Global stability and persistence of three species with Holling type-IV functional response

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Abstract

In this paper a prey-predator model involving Holling type IV functional response is is proposed and studied. The existence of all possible equilibrium points is carried out. The local stability analysis of the system is carried out. The global dynamics of the system is investigated with the help of the Lyapunov function. Finally, the numerical simulation is used to study the global dynamical behavior of the system. It is observed that, the system has either stable point or periodic dynamics.

Keywords: prey-predator model, Holling type IV functional response, stability analysis, Lyapunov function.

1. Introduction

Variety of the mathematical models for interacting species incorporating different factors to suit the varied requirements are available in literature, a successful model is one that meets the objectives, explains what is currently happening and predicts what will happen in future. The first major attempt to predict the evolution and existence of species mathematically is due to the American physical chemist Lotka (1925) and independently by the Italian mathematician Volterra (1926), see [1], which constitute the main theme of the deterministic theory of population-dynamics in theoretical biology even today. Over the last few decades, many models for two or more interacting species have been proposed on the basis of Lotka-Volterra models by taking into account the effects of crowding, age structure, time delay, functional response, switching, etc. [2,3,4].

Keeping the above in view, in this paper consideration is given to analyze and study the dynamical behavior and persistence of prey-predator model with Holling type-IV functional response have been proposed and studied.

2. Mathematical model formulation

Let $x(t)$ and $y(t)$ are the density of two predator species at time t, $z(t)$ be the density of prey species at time t that consumes the prey species according to Holling type IV functional response then the dynamics of a prey–predator model can be represented by the following system of ordinary differential equations.

$$\frac{dx}{dt}=\frac{e_1 w_1 \gamma_1 xz}{\alpha_1}-\mu_1 x-\delta_1 x^2 = xf_1(x,y,z)$$
$$\frac{dy}{dt}=\frac{e_2 w_2 \gamma_2 yz}{\alpha_2}-\mu_2 y-\delta_2 y^2 = yf_2(x,y,z) \qquad (1)$$
$$\frac{dz}{dt}=(a-bz)z-\frac{w_1 \gamma_1 xz}{\alpha_1}-\frac{w_2 \gamma_2 yz}{\alpha_2} = zf_3(x,y,z)$$

where $\alpha_1 = z^2+\gamma_1 z+\gamma_1\beta_1$ and $\alpha_2 = z^2+\gamma_2 z+\gamma_2\beta_2$, with $x(0)\geq 0$, $y(0)\geq 0$ and $z(0)\geq 0$.

Note that all the parameters of system (1) are assumed to be positive constants and can be described as following: a is the intrinsic growth rate of the prey population; $\mu_i, i=1,2$ are the death rates of the predator population; the parameter b is the strength of intra-specific competition among the prey species; the parameter $\beta_i, i=1,2$ can be interpreted as the half-saturation constant in the absence of any inhibitory effect; the parameter $\gamma_i, i=1,2$ are a direct measure of the predator immunity from the prey; $w_i, i=1,2$ are the maximum attack rate of the prey by a predator; $e_i, i=1,2$ represent the conversion rate. Finally, $\delta_i, i=1,2$ are the strength of intra-specific competition among the predator species . The initial condition for system (1) may be taken as any point in the region $R_+^3=\{(x,y_1,y_2): x\geq 0, y_1\geq 0, y_2\geq 0\}$. Obviously, the interaction functions in the right hand side of system (1) are continuously differentiable functions on R_+^3, hence they are Lipschitizian. Therefore the solution of system (1) exists and is unique. Further, all the solutions of system (1) with non-negative initial condition are uniformly bounded as shown in the following theorem.

Theorem 1. System (1) is dissipative system on R_+^3.

Proof. It is well known that the dynamical system is dissipative if and only if it is uniformly bounded. Now according to the first equation of system (1) we have

$$\frac{dz}{dt}\leq (a-bz)z$$

Thus by solving the differential inequality:

$$\lim_{t\to\infty} Sup. z(t)\leq \frac{a}{b} \Rightarrow z(t)\leq \frac{a}{b}, \forall t>0$$

Now, consider the function:

$$W=\frac{1}{e_1}x+\frac{1}{e_2}y+z$$

Then

$$\frac{dW}{dt}\leq az-\frac{\mu_1}{e_1}x-\frac{\mu_2}{e_2}y$$

$$\frac{dW}{dt}\leq (a+1)z-(z+\frac{\mu_1}{e_1}x+\frac{\mu_2}{e_2}y)$$

$$\frac{dW}{dt} \le m - \omega W$$

where $m = a + 1$ and $\omega = \min\{1, \mu_1, \mu_2\}$. Therefore, by solving the last differential inequality it is observed that

$$\lim_{t\to\infty} Sup.W(t) \le \frac{m}{\omega} \Rightarrow W(t) \le \frac{m}{\omega}, \forall t > 0$$

Thus all solutions of system (1) are uniformly bounded, and hence the system is dissipative.

