Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.5, No.12, 2015

Gibbs Phenomenon Resolution in 2&3 Dimensional Space via Mupad Micro Models

Dr. Faisal Abdulateef Shaghati
Open Educational College Mathematics Department Iraq - Baghdad - Al adhamiya - near the judicial institute

Abstract
This paper aims to reconstruct Mupad models to treat & resolve the overshoots & tautologically undershoots that usually appears while approximating continuous or piecewise continuous function through by Fourier series, i.e. inexperienced resolution for Gibbs phenomenon. These models have to go from harmonic to square wave as well as triangle wave ones in 2D,3D planes & vice versa. The author will come across advantages & disadvantages of Fourier series as well as a new reformulation for Dirchlet's Theorem; in addition to that an extended accuracy for Gibbs Constant will find itself among the lines of this work.
Keywords: Gibbs Phenomenon, Gibbs Constant, Fourier series, overshoots, undershoots, square wave, Triangle wave, Dirchlet's Theorem.

Introduction:

Gibbs phenomenon was coming by the name of distinguish great American scientist who is JOSAIH WILLARD GIBBS; therefore it seems so relevant to preface this paper with appropriate foreword on his scientific innovations & impacts at the science entity & progress. One reading tens biographies of great scientists will find a few of them were interesting & working in multiple scientific subjects, but it was not in the most significance pure & applied fields of science as Physics, Chemistry & Mathematics as J.W.GIBBS was. Then the most shining enlightenments for GIBBS scientific contributions was on physical chemistry , statistical mechanics, physics & mathematics, so GIBBS'S thermodynamics found many applications during the early 20^{th} Century, from the electrochemistry to the developments of Haber process for the synthesis of ammonia[13], also GIBBS works on statistical ensembles as presented in his 1902 textbook, had had a great impact in both theoretical physics & pure mathematics, thus Einstein had been considered GIBBS interpretations were to a large extent superior to his own[28][26][21]. Moreover GIBBS early papers on the use of graphical methods in thermodynamics reflect a powerfully original foundation of what mathematicians later called vector analysis [28]. Mathematician Norbert Wiener cited GIBBS theoretical approach to use probability in the formulation of statistical mechanics as the first great revolution of twentieth Century Physics and as a major influence on his conception of cybernetics [27], then significance part of GIBBS works was on the application of Maxwell's equations in physical optics.
Therefore several mathematical concepts hold GIBBS name, for instance GIBBS lemma in game theory, GIBBS inequality & GIBBS algorithm in information theory, GIBBS entropy & GIBBS paradox in Statistical Mechanics as well as GIBBS Phenomenon in mathematical analysis.

Also it seems worthwhile to remain a trait of GIBBS personality, J.WILLARD. GIBBS was distinguished by high level morality, HE was generous & human kindly gentleman, in spite of his relative richness and elite Ian position, one see him preferred simplicity in his own life, so J.W.GIBBS was seen lovely, habitually & peacefully driving a horse carriage at the streets of his quiet city New Haven, in accordance to Lynde Wheeler the GIBBS student at Yale, in his later years GIBBS was always neatly dressed, usually wore a felt hat along the road and naturally behave as moderate normal human, his former student Henry A. Bum stead referred to GIBBS style as genial and kindly in relation with his team & others, never showing impatience or irritation, HE was ideal unselfish Christian gentleman. In minds of those who knew him, the greatness of his intellectual achievements will never overshadow the beauty & dignity of his personality.

Therefore, in full meaning brief, several scientists nominated by Noble Prize remained the impact of his theories & applications on their achievements and innovations, among them Einstein who praised him as the greatest scientific mind in American history[1][22][25].
Hereafter, in 1848 Henry Wilbraham had had published a paper that was for the first time talking and analyzed GIBBS phenomenon, but that paper hadn't brought attention of mathematicians at those times. In 1898 Albert Michelson observed the phenomenon via a device that he developed to compute and resynthesize the Fourier series, but as many as he input Fourier terms, its resulted graph contained repeated humps at the left or/and the right of the discontinuities points, therefore Michelson had been taken by a doubt that his device is not well enough to achieve the task, but J.WILLARD GIBBS in 1899 pointed out that the oscillations were a mathematical phenomenon and would always occur when synthesizing a discontinuous function with a Fourier

series, therefore about seven years after, in 1906 Maxime Bocher introduced mathematical analysis of the phenomenon & named it GIBBS phenomenon [8,9].

