)^3$

$B_{35}(x) = 10x^3(1-x)^2$

$B_{45}(x) = 5x^4(1-x)$

$B_{55}(x) = x^5$

and the first seven (B-Polynomials) of degree six over [0,1] are given

$B_{06}(x) = (1-x)^6$

$B_{16}(x) = 6x(1-x)^5$

$B_{26}(x) = 15x^2(1-x)^4$

$B_{36}(x) = 20x^3(1-x)^3$

$B_{46}(x) = 15x^4(1-x)^2$

$B_{56}(x) = 6x^5(1-x)$

$B_{66}(x) = x^6$

3- (B-Polynomials) Approximation:

A function u(x) equation, square integrable in (0,1), many be expressed in terms of Bernstein basis [7]. In practice, only the first (m+1)-terms Bernstein polynomials are considered. Hence if we write

$$u(x) \approx \sum_{k=0}^{m} c_k B_{km}(x) = C^T \varnothing(x) \qquad (9)$$

where $\varnothing^T(x) = [B_{0m}(x), B_{1m}(x), \ldots, B_{mm}(x)]$ and

$C^T = [c_0, c_1, \ldots, c_m]$ can be calculated by:

$$C^T = \left(\int_0^1 u(x)\emptyset^T(x)dx\right) Q^{-1}, \qquad (10)$$

where Q is an $(m+1)\times(m+1)$ matrix and is said dual matrix of $\varnothing(x)$

$$Q(x) = (\varnothing(x), \varnothing(x)) = \int_0^1 \emptyset(x)\,\emptyset^T(x)dx$$

$$= \int_0^1 \left(A\Delta_n(x)\right)\left(A\Delta_n(x)\right)^T dx$$

$$= A\left[\int_0^1 \Delta_n(x)\Delta_n^T(x)dx\right]A^T$$

$$= AHA^T, \qquad (11)$$

A is defined by eqs.(7) and H is a Hilbert matrix

$$H = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{m+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{m+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m+1} & \frac{1}{m+2} & \cdots & \frac{1}{2m+1} \end{bmatrix} \text{ and}$$

$$Q = \begin{bmatrix} Q_1 & 0 & \ldots & 0 \\ 0 & Q_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & Q_m \end{bmatrix}$$

The elements of the dual matrix Q, are given explicity by

$$(Q_m)_{k+1,i+1} = \int_0^1 B_{km}(x)B_{im}(x)dx$$

$$= \binom{m}{k}\binom{m}{i}\int_0^1 (1-x)^{2m-(k+i)}x^{k+i}dx \qquad (12)$$

where $k, i = 0,1,\ldots,m$

4- The Derivative for Orthonormal (B-Polynomials):

The representation of the orthonormal Bernstein Polynomials, denoted by $b_{i5}(x), b_{i6}(x)$ here, was discovered by analyzing the resulting orthonormal polynomials after applying the Gram-Schmidt process on sets of Bernstein polynomials of degree five and six.
We get the following sets of orthonormal polynomials [10],[11].

$$b_{05}(x) = \sqrt{11}(1-x)^5$$
$$b_{15}(x) = 6\left[5t(1-x)^4 - \frac{1}{2}(1-x)^5\right]$$
$$b_{25}(x) = \frac{18\sqrt{7}}{5}\left[10(1-x)^3t^2 - 5(1-x)^4t + \frac{5}{18}(1-x)^5\right]$$
$$b_{35}(x) = \frac{28}{\sqrt{5}}\left[10(1-x)^2x^3 - 15(1-x)^3x^2 + \frac{30}{7}(1-x)^4x - \frac{5}{28}(1-x)^5\right]$$

$$b_{45}(x) = 7\sqrt{3}\left[5(1-x)x^4 - 20(1-x)^2x^3 + 18(1-x)^3x^2 - 4(1-x)^4x + \frac{1}{7}(1-x)^5\right]$$
$$b_{55}(x) = 6\left[x^5 - \frac{25}{5}(1-x)x^4 + \frac{100}{3}(1-x)^2x^3 - 25(1-x)^3x^2 + 5(1-x)^4x - \frac{1}{6}(1-x)^5\right]$$