3. Existence of equilibrium points and stability analysis.

The system (1) have at most five non-negative equilibrium points, two of them namely $E_0 = (0,0,0)$, $E_z = (0,0,\frac{a}{b})$ always exist. While the existence of other equilibrium points are shown in the following:

The second predator free equilibrium point $E_{xz} = (\hat{x},0,\hat{z})$ exists in $Int.R_+^2$ of $xz-$plane, where

$$\hat{x} = \frac{1}{\delta_1}\left(\frac{e_1 w_1 \gamma_1 \hat{z}}{\hat{\alpha}_1} - \mu_1\right) \qquad (3.1)$$

where $\hat{\alpha}_1 = \hat{z}^2 + \gamma_1 \hat{z} + \gamma_1 \beta_1$, while $\hat{z}$ represents the positive root to the following equation:

$$h_1 z^5 + h_2 z^4 + h_3 z^3 + h_4 z^2 + h_5 z + h_6 = 0 \qquad (3.2)$$

where $h_1 = -\delta_1 b < 0$, $h_2 = \delta_1(a - 2b\gamma_1)$, $h_3 = \delta_1\gamma_1[2(a - b\beta_1) - b\gamma_1]$, $h_4 = \delta_1\gamma_1[a(2\beta_1 + \gamma_1) - (2b\gamma_1\beta_1 - \mu_1 w_1)]$,

$h_5 = \gamma_1^2[\delta_1\beta_1(2a - b\beta_1) - w_1(w_1 e_1 - \mu_1\delta_1)]$ and $h_6 = \delta_1\gamma_1\beta_1[a\gamma_1\beta_1 + \mu_1 w_1)] > 0$.

Obviously, Eq. (3.2) has a unique positive root say $\hat{z}$ provided that one set of the following sets of conditions hold.

$$h_2 < 0,\ h_3 < 0 \ and\ h_5 > 0 \qquad (3.3a)$$

$$h_2 < 0, h_4 > 0 \ and\ h_5 > 0 \qquad (3.3c)$$

Therefore, by substituting $\hat{z}$ in Eq. (3.1), system (1) has a unique equilibrium point in the $Int.R_+^2$ of xz-plane given by $E_{xz} = (\hat{x},0,\hat{z})$, provided that

$$\frac{e_1 w_1 \gamma_1 \hat{z}}{\hat{\alpha}_1} > \mu_1 \qquad (3.4)$$

The first predator free equilibrium point $E_{yz} = (0,\tilde{y},\tilde{z})$ exists in $Int.R_+^2$ of yz-plane, where

$$\tilde{y} = \frac{1}{\delta_2}\left(\frac{e_2 w_2 \gamma_2 \tilde{z}}{\tilde{\alpha}_2} - \mu_2\right) \qquad (3.5)$$

where $\tilde{\alpha}_2 = \tilde{z}^2 + \gamma_2 \tilde{z} + \gamma_2 \beta_2$, while $\tilde{z}$ represents the positive root to the following equation:

$$d_1 z^5 + d_2 z^4 + d_3 z^3 + d_4 z^2 + d_5 z + d_6 = 0 \qquad (3.6)$$

where $d_1 = -\delta_2 b < 0$, $d_2 = \delta_2(a - 2b\gamma_2)$, $d_3 = \delta_2\gamma_2[2(a - b\beta_2) - b\gamma_2]$, $d_4 = \delta_2\gamma_2[a(2\beta_2 + \gamma_2) - (2b\gamma_2\beta_2 - \mu_2 w_2)]$ $d_5 = \gamma_2^2[\delta_2\beta_2(2a - b\beta_2) - w_2(w_2 e_2 - \mu_2\delta_2)]$ and $d_6 = \delta_2\gamma_2\beta_2[a\gamma_2\beta_2 + \mu_2 w_2)] > 0$. Obviously, Eq. (3.6) has a unique positive root say $\tilde{z}$ provided that one set of the following sets of conditions hold.

$$d_2 < 0,\ d_3 < 0 \ and\ d_5 > 0 \tag{3.7a}$$

$$d_2 < 0, d_4 > 0 \ and\ d_5 > 0 \tag{3.7b}$$

Therefore, by substituting $\tilde{z}$ in Eq. (3.5), system (1) has a unique equilibrium point in the $Int.R_+^2$ of yz-plane given by $E_{yz} = (0, \tilde{y}, \tilde{z})$, provided that

$$\frac{e_2 w_2 \gamma_2 \tilde{z}}{\tilde{\alpha}_2} > \mu_2 \tag{3.8}$$

Finally, the coexistence equilibrium point $E_{xyz} = (x^*, y^*, z^*)$ exists in $Int.R_+^3$**,** where

$$x^* = \frac{1}{\delta_1}\left(\frac{e_1 w_1 \gamma_1 z^*}{\alpha_1^*} - \mu_1\right) \tag{3.9}$$

and

$$y^* = \frac{1}{\delta_2}\left(\frac{e_2 w_2 \gamma_2 z^*}{\alpha_2^*} - \mu_2\right) \tag{3.10}$$