There are several interpretations & approaches to the famous phenomenon that is the subject of this work, one of them analyze it not as a consequence of using trigonometric polynomial as approximating function, but it is a mathematical result to that Fourier series are best square approximation; and it is naturally appears where one uses least square approximations for a function within jump discontinuities, even if the approximating functions are not trigonometric ones [16]. Other viewpoints that strongly support this analysis say that the phenomenon above is not the special quirk occurs only in trigonometric Fourier series, a similar phenomenon exists in other classical orthogonal series [17, 12], Spline expansions [16], wavelet series[11,9], and even in sampling approximations [5]. Moreover there is strong similarity in Runge's function approximation.

As it remained at all resources, as the number of Fourier series' terms becomes larger & larger, the approximation error appears smaller and smaller whereas the overshoots are reduced in width as well as in height but oscillations doesn't disappear, then at the square wave instance the Fourier series exceeds the height pi/4 by the following quantity that equals $1/2\int_0^{pi}\frac{\sin(t)}{t}*dt-\frac{pi}{4}=\frac{pi}{2}*(0.089490\ldots)=$ 1.851937052 …, then while by it is valid for every continuous or piecewise continuous approximated function through by Fourier series, it's called GIBBS constant. For matter of consistency, here is an extended accuracy for it via Mupad micro model as follow,

```
% Mupad Micro model for GIBBS constant
DIGITS: = 30: float (int (sin(x)/x, (x = 0.. PI)))
1.85193705198246617036105337016
```

Therefore resolving GIBBS phenomenon is benefit able in signal processing, while it removes the signal clipping along the overshoots & undershoots, also in spinal MR imaging & N. Medicine whereas it treats the certain artifact that causes presence of adjacent regions of markedly differing signal intensity, also to smooth the noise while maximally reducing the repeated humps[29][6]. For theoretical & applicable reasons, the author would like to reconstruct appropriate Mupad models to resolve GIBBS phenomenon in two as well as three dimensional space & totally removing the occurred oscillations, it will be appear at the ascending paragraphs of this article.

2. Mupad models for Fourier series

Fourier series is essential mathematical tool in transformation processing that frequently required in scientific application as engineering, physics & other fields, it plays key role especially in system's control theory, also in analysis of signal processing, coupled dynamical systems, Mechanics, heat transfer in solid bodies as well as electronic, optical signaling & imaging systems. Fourier series has specific advantages in comparison with other series, thus it can approximate piecewise continuous function i.e. those functions within finite number of discontinuous points, while it is not possible via Taylor series where is it requires analytic function at all points, also Fourier series can expand periodic as well as non-periodic function since the function has periodic extension outside its closed interval, also expansion of piecewise continuous functions through by Fourier series offers all modes of oscillations that physicists can make use of them for their specifically significance tasks, also Fourier series of piecewise continuous functions is not uniformly convergent at all points if not verified continuity & convergence to left-right limits' average of the piecewise function at each point of discontinuity, in addition to that integration the terms of convergence Fourier series in opposite to their differentiation is possible &valid, for instance it appears while doing Fourier series summation, while by the derivative of first term (the fundamental term) at many cases omit it out of the series[3][10][30][20].

Therefore infinite series & usually among them Fourier series are unique mathematical form to evaluate or approximate complicated functions, but there is no a rule that determine the number of terms that to be truncated aiming at the best approximated value, whereas the upper value of the series summation equal infinity, it differs in accordance to the series as well as the desired value one aims to calculate, except that at the case of so large number of terms the process of calculation as well as graphing seems so tedious especially without electronic device, then even so programming language & specifically programmed Functions have to be a valuable to overcome the task, nowadays the technological & scientific progress offer high level auxiliary alternatives for computational purposes, the author find the best choice for his mathematical aims including this paper in MAT LAB & Mupad as its integrated part, then he pretends Mupad models reconstruction to resolve GIBBS' phenomenon, but meanwhile it is subjectively related within Fourier series one would like to introduce some models for continuous or piecewise continuous functions transformed through by Fourier series.