and

$$b_{06}(x) = \sqrt{13}(1-x)^6$$
$$b_{16}(x) = \sqrt{44}\left[6t(1-x)^5 - \frac{1}{2}(1-x)^6\right]$$
$$b_{26}(x) = 11\left[15(1-x)^4x^2 - 6(1-x)^5x + \frac{3}{11}(1-x)^6\right]$$
$$b_{36}(x) = \sqrt{252}\left[20(1-x)^3x^3 - \frac{45}{2}(1-x)^4x^2 + 5(1-x)^5x - \frac{11}{66}(1-x)^6\right]$$
$$b_{46}(x) = \frac{42}{\sqrt{5}}\left[15(1-x)^2x^4 - 40(1-x)^3x^3 + \frac{180}{7}(1-x)^4x^2 - \frac{30}{7}(1-x)^5x + \frac{5}{42}(1-x)^6\right]$$
$$b_{56}(x) = \frac{28}{\sqrt{3}}\left[6t^5(1-x) - \frac{75}{2}(1-x)^2x^4 + 60(1-x)^3x^3 - 30(1-x)^4x^2 + \frac{30}{7}(1-x)^5x - \frac{3}{28}(1-x)^6\right]$$
$$b_{66}(x) = 7\left[x^6 - 18(1-x)x^5 + 75(1-x)^2x^4 - 100(1-x)^3x^3 + 45(1-x)^4x^2 - 6t(1-x)^5 + \frac{1}{7}(1-x)^6\right]$$

In addition, we have determind the explicit representation for the orthonormal Bernstein polynomials as [16]

$$b_{km}(x) = \left(\sqrt{2(m-k)+1}\right)(1-x)^{m-k}\sum_{i=0}^{k}(-1)^i\binom{2m+1-i}{k-i}\binom{k}{i}x^{k-i} \qquad (13)$$

The eqs(13) can be written in terms of the Bernstein basis functions as

$$b_{km}(x) = \left(\sqrt{2(m-k)+1}\right)\sum_{i=0}^{k}(-1)^i\frac{\binom{2m+1-i}{k-i}\binom{k}{i}}{\binom{m-i}{k-i}}B_{k-i,m-i}(x) \qquad (14)$$

Any generalized Bernstein basis polynomials of degree m can be wretten as a linear combination of the generalized Bernstein basis polynomials of degree m+1

$$B_{k,m}(x) = \frac{m-k+1}{m+1}B_{k,m+1}(x) + \frac{k+1}{m+1}B_{k+1,m+1}(x) \qquad (15)$$

By utilizing eqs(15),the following functions can be written as

$$B_{k,m-1}(x) = \frac{m-k}{m}B_{k,m}(x) + \frac{k+1}{m}B_{k+1,m}(x) \qquad (16)$$

and

$$B_{k-1,m-1}(x) = \frac{m-k+1}{m}B_{k-1,m}(x) + \frac{k}{m}B_{k,m}(x) \qquad (17)$$

Substituting these eqs(16) and (17) into the right hand side of the eqs(4), we get the following derivatives of Bernstein basis polynomials

$$\frac{d}{dx}B_{k,m}(x) = (m-k+1)B_{k-1,m}(x) + (2k-m)B_{k,m}(x) - (k+1)B_{k+1,m}(x) \qquad (18)$$

In [10], the derivative of the orthonormal (B-Polynomials)of degree five are introducedas given

$$b'_{05} = \left[-5\sqrt{11}\,B_{05} - \sqrt{11}B_{15}\right]$$

$$b'_{15} = [45B_{05} - 15B_{15} - 2B_{25}]$$

$$b'_{25} = \left[23\sqrt{7}\,B_{05} + \frac{121\sqrt{7}}{5}B_{15} + \frac{18\sqrt{7}}{5}B_{25} - \frac{54\sqrt{7}}{5}B_{35}\right]$$