While, z^* represents the positive root of each of the following equation:

$$Q_1 z^9 + Q_2 z^8 + Q_3 z^7 + Q_4 z^6 + Q_5 z^5 + Q_6 z^4 + Q_7 z^3 + Q_8 z^2 + Q_9 z + Q_{10} = 0$$
(3.11)

where:

$$Q_1 = -b\delta_1\delta_2$$
$$Q_2 = \delta_1\delta_2[a - 2b(\gamma_1 + \gamma_2]$$
$$Q_3 = \delta_1\delta_2[2a(\gamma_1 + \gamma_2) - b(\gamma_1[2\beta_1 + \gamma_1 + 4\gamma_2] + \gamma_2[2\beta_2 + \gamma_2]))]$$

$$Q_4 = \delta_1\delta_2[a(\gamma_1[2\beta_1 + \gamma_1 + 4\gamma_2] + \gamma_2[2\beta_2 + \gamma_2]) - 2b(\gamma_1^2\beta_1 + \gamma_1\gamma_2[2\beta_1 + \gamma_1 + 2\beta_2 + \gamma_2] + \gamma_2^2\beta_2)]$$

$$Q_5 = \delta_1\delta_2[2a(\gamma_1^2\beta_1 + \gamma_1\gamma_2[2\beta_1 + \gamma_1 + 2\beta_2 + \gamma_2] + \gamma_2^2\beta_2) - b(\gamma_1^2\beta_1^2 + 2\gamma_1\gamma_2[2\gamma_1\beta_1 + 2\beta_1\beta_2 + \gamma_1\beta_2 + \gamma_2\beta_1 + \gamma_1\gamma_2 + 2\gamma_2\beta_2] + \gamma_2^2\beta_2^2)] - w_1\gamma_1\delta_2[e_1w_1\gamma_1 - \mu_1(2\gamma_2 + \gamma_1)] - w_2\gamma_2\delta_1[e_2w_2\gamma_2 - \mu_2(2\gamma_1 + \gamma_2)]$$

$$Q_6 = \delta_1\delta_2[a(\gamma_1^2\beta_1^2 + 2\gamma_1\gamma_2[2\gamma_1\beta_1 + 2\beta_1\beta_2 + \gamma_1\beta_2 + \gamma_2\beta_1 + \gamma_1\gamma_2 + 2\gamma_2\beta_2] + \gamma_2^2\beta_2^2) - 2b(\gamma_1^2\gamma_2\beta_1^2 + \gamma_1\gamma_2^2[\gamma_1\beta_1 + 2\beta_1\beta_2 + \gamma_1\beta_2 + \beta_2^2])] - w_1\gamma_1\gamma_2\delta_2[2(e_1w_1\gamma_1 + \beta_2\mu_1) - \mu_1(\gamma_2 + 2\gamma_1)] - w_2\gamma_2\gamma_1\delta_1[2(e_2w_2\gamma_2 + \beta_1\mu_2) - \mu_2(\gamma_1 + 2\gamma_2)]$$

$$Q_7 = \delta_1\delta_2[2a\gamma_1\gamma_2(\gamma_1\beta_1^2 + \gamma_2[\gamma_1\beta_1 + 2\beta_1\beta_2 + \gamma_1\beta_2 + \beta_2^2]) - b\gamma_1\gamma_2(\beta_1\beta_2[4\gamma_1 + 2\gamma_1\beta_1 + 4\gamma_1\gamma_2 + 2\gamma_2\beta_2] + \gamma_1\gamma_2[\beta_1^2 + \beta_2^2])] - w_1\gamma_1\gamma_2\delta_2[e_1w_1\gamma_1(2\beta_2 + \gamma_2) - \mu_1(2\beta_2[\gamma_1 + \gamma_2] + \gamma_1[\gamma_2 + 2\beta_1])] - w_2\gamma_1\gamma_2\delta_1[e_2w_2\gamma_2(2\beta_1 + \gamma_1) - \mu_2(2\beta_1[\gamma_1 + \gamma_2] + \gamma_2[\gamma_1 + 2\beta_2])]$$

$$Q_8 = \delta_1\delta_2([a(2\gamma_1^2\gamma_2\beta_1\beta_2[2 + \beta_1] + \gamma_1^2\gamma_2^2[\beta_1^2 + 4\beta_1\beta_2 + \beta_2^2] + 2\gamma_1\gamma_2^2\beta_1\beta_2^2] - 2b\gamma_1^2\gamma_2^2\beta_1\beta_2[\beta_1 + \beta_2]) - w_1\gamma_1\gamma_2\delta_2[\gamma_2\beta_2(2e_1w_1\gamma_1 - \mu_1[\beta_2 + 2\gamma_1]) - \mu_1\gamma_1\beta_1(2\beta_2 + \gamma_2)] - w_2\gamma_1\gamma_2\delta_1[\gamma_1\beta_1(2e_2w_2\gamma_2 - \mu_2[\beta_1 + 2\gamma_2]) - \mu_2\gamma_2\beta_2(2\beta_1 + \gamma_1)]$$

$$Q_9 = \delta_1\delta_2[2a\gamma_1^2\gamma_2^2\beta_1\beta_2(\beta_1 + \beta_2) - b\gamma_1^2\gamma_2^2\beta_1^2\beta_2^2 - w_1\gamma_1^2\gamma_2^2\beta_2^2\delta_2[e_1w_1 + \mu_1] - w_2\gamma_1^2\gamma_2^2\beta_1^2\delta_1[e_2w_2 + \mu_2]$$

$$Q_{10} = \gamma_1^2\gamma_2^2[\delta_2\beta_1\beta_2^2(a\delta_1\beta_1 + w_1\mu_1) + w_2\delta_1\beta_1^2\beta_2\mu_2]$$