For that it is little bit lines before the aimed models, Fourier series is mathematical formula that can express any periodic or non-periodic function within periodic extension as summation of many or so many to infinity terms of sine's or cosine's, here is an essential advantage of Fourier series that it transforms the function target to easier mathematical harmonic function, then one can treat & make use of it incomparably better than its original form, then f(t) can be reformulated as follows:

$f(t) = a_{0/2} + \sum_{n=1}^{\infty}(a_n \cos \mathrm{nt} + b_n \sin \mathrm{nt})$

Or it can more easily be written as bellows:

$f(t) = a_{0/2} + \sum_{n=1}^{\infty} a_n \cos \mathrm{nt} + \sum_{n=1}^{\infty} b_n \sin \mathrm{nt}$

Then here is the most significance point, which kind of functions can be expanded through by Fourier series & is the Fourier series of f (t) converges to f (t)? it seems somewhat unusual or answered question, but also it may seem for someone that it is essential to analyze it and make it clear that what is usually referred to it as discontinuous function have to be transformed through by Fourier series is not that kind of functions considered in calculus as discontinuous ones, while by they are multiple functions joined by discontinuous point or disjoined by finite points, also they are functions that constructed by several continuous functions of different formulae, thus one can referred to them as piecewise continuous functions, for instance saw-tooth wave or square wave function, as the following theorem reformulate it:

THEOREM: If f (t) is continuous or piecewise continuous function that is a bounded periodic or non-periodic within periodic extension out of its closed interval, such that in any one period it has at most a finite number of local maxima & minima & finite number of discontinuity points, then the Fourier series of f(t) converges to f (t) at all points where f (t) is continuous and converges to the average of the right &left limits of f (t) at each point where f (t) is finite discontinuous.

The Theorem above includes what's so called Dirchlet's conditions, that it will be mutually verified within the elaborated models, the following sum is a Fourier representation of a periodic step function:

$$f(x) = \sum_{n=1}^{\infty} \frac{(\sin((2k-1)x)}{2k-1}$$

Then one would like to trace the convergence of the partial sums for certain values of n

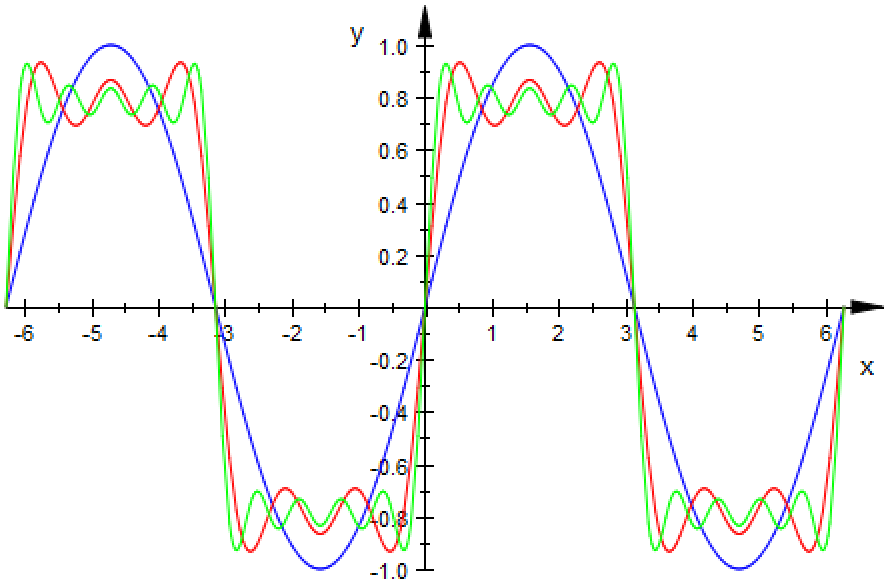
$$f(x) = \sum_{k=1}^{n} \frac{(\sin((2k-1)x)}{2k-1}$$

Thus here is the following modified model to do the task using the Mupad Function floor (n) for partial sum for Fourier series of a periodic step function[23,24,19,15],

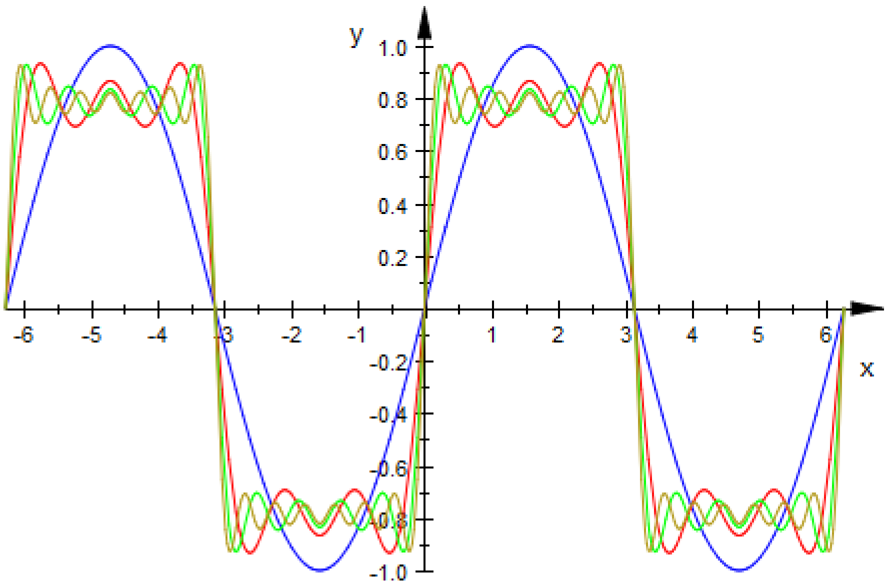
```
f _ n := sum(sin((2*k-1)*x)/(2*k-1), k = 1..floor(n))

plotfunc2d (f _ n  $ n = 1…5 step 2, x = -2*PI … 2*PI, LegendVisible = FALSE)
```