$b'_{35} = \left[\frac{145}{\sqrt{5}}B_{05} - \frac{235}{\sqrt{5}}B_{15} + \frac{78}{\sqrt{5}}B_{25} + \frac{154}{\sqrt{5}}B_{35} - \frac{113}{\sqrt{5}}B_{45}\right]$

$b'_{45} = \left[-33\sqrt{3}B_{05} + \frac{331\sqrt{3}}{5}B_{15} - \frac{217}{5}B_{25}\frac{126\sqrt{3}}{5}B_{35} + 77\sqrt{3}B_{45} - 35\sqrt{3}B_{55}\right]$

$b'_{55} = [35B_{05} - 55B_{15} + 63B_{25} + 35B_{35} - 119B_{45} + 105B_{55}]$

and we introduce the derivative of the orthonormal (B-Polynomials) for degree six

$b'_{06} = \left[-6\sqrt{11}\,B_{06} - \sqrt{13}B_{16}\right]$

$b'_{16} = \left[8\sqrt{11}B_{06} - 7\sqrt{11}B_{16} - 4\sqrt{11}B_{26}\right]$

$b'_{26} = [-84\,B_{06} + 96B_{16} + B_{26} - 3B_{36}]$

$b'_{36} = \left[36\sqrt{7}B_{06} - 64\sqrt{7}B_{16} + 32\sqrt{7}B_{26} + 27\sqrt{7}B_{36} - 24\sqrt{7}B_{46}\right]$

$b'_{46} = \left[-42\sqrt{5}B_{06} + 95\sqrt{5}B_{16} - 150.263681B_{26} - 40.24922359B_{36} + 187.8297101B_{46} - 42\sqrt{5}B_{56}\right]$

$b'_{56} =$
$\left[46\sqrt{3}B_{06} - 206.1140461B_{16} + 235.5589098B_{26} - 14\sqrt{3}B_{36} - 242.4871131B_{46} + 266.7358244B_{56} - 56\sqrt{3}B_{66}\right]$

$b'_{66} = [-48B_{06} + 132B_{16} - 168B_{26} + 42B_{36} + 168B_{46} - 252B_{56} + 168B_{66}]$

5. Second kind integral equations:

In this section, we use Orthonormal Polynomials for solving second kind Fredholm and Volterra integral equations.

1- Fredholm integral equation (FIE):

Where in eq(1) $g(x) \in L^2[0,1], k(x,t) \in L^2([0,1] \times [0,1])$ are known and $u(t)$ is unknown function to be determined.

First we assume the unknown functions

$$u_i(x) = C_i^T B(x), i = 1,2,\dots,n \tag{19}$$

by substituting (19)in (1) we have:

$$C_i^T B(x) = g_i(x) + \int_0^1 k_{i,j}(x,t)\, C_i^T B(t)dt$$

$$C_i^T B(x) - \int_0^1 k_{i,j}(x,t)\, C_i^T B(t)dt = g_i(x) \tag{20}$$

Pick distinct node points $t_1, t_2, \dots, t_n \in [0,1]$

This leads to determining $\{\, c_1, c_2, \dots, c_n\}$ as the solution of linear system

$$\sum_{i=1}^{n} C_j \left[B(x_i) - \int_0^1 k(x_j, t_i)\, B(t_i)dt\right] = g(x_i) \tag{21}$$

In this paper Collocation points are $t_i = \frac{i}{n}\ \ for\ i = 1,2,\dots,n$ so that we have a system of linear equations

$L_n X = l_n$ where

$L_n = \left|B(x_i) - \int_0^1 k(x_j, t_i)\, B(t_i)dt\right|_{i=0}^{n} \qquad j = 1,2,\dots,n$

$l_n = [g(x_i)], \qquad i = o, 1, \dots, n$

2- Volterra integral equation (VIE):

Similarly above section by using Collocation points $t_i = \frac{i}{n}\ \ for\ i = 1,2,\dots,n$

$L_n = \left|B(x_i) - \int_0^x k(x_j,t_i)\,B(t_i)dt\right|_{i=0}^{n} \qquad j = 1,2,\dots,n$

$l_n = [f(x_i)], \qquad i = o,1,\dots,n$

6.Numerical Results:

In this section VIE, FIE is considered and solved by the introduced method.