So by using Descartes rule of signs, Eq. (3.11)has a unique positive root say x^* provided that one set of the following sets of conditions hold:

$$Q_2 < 0, Q_3 < 0, Q_4 < 0, Q_5 < 0, Q_6 < 0, Q_7 < 0 \text{ and } Q_8 < 0 \quad (3.12a)$$

$$Q_2 < 0, Q_3 < 0, Q_4 < 0, Q_5 < 0, Q_6 < 0, Q_7 < 0 \text{ and } Q_9 > 0 \quad (3.12b)$$

$$Q_2 < 0, Q_3 < 0, Q_4 < 0, Q_5 < 0, Q_6 < 0, Q_8 > 0 \text{ and } Q_9 > 0 \quad (3.12c)$$

$$Q_2 < 0, Q_3 < 0, Q_4 < 0, Q_5 < 0, Q_7 > 0, Q_8 > 0 \text{ and } Q_9 > 0 \quad (3.12d)$$

$$Q_2 < 0, Q_3 < 0, Q_4 < 0, Q_6 > 0, Q_7 > 0, Q_8 > 0 \text{ and } Q_9 > 0 \quad (3.12e)$$

$$Q_2 < 0, Q_3 < 0, Q_5 > 0, Q_6 > 0, Q_7 > 0, Q_8 > 0 \text{ and } Q_9 > 0 \quad (3.12f)$$

$$Q_2 < 0, Q_4 > 0, Q_5 > 0, Q_6 > 0, Q_7 > 0, Q_8 > 0 \text{ and } Q_9 > 0 \quad (3.12g)$$

$$Q_3 > 0, Q_4 > 0, Q_5 > 0, Q_6 > 0, Q_7 > 0, Q_8 > 0 \text{ and } Q_9 > 0 \quad (3.12h)$$

Therefore, by substituting z^* in Eqs. (3.9) and (3.10), system (1) has a unique equilibrium point in the $Int.R_+^3$ by $E_{xyz} = (x^*, y^*, z^*)$, provided that

$$\frac{e_1w_1\gamma_1z^*}{\alpha_1^*} > \mu_1 \quad (3.13a)$$

$$\frac{e_2 w_2 \gamma_2 z^*}{\alpha_2^*} > \mu_2 \tag{3.13b}$$

4. Local stability analysis of system (1):

In this section the stability analysis of the above mentioned equilibrium points of system (1) are investigated analytically.

The Jacobian matrix of system (1) at the equilibrium point $E_0 = (0,0,0)$ can be written as $J_0 = J(E_0) = [c_{ij}]_{3\times3}; i, j = 1,2,3$, where $c_{11} = -\mu_1$, $c_{22} = -\mu_2$, $c_{33} = a$ and zero otherwise. Then the eigenvalues of J_0 are:

$$\lambda_{01} = -\mu_1 < 0,\ \lambda_{02} = -\mu_2 < 0,\ \lambda_{03} = a > 0$$

Therefore, the equilibrium point E_0 is a saddle point.

The Jacobian matrix of system (1) at the equilibrium point $E_z = (0,0,\frac{a}{b})$ can be written as $J_z = J(E_z) = [l_{ij}]_{3\times3}; i, j = 1,2,3$, where $l_{11} = \frac{a e_1 w_1 \gamma_1 b}{a^2 + (a+\beta_1 b) b \gamma_1} - \mu_1$, $l_{22} = \frac{a e_2 w_2 \gamma_2 b}{a^2 + (a+\beta_2 b) b \gamma_2} - \mu_2$, $l_{31} = -\frac{a w_1 \gamma_1 b}{a^2 + (a+\beta_1 b) b \gamma_1}$, $l_{32} = -\frac{a w_2 \gamma_2 b}{a^2 + (a+\beta_2 b) b \gamma_2}$, $l_{33} = -a$ and zero otherwise. Hence, the eigenvalues of J_z are:

$$\breve{\lambda}_1 = \frac{a e_1 w_1 \gamma_1 b}{a^2 + (a+\beta_1 b) b \gamma_1} - \mu_1,\ \breve{\lambda}_2 = \frac{a e_2 w_2 \gamma_2 b}{a^2 + (a+\beta_2 b) b \gamma_2} - \mu_2,\ \breve{\lambda}_3 = -a$$

Clearly, E_z is locally asymptotically stable in the R_+^3 if the following two conditions are satisfied

$$\mu_1 > \frac{a e_1 w_1 \gamma_1 b}{a^2 + (a+\beta_1 b) b \gamma_1} \tag{4.1a}$$

$$\mu_2 > \frac{a e_2 w_2 \gamma_2 b}{a^2 + (a+\beta_2 b) b \gamma_2} \tag{4.1b}$$