Here is the first instruction has to be accompanied with every Mupad micro model at this paper.

Therefore the model above can produce the required models within odd number of truncated terms in accordance to one desire, but it seems reasonable to choose as small as possible number as the following models within only one, three, five terms,



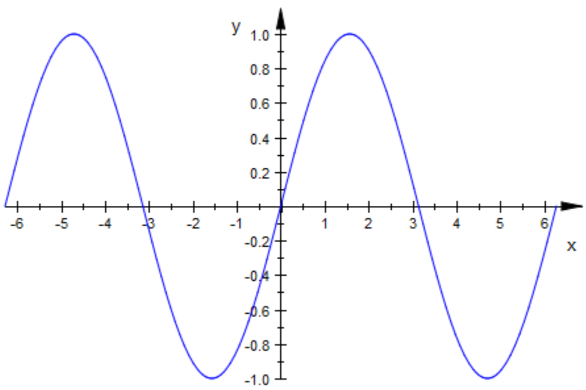
Then the scene is to considerable extent is readable, that doesn't due to colors but to that the number of overshoots equals the nth term of the partial sum, then one can observe the sinusoidal harmonic function within one hump is resulted for n equals one, while those within three or five humps (overshoots) are consequently represented the partial sum of the Fourier series when by n equals three & five, also it's obvious that the overshoots' height except the fundamental harmonic inversely proportional to the terms number, i.e. as the terms number becomes larger the overshoots seem shorter, that creates an impression that overshoots are going to decay and may disappear, then it calls one to increase the terms number aiming at better conclusion, then gradually one may let n equals 7 terms, although it's so small number with respect to the original upper bound i.e. infinity; it requires nothing more than replacing the number 5 by 7 at steepness Function of the second instruction of the micro model above, but in spite of small numbers of terms while additionally their model well colored, the output of the modified model didn't offer an enhancement to continue in the process of terms number increasing as one will see at the ascending page, while the writer of these lines through by his very eyes plus glasses couldn't recognize the representations of the Fourier series functions of relatively little bit terms' truncation, this case bring to one attention the Theory of Experience & Mistake for the American psychologist JHON DEWY, hence
One may sometimes learn from his own mistakes, although there isn't mathematical contradiction. For matter of consistency let's take a glance at the model of the dialogue above,



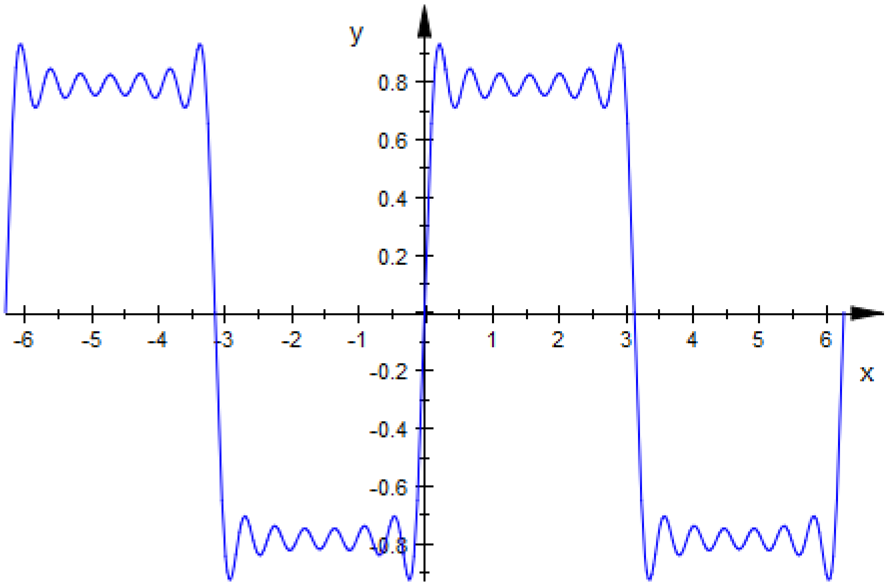
Therefore it's aesthetically pleasing, but one is also looking for better readability of its mathematical construction whereas it seems somewhat complicated, ambiguous & doesn't allow one to analyze it perfectly, then the author preferred to disconnect this overlapping and construct another model that would come in accordance within the mathematical analysis of Fourier series rule, that is as larger number of terms of Fourier series can be truncated as better approximation will be approached.