Example 1: Consider the following FIE

$$u(x) = \sin x + \int_0^1 (1 - x\cos xt)\,u(t)dt \tag{22}$$

the exact solution u(x)=1. Solving eqs(20) and (21) we get the values of $C^{\ T}$
C=[0.91402903 0.10091438 3.48867166 −2.7619172 3.51373945 0.15379915 0.98896685]^T
Table1 shows the numerical results for this example.

Table 1:some numerical results for example 1

x	Approximat solution $b_n(x)$	Approximat solution $B_n(x)$	$AbsouteError$ $\lvert exact - B_{n6}\rvert$
0	0.91402333	0.99993256	6.7440e-005
0.1	0.93422123	0.99993267	6.7330e-005
0.2	0.96878356	0.99994532	5.4680e-005
0.3	0.96878320	0.99994444	5.5560e-005
0.4	0.97843933	0.99995324	4.6760e-005
0.5	0.93270466	0.99995417	4.5830e-005
0.6	0.99388042	0.99996618	3.3820e-005
0.7	0.99963715	0.99997654	2.3460e-005
0.8	0.98888323	0.99998790	1.2100e-005
0.9	0.99999668	0.99999999	1.0000e-008
1	0.99999998	1.00000000	0.00000000
$M.S.E$ =6.7440e-005 $L.S.E$=2.1286e-008			

Example 2: Consider the following FIE

$$u(x) = e^{-x} - \int_0^1 xe^t u(t)dt$$

the exact solution $u(x) = e^{-x} - \frac{x}{2}$. Solving eqs(20) and (21) we get the values of $C^{\ T}$
C=[1 1.13944655 −0.05603088 0.85196593 −0.13491534 0.11655846 − 0.16383600]^T
Table2 shows the numerical results for this example.

‘

Table2:some numerical results for example 2

x	Exact solution	Approximat solution $b_n(x)$	Approximat solution $B_n(x)$	$AbsouteError$ $\|exact - B_{n6}\|$
0	1	1	1	0.00000000
0.1	0.85483742	0.94188967	0.85488967	0.00005225
0.2	0.71873075	0.76843118	0.71811760	0.00008101
0.3	0.59081822	0.59503874	0.59081874	0.00000052
0.4	0.47032005	0.46240281	0.47032115	0.00000110
0.5	0.35653066	0.35230187	0.35653066	0.00352200
0.6	0.24881164	0.24605093	0.24880053	0.00001111
0.7	0.14658530	0.13907957	0.14658510	0.00000020
0.8	0.04932896	0.04047384	0.04932895	0.00000001
0.9	-0.04343034	-0.04663478	-0.04343155	0.00000121
1	-0.13212056	-0.16383600	-0.13212056	0.00000000
	$M.S.E$ =0.00352200 $L.S.E$=0.00000000			