However, E_z is a saddle point in the R_+^3 if at least one of the following two conditions are satisfied

$$\mu_1 < \frac{a e_1 w_1 \gamma_1 b}{a^2 + (a+\beta_1 b) b \gamma_1} \tag{4.1c}$$

$$\mu_2 < \frac{a e_2 w_2 \gamma_2 b}{a^2 + (a+\beta_2 b) b \gamma_2} \tag{4.1d}$$

So, the Jacobian matrix of system (1) at the equilibrium point $E_{xz} = (\hat{x}, 0, \hat{z})$ in $xz-$plane, can be written in the form: $J_{xz} = J(E_{xz}) = [f_{ij}]_{3\times3}; i, j = 1,2,3$, where

$f_{11} = -\delta_1 \hat{x}$, $f_{13} = \frac{e_1 w_1 \gamma_1 \hat{x}(\gamma_1 \beta_1 - \hat{z}^2)}{\hat{\alpha}_1^2}$, $f_{22} = \frac{e_2 w_2 \gamma_2 \hat{z}}{\hat{\alpha}_2} - \mu_2$, $f_{31} = \frac{-w_1 \gamma_1 \hat{z}}{\hat{\alpha}_1}$, $f_{32} = \frac{-w_2 \gamma_2 \hat{z}}{\hat{\alpha}_2}$, $f_{33} = \hat{z}\left(-b + \frac{w_1 \gamma_1 \hat{x}(2\hat{z} + \gamma_1)}{\hat{\alpha}_1^2}\right)$ and zero otherwise. Clearly, the eigenvalues of J_{xz} are given by:

$$\hat{\lambda}_1 + \hat{\lambda}_3 = -\delta_1 \hat{x} + \hat{z}\left(-b + \frac{w_1 \gamma_1 \hat{x}(2\hat{z} + \gamma_1)}{\hat{\alpha}_1^2}\right)$$

$$\hat{\lambda}_1 . \hat{\lambda}_3 = -\delta_1 \hat{x}\hat{z}\left(-b + \frac{w_1 \gamma_1 \hat{x}(2\hat{z} + \gamma_1)}{\hat{\alpha}_1^2}\right) + \frac{w_1 \gamma_1 \hat{z}}{\hat{\alpha}_1}\left(\frac{e_1 w_1 \gamma_1 \hat{x}(\gamma_1 \beta_1 - \hat{z}^2)}{\hat{\alpha}_1^2}\right)$$

$$\hat{\lambda}_2 = \frac{e_2 w_2 \gamma_2 \hat{z}}{\hat{\alpha}_2} - \mu_2$$

Consequently, E_{xz} is locally asymptotically stable in the R_+^3 if the following conditions are satisfied.

$$b > \frac{w_1 \gamma_1 \hat{x}(2\hat{z} + \gamma_1)}{\hat{\alpha}_1^2} \quad (4.2a)$$

$$\gamma_1 \beta_1 > \hat{z}^2 \quad (4.2b)$$

$$\frac{e_2 w_2 \gamma_2 \hat{z}}{\hat{\alpha}_2} < \mu_2 \quad (4.2c)$$

However, E_{xz} will be unstable point in the R_+^3 if we reversed any one of the above conditions.

The Jacobian matrix of system (1) at the equilibrium point $E_{yz} = (0, \tilde{y}, \tilde{z})$ in $yz-$plane, can be written in the form: $J_{yz} = J(E_{yz}) = [g_{ij}]_{3\times 3}; i, j = 1,2,3$, where $g_{11} = \frac{e_1 w_1 \gamma_1 \tilde{z}}{\tilde{\alpha}_1} - \mu_1$, $g_{22} = -\delta_2 \tilde{y}$, $g_{23} = \frac{e_2 w_2 \gamma_2 \tilde{y}(\gamma_2 \beta_2 - \tilde{z}^2)}{\tilde{\alpha}_2^2}$, $g_{31} = \frac{-w_1 \gamma_1 \tilde{z}}{\tilde{\alpha}_1}$, $g_{32} = \frac{-w_2 \gamma_2 \tilde{z}}{\tilde{\alpha}_2}$, $g_{33} = \tilde{z}\left(-b + \frac{w_2 \gamma_2 \tilde{y}(2\tilde{z} + \gamma_2)}{\tilde{\alpha}_2^2}\right)$ and zero otherwise. Clearly the eigenvalues of J_{yz} are given by:

$$\tilde{\lambda}_1 = \frac{e_1 w_1 \gamma_1 \tilde{z}}{\tilde{\alpha}_1} - \mu_1$$

$$\tilde{\lambda}_2 + \tilde{\lambda}_2 = -\delta_2 \tilde{y} + \tilde{z}\left(-b + \frac{w_2 \gamma_2 \tilde{y}(2\tilde{z} + \gamma_2)}{\tilde{\alpha}_2^2}\right)$$

$$\hat{\lambda}_2 . \hat{\lambda}_3 = -\delta_2 \tilde{y}\tilde{z}\left(-b + \frac{w_2 \gamma_2 \tilde{y}(2\tilde{z} + \gamma_2)}{\tilde{\alpha}_2^2}\right) + \frac{w_2 \gamma_2 \tilde{z}}{\tilde{\alpha}_2}\left(\frac{e_2 w_2 \gamma_2 \tilde{y}(\gamma_2 \beta_2 - \tilde{z}^2)}{\tilde{\alpha}_2^2}\right)$$