3. Toward GIBBS phenomenon' resolution

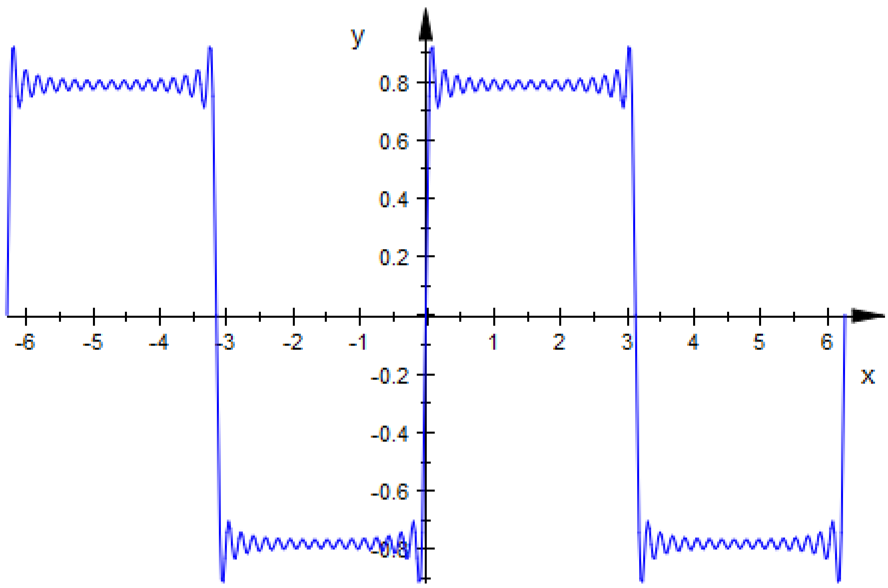
in mathematical modeling a little modifying may create wonders as the following micro models:
plotfunc2d (f_ n $ n = 1…1, x = -2*PI…2*PI, LegendVisible = FALSE)



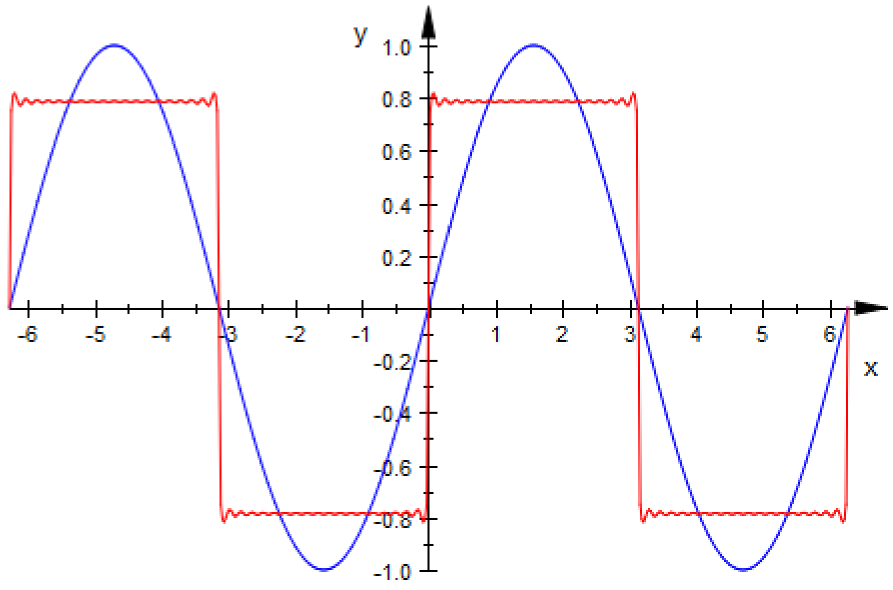
The previous micro model is working on the base of the first model that produces the partial sum of the Fourier series, then one only has to reasonably modified the number of terms he likes aiming at overshoots disappearing, here are the serial occurrence of the outputs appearing via the Mupad micro model above, it is unusual & inexperienced going from one term to six hundreds & one terms within specific choices taking in consideration the most effective number of terms, let's take a glance:



Hence it is the model for a partial sum of Fourier series at the case of 7 terms truncation and then 7 humps, it's obviously seen to convince one by the equality rule between terms & overshoots' number. The ascending model will ensure the inverse proportion between the overshoots' height & terms' number as it appears when by 17 terms truncation:

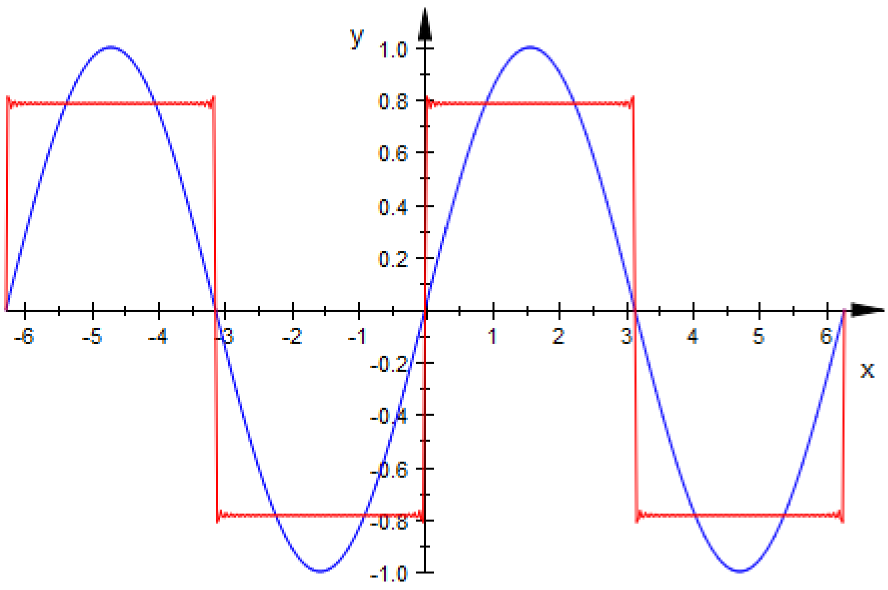


Hereafter a unique model that includes the 1st & 101st models at the same plane showing the progress of the oscillations treatment through by increasing the terms' number. Thus the oscillations will be also gradually being removed while one increases the terms' steepness by 1ooth time for each model as follows:

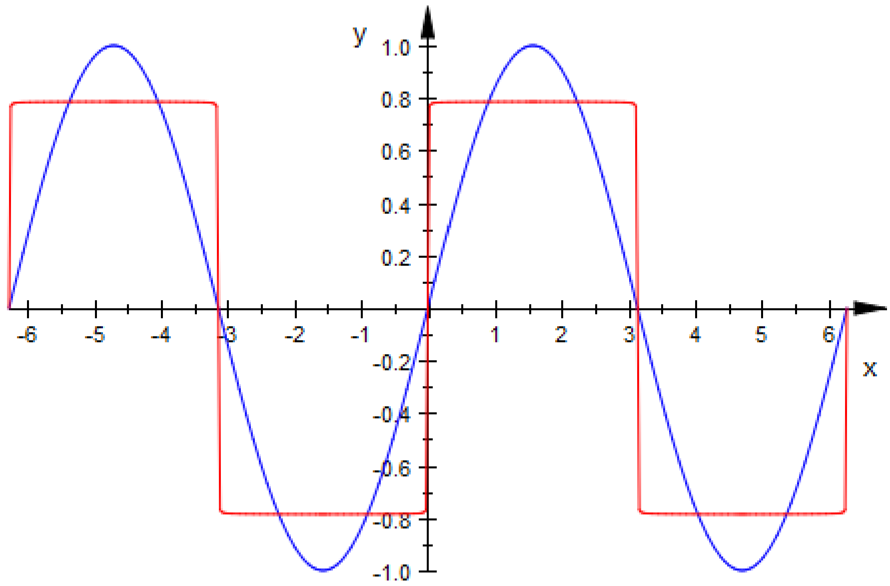


And so on the overshoots & undershoots will being disappeared for each 100 steps more as the model below:

plotfunc2d (f_ n $ n = 1 .. 201 step 200, x = -2*PI..2*PI, LegendVisible = FALSE)