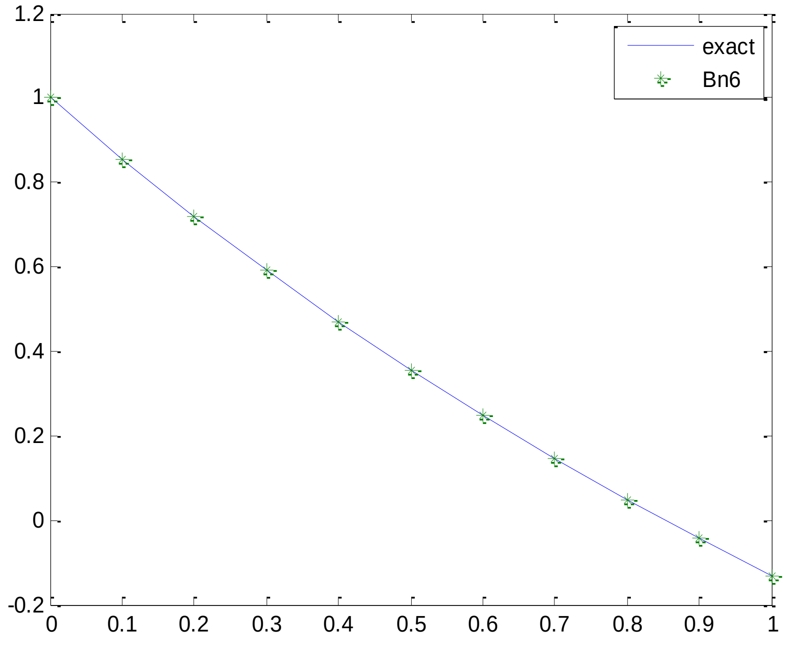

Fig1 of example1

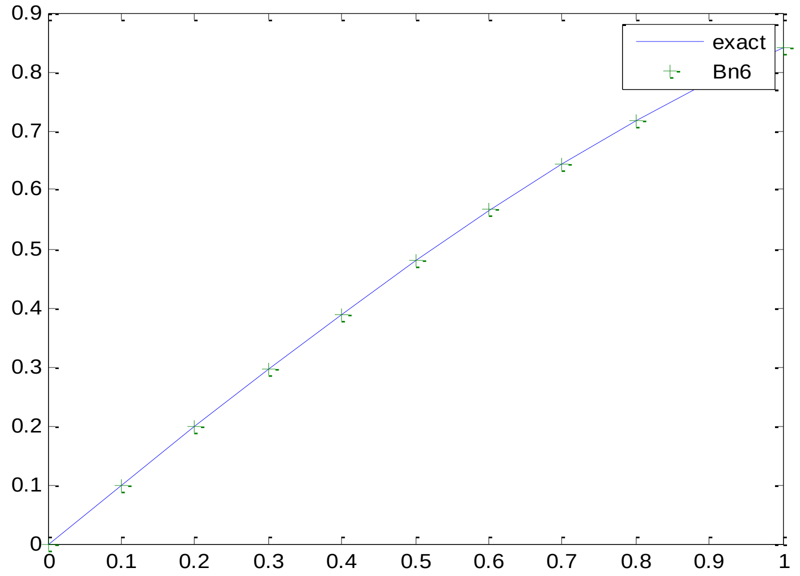

Fig 2 of example3

Example 3: Consider the following VIE

$$u(x) = x - \int_0^x (x-t)u(t)dt \qquad (23)$$

The exact solution $u(x) = \sin x$. Table(3) shows the numerical results for this example(3)

C=[0 0.15606151 0.37680718 0.40682330 0.73751386 0.68080388 0.84172197]^T

Table3:some numerical results for example 3

x	Exact solution	Approximat solution $b_n(x)$	Approximat solution $B_n(x)$	$AbsouteError$ $\lvert exact - B_{n6}\rvert$
0	0.00000000	0.00000000	0.00000000	0.00000000
0.1	0.09983342	0.09924030	0.09983389	0.00000059
0.2	0.19866933	0.19972478	0.19865878	0.00001055
0.3	0.29552021	0.29617066	0.29552053	0.00000065
0.4	0.38941834	0.38930420	0.38941820	0.00000014
0.5	0.47942554	0.47990931	0.47942560	0.00000048
0.6	0.56464247	0.56604342	0.56604342	0.00140095
0.7	0.64421769	0.64342047	0.64342047	0.00007972
0.8	0.71735609	0.70896119	0.71732785	0.00008394
0.9	0.78332691	0.76751046	0.78331110	0.00001581
1	0.84147098	0.84172197	0.84147073	0.00000025
	$M.S.E$ =0.00140095 $L.S.E$=0.00000000			

Conclusion:

In this work, VIE,FIE have been solved by using Bernstein basis polynomials of degree m in collocation method. Comparison of the approximate solutions and the exact solutions show that the proposed method is efficient tool. Illustrative examples are included to demonstrate the validity and applicability of the technique.

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