Consequently, E_{yz} is locally asymptotically stable in the R_+^3 if the following conditions are satisfied:

$$\frac{e_1 w_1 \gamma_1 \tilde{z}}{\tilde{\alpha}_1} < \mu_1 \tag{4.3a}$$

$$b > \frac{w_2 \gamma_2 \tilde{y}(2\tilde{z} + \gamma_2)}{\tilde{\alpha}_2^2} \tag{4.3b}$$

$$\gamma_2 \beta_2 > \tilde{z}^2 \tag{4.3c}$$

Moreover, E_{yz} is unstable point in the R_+^3 if we reversed any one of the above conditions.

Finally, the Jacobian matrix of the system (1) at the coexistence equilibrium point $E_{xyz} = (x^*, y^*, z^*)$ in the $Int.R_+^3$ can be written as:

$$J_{xyz} = J(E_{xyz}) = \lfloor a_{ij} \rfloor_{3\times 3}; i, j = 1,2,3 \tag{4.4}$$

where $a_{11} = -\delta_1 x^*$, $a_{13} = \frac{e_1 w_1 \gamma_1 x^* (\gamma_1 \beta_1 - z^{*2})}{\alpha_1^{*2}}$, $a_{22} = -\delta_2 y^*$,

$a_{23} = \frac{e_2 w_2 \gamma_2 y^* (\gamma_2 \beta_2 - z^{*2})}{\alpha_2^{*2}}$, $a_{31} = \frac{-w_1 \gamma_1 z^*}{\alpha_1^*}$, $a_{32} = \frac{-w_2 \gamma_2 z^*}{\alpha_2^*}$,

$a_{33} = z^* \left(-b + \frac{w_1 \gamma_1 y^* (2z^* + \gamma_1)}{\alpha_1^{*2}} + \frac{w_2 \gamma_2 y^* (2z^* + \gamma_2)}{\alpha_2^{*2}} \right)$ and zero otherwise. Therefore the characteristic equation of J_{xyz} is

$$\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0 \tag{4.5}$$

where

$$A_1 = -(a_{11} + a_{22} + a_{33})$$
$$A_2 = a_{11}a_{22} + a_{11}a_{33} - a_{13}a_{31} - a_{13}a_{31} + a_{22}a_{33} - a_{23}a_{32}$$
$$A_3 = a_{11}(a_{23}a_{32} - a_{22}a_{33}) + a_{13}a_{22}a_{31}$$

And

$$\begin{aligned} \Delta &= A_1 A_2 - A_3 \\ &= -(a_{11} + a_{22})[a_{11}a_{22} + a_{33}(a_{11} + a_{22} + a_{33})] \\ &\quad + a_{13}a_{31}[a_{11} + a_{33}] + a_{23}a_{32}(a_{22} + a_{33}) \end{aligned}$$

Therefore, in the following theorem, the local stability conditions for the positive equilibrium point E_{xyz} in the $Int.R_+^3$ are established.

Theorem 2. Assume that E_{xyz} exists in the $Int.R_+^3$ and the following conditions are satisfied;

$$b > \frac{w_1\gamma_1 y^*(2z^*+\gamma_1)}{\alpha_1^{*2}} + \frac{w_2\gamma_2 y^*(2z^*+\gamma_2)}{\alpha_2^{*2}} \quad (4.6a)$$

$$z^{*2} < \min.\{\gamma_1\beta_1, \gamma_2\beta_2\} \quad (4.6b)$$

Then it is locally asymptotically stable.

Proof: According to the Routh-Hawirtiz criterion the characteristic equation (4.5) has roots with negative real parts if and only if $A_1 > 0$, $A_3 > 0$ and $\Delta > 0$.

Note that, it is easy to verify that, condition (4.6a) guarantees that $A_1 > 0$; while conditions (4.6a)and (4.6b) ensure the positivity of A_3 (i.e. $A_3 > 0$) and Δ. Hence, all the roots (eigenvalues) of the J_{xyz} have negative real parts. Therefore E_{xyz} is locally asymptotically stable in the $Int.R_+^3$ and hence the proof is complete. ■

Now, before go further to study the global dynamical behavior of system (1) in the $Int.R_+^3$, we will discuss the dynamical behavior of system (1) in the interior of the boundary planes as shown in the following theorems.

Theorem 3. Suppose that the equilibrium points E_{xz} and E_{yz} are locally asymptotically stable in the $Int.R_+^2$ of $xz-$ and $yz-$planes provided that

$$b > \frac{w_1\gamma_1 x(2z+\gamma_1)}{\alpha_1^2} \quad (4.7a)$$

$$b > \frac{w_2\gamma_2 y(2z+\gamma_2)}{\alpha_2^2} \quad (4.7b)$$

respectively, then E_{xz} and E_{yz} are a globally asymptotically stable in $Int.R_+^2$ of $xz-$ and $yz-$planes respectively.