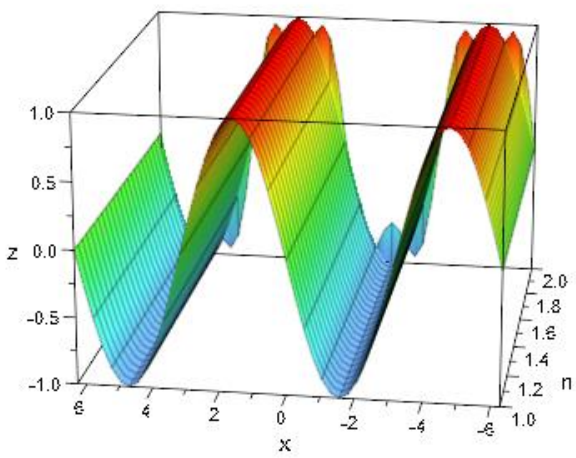


Moreover about to be invisible oscillations appears for n equals 301,401 & 501 term, while it totally disappear altogether with its 'small ears' for n equals 601 term as bellows:
plotfunc2d (f_ n $ n =1… 601 step 600, x = -2*PI...2*PI, LegendVisible = FALSE)

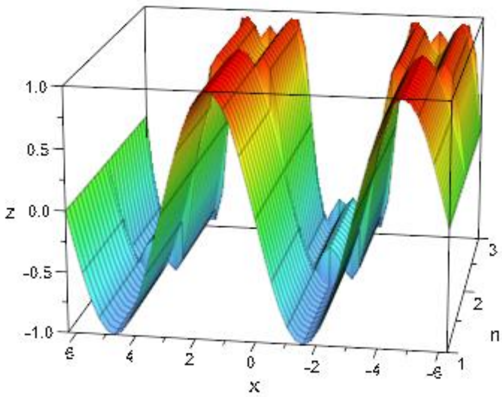


Interesting that is the final number of terms is also valid for triangle wave, may be more interesting that process can be achieved ad hoc by one micro model for every continuous or piecewise continuous approximated through by Fourier series, moreover it may be most interesting that is also valid in three Dimensional space as the following micro models consist it:

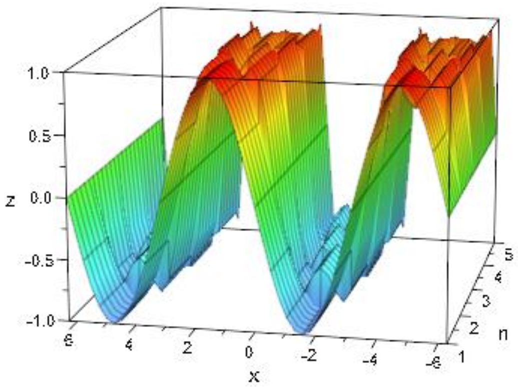
plotfunc3d (f_n, n = 1…2, x= -2*PI …2*PI, Submesh = [5, 1], FillColorType = Rainbow)



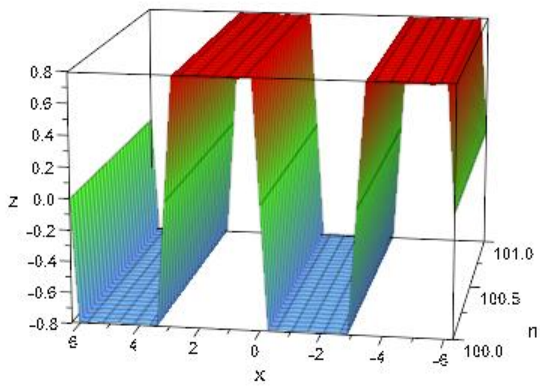
plotfunc3d (f_n, n = 1…3, x= - 2*PI...2*PI, Submesh = [5, 1], FillColorType = Rainbow)



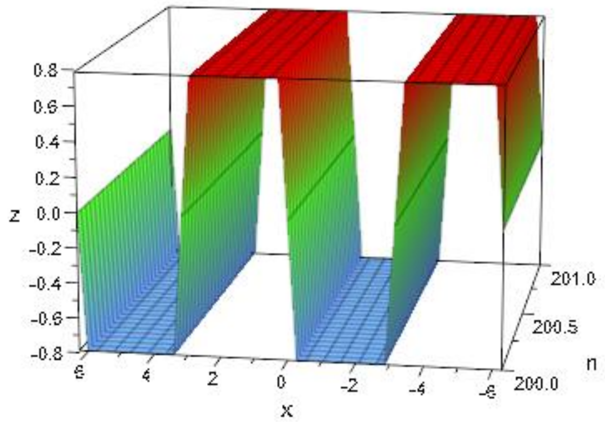
plotfunc3d (f_n, n = 1…5, x= -2*PI…2*PI, Submesh = [5, 1], FillColorType = Rainbow)