Proof. The proof follows directly by using Bendixson-Dulic criterion with Dulic function $1/xz$ and $1/yz$, then by using Poincare-Bendixson theorem.

5. Global dynamical behavior of system (1).

In this section the global dynamics of system (1) near the equilibrium points E_z, E_{xz}, E_{yz} and E_{xyz} are investigated with the help of Lyapunov function as shown in the following theorems.

In the following theorem the global stability condition of $E_z = (0,0,\breve{z})$ with $\breve{z} = \frac{a}{b}$ is established.

Theorem 4. Suppose that the equilibrium point $E_z = (0,0,\breve{z})$ is locally asymptotically stable and let the following condition holds.

$$\breve{z} \le \min.\left\{\frac{\mu_1\beta_1}{e_1 w_1}, \frac{\mu_2\beta_2}{e_2 w_2}\right\} \quad (5.1)$$

Then it is a globally asymptotically stable point.

Proof: Consider the following positive definite Lyapunov function about E_z

$$V_1(x,y,z) = \frac{x}{e_1} + \frac{y}{e_2} + \left(z - \breve{z} - \breve{z}\ln\frac{z}{\breve{z}} \right)$$

Clearly, V_1 is a continuously differentiable real valued function defined on R^3_+. Further, we have

$$\frac{dV_1}{dt} \le -\left(\frac{\mu_1}{e_1} - \frac{w_1\gamma_1\breve{z}}{\gamma_1\beta_1} \right)x - \left(\frac{\mu_2}{e_2} - \frac{w_2\gamma_2\breve{z}}{\gamma_2\beta_2} \right)y - b(z-\breve{z})^2$$

Therefore, $\frac{dV_1}{dt} < 0$ under condition (5.1), and hence V_1 is strictly Lyapunov function. Therefore, E_z is globally asymptotically stable in the R^3_+.

Theorem 5. Suppose that the equilibrium point $E_{xz} = (\hat{x}, 0, \hat{z})$ is locally asymptotically stable and let the following condition holds.

$$b > \frac{w_1\gamma_1\hat{x}(z+\hat{z}+\gamma_1)}{\alpha_1\hat{\alpha}_1} \quad (5.2a)$$

$$\left(\frac{e_1 w_1 \gamma_1(\gamma_1\beta_1 - z\hat{z})}{\alpha_1\hat{\alpha}_1} - \frac{w_1\gamma_1}{\alpha_1} \right)^2 < \delta_1\left(b - \frac{w_1\gamma_1\hat{x}(z+\hat{z}+\gamma_1)}{\alpha_1\hat{\alpha}_1} \right) \quad (5.2b)$$

$$\frac{\mu_2}{e_2} \ge \frac{w_2\hat{z}}{\beta_2} \quad (5.2c)$$

Then it is a globally asymptotically stable point.

Proof. Consider the following positive definite Lyapunov function about E_{xz}

$$V_2(x,y,z) = \left(x - \hat{x} - \hat{x}\ln\frac{x}{\hat{x}} \right) + \frac{y}{e_2} + \left(z - \hat{z} - \hat{z}\ln\frac{z}{\hat{z}} \right)$$

Clearly, V_2 is a continuously differentiable real valued function defined on R^3_+. Further, we have

$$\frac{dV_2}{dt} \le -\left[\sqrt{\delta_1}(x-\hat{x}) - \sqrt{b - \frac{w_1\gamma_1\hat{x}(z+\hat{z}+\gamma_1)}{\alpha_1\hat{\alpha}_1}}(z-\hat{z}) \right]^2 - \left(\frac{\mu_2}{e_2} - \frac{w_2\hat{z}}{\beta_2} \right)y$$

According to the above, conditions (5.2a)-(5.2c) guarantee that $\frac{dV_2}{dt} < 0$ for any point in R^3_+, and hence E_{xz} is globally asymptotically stable in R^3_+. ■

Theorem 6. Suppose that the equilibrium point $E_{yz} = (0, \tilde{y}, \tilde{z})$ is locally asymptotically stable and let the following condition holds.

$$b > \frac{w_2\gamma_2\tilde{y}(z+\tilde{z}+\gamma_2)}{\alpha_2\tilde{\alpha}_2} \tag{5.3a}$$

$$\left(\frac{e_2 w_2\gamma_2(\gamma_2\beta_2 - z\tilde{z})}{\alpha_2\tilde{\alpha}_2} - \frac{w_2\gamma_2}{\alpha_2}\right)^2 < \delta_2\left(b - \frac{w_2\gamma_2\tilde{y}(z+\tilde{z}+\gamma_2)}{\alpha_2\tilde{\alpha}_2}\right) \tag{5.3b}$$

$$\frac{\mu_1}{e_1} \geq \frac{w_1\tilde{z}}{\beta_1} \tag{5.3c}$$

Then it is a globally asymptotically stable point.