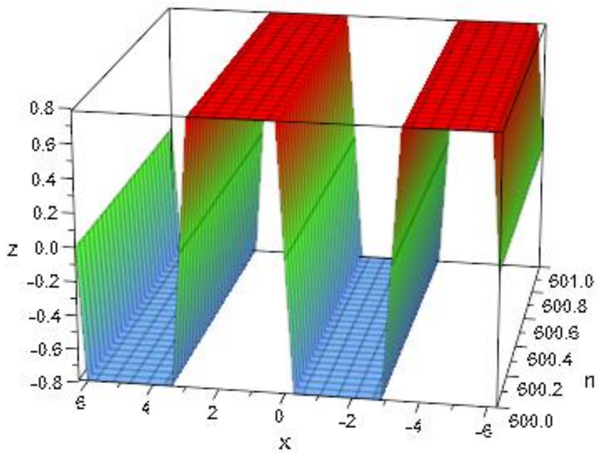
plotfunc3d (f_n, n = 100…101, x= -2*PI…2*PI, Submesh = [5, 1], FillColorType = Rainbow)



Plotfunc3d (f_n, n = 200…201, x= -2*PI…2*PI, Submesh = [5, 1], FillColorType = Rainbow)



plotfunc3d (f_n, n = 600…601, x= -2*PI...2*PI, Submesh = [5, 1], FillColorType = Rainbow)



In two as well as in three dimensional spaces, the oscillations disappear and seem to be gradually invisible & uniformly convergence to the point or points of finite discontinuity, that is seen at the function' path on the plane, or at the upper & lower rectangles on the 3d space, when by the partial sum of Fourier series for 101terms keeps 100 steps across its path to 601st term whereas no overshoots or undershoots being occur. Therefore it seems that finite sum of continuous functions can be piecewise continuous and hence doesn't exhibit the GIBBS phenomenon.

REFRRENCES

1. American Physical Society "J. Willard Gibbs"(http://www. /historicsites/gibbs.cfm)., 2012
2. Bernie. D. Shizgal, Jung Jae Hun, Towards the Resolution of Gibbs Phenomenon
3. Bill Cox, Understanding Engineering Mathematics, 2001
4. C. Ray Wylie, Advanced Engineering Mathematics, 1975
5. G. G. Walter & H-T. Shim, Phenomenon for sampling series & what to do about, 1998
6. Gengsheg L. Zeng & Richard J. Allred, Part -Image for Reduction of Gibbs Phenomenon, 2009
7. Gibbs phenomenon, Lynda.com 2015
8. Gibbs phenomenon, Wikipedia 2015
9. Gibbs phenomenon, Wikiversity 2015
10. H. k. DASS & Dr. RAMA VERMA, Mathematical Physics, 2011
11. H. T. Shim & H. Volkmer, On Gibbs' phenomenon for wavelet expansions, 1966
12. H. t. Shim, sum ability for Meyer wavelets, 2002
13. Haber Fritz, Theoretical Development of chemistry, 1925

14. HONG-TAE SHIM & CHIN-HONG PARK, Gibbs phenomenon for Trig Interpolation, 2004
15. http//www.at mos. Washington Edu.2015, Square waves from sine wave 2015
16. J. Foster F. B. Richards, Gibbs Phenomenon for Piecewise- Linear Approximation,1991
17. Jan GAO, Difference of two sets & estimation of Clarke generalized Jacobian via quasi diff., 2001
18. Josiah Willard Gibbs, Wikipedia 2015
19. Mathematics Stock Exchange, How to plot Fourier series in Mat Lab 2015
20. N. p .Bali & Dr. Manish Goyal, Engineering Mathematics, 2011
21. Navarro, Luis. Gibbs, Einstein and Foundation of Statistical Mechanics, 1998
22. O'Connor, JHON J.; Robertson, Edmund F. ; " Josiah Willard Gibbs", 1997
23. PROFESSORS TEAM, Advanced Graphics in MUPAD, MATLAB 2014
24. PROFESSORS TEAM, THE LANGUAGE OF TECHNICAL COMPUTING, MATLAB 2014
25. Tiselius, Arne. Award Ceremony Speech, Nobel Prize in Chemistry, 1949
26. Wiener, Norbert, Cybernetics, 1961
27. Wiener, Norbert. The Human Use of Human Beings, 1950
28. Wightman 1979, pp. x-xxxiv
29. Wightman Arthur S.; On the Prescience of J. Willard Gibbs, 1990
30. Wikipedia, Overview on Fourier series & Gibbs phenomenon 2015