Proof. Consider the following positive definite Lyapunov function about E_{yz}

$$V_3(x,y,z) = \frac{x}{e_1} + \left(y - \tilde{y} - \tilde{y}\ln\frac{y}{\tilde{y}}\right) + \left(z - \tilde{z} - \tilde{z}\ln\frac{z}{\tilde{z}}\right)$$

Clearly, V_3 is a continuously differentiable real valued function defined on R_+^3. Further, we have

$$\frac{dV_3}{dt} \leq -\left[\sqrt{\delta_2}(y-\tilde{y}) - \sqrt{b - \frac{w_2\gamma_2\tilde{y}(z+\tilde{z}+\gamma_2)}{\alpha_2\tilde{\alpha}_2}}(z-\tilde{z})\right]^2 - \left(\frac{\mu_1}{e_1} - \frac{w_1\tilde{z}}{\beta_1}\right)x$$

According to the above, conditions (5.3a)-(5.3c) guarantee that $\frac{dV_3}{dt} < 0$ for any point in R_+^3, and hence E_{yz} is globally asymptotically stable in R_+^3. ■

Finally the global stability of the coexistence equilibrium point of system (1) is investigated in the following theorem.

Theorem 7. Suppose that the equilibrium point $E_{xyz} = (x^*, y^*, z^*)$ is locally asymptotically stable and let the following condition holds.

$$r_{33} > 0 \tag{5.6a}$$

$$r_{13}^2 < r_{11}r_{33} \tag{5.6b}$$

$$r_{23}^2 < r_{22}r_{33} \tag{5.6c}$$

here we have:

$$r_{11}=\delta_1,\ r_{13}=\frac{e_1 w_1\gamma_1(\gamma_1\beta_1-zz^*)}{\alpha_1\alpha_1^*}-\frac{w_1\gamma_1}{\alpha_1},$$

$$r_{22}=\delta_2,\ r_{23}=\frac{e_2 w_2\gamma_2(\gamma_2\beta_2-zz^*)}{\alpha_2\alpha_2^*}-\frac{w_2\gamma_2}{\alpha_2},$$

$$r_{33}=b-\frac{w_1\gamma_1 x^*(z+z^*+\gamma_1)}{\alpha_1\alpha_1^*}-\frac{w_2\gamma_2 y^*(z+z^*+\gamma_2)}{\alpha_2\alpha_2^*}$$

Then it is a globally asymptotically stable point.

Proof: Consider the following positive definite Lyapunov function about E_{xyz}

$$V_4(x,y,z)=\left(x-x^*-x^*\ln\frac{x}{x^*}\right)+\left(y-y^*-y^*\ln\frac{y}{y^*}\right)+\left(z-z^*-z^*\ln\frac{z}{z^*}\right)$$

Clearly, V_4 is a continuously differentiable real valued function defined on $Int.R_+^3$. Further, we have

$$\frac{dV_4}{dt}\le -r_{11}(x-x^*)^2-r_{13}(x-x^*)(z-z^*)-r_{22}(y-y^*)^2$$
$$-r_{23}(y-y^*)(z-z^*)-r_{33}(z-z^*)^2$$

Obviously, due to conditions (5.6a)-(5.6c), we get that

$$\frac{dV_4}{dt}\le-\left[\sqrt{r_{11}}(x-x^*)-\sqrt{\frac{r_{33}}{2}}(z-z^*)\right]^2-\left[\sqrt{r_{22}}(y-y^*)-\sqrt{\frac{r_{33}}{2}}(z-z^*)\right]^2$$

Clearly $\frac{dV_4}{dt}<0$, therefore the origin and then E_{xyz} is locally asymptotically stable point in the $Int.R_+^3$ and hence the proof is complete. ■

6. Persistence Analysis

In this section, the persistence of system (1) is studied. It is well known that the system is said to be persistence if and only if each species persists. Mathematically this is meaning that the solution of system (1) do not have omega limit set in the boundaries of R_+^3 [5]. Therefore, in the following theorem, the necessary and sufficient conditions for the uniform persistence of the system (1) are derived.

Theorem 8. Assume that there are no periodic dynamics in the boundary planes xz and yz respectively. Further, if in addition to conditions (4.1c), (4.1d) the following conditions are hold.

$$\frac{e_2 w_2\gamma_2\hat{z}}{\hat{\alpha}_2}>\mu_2 \tag{6.1}$$

$$\frac{e_1 w_1 \gamma_1 \tilde{z}}{\tilde{\alpha}_1} > \mu_1 \tag{6.2}$$

Then, system (1) is uniformly persistence.

Proof: Consider the function $\sigma(x,y,z) = x^{p_1} y^{p_2} z^{p_3}$, where $p_i; i = 1,2,3$ are an undetermined positive constants. Obviously $\sigma(x,y,z)$ is a C^1 positive function defined in R_+^3, and $\sigma(x,y,z) \to 0$ if $x \to 0$ or $y \to 0$ or $z \to 0$. Consequently we obtain

$$\Psi(x,y,z) = \frac{\sigma'(x,y,z)}{\sigma(x,y,z)} = p_1 f_1 + p_2 f_2 + p_3 f_3$$

Here $f_i; i = 1,2,3$ are given in system (1). Therefore

$$\begin{aligned}\Psi(x,y,z) = {} & p_1\left(\frac{e_1 w_1 \gamma_1 z}{\alpha_1} - \mu_1 - \delta_1 x\right) \\ & + p_2\left(\frac{e_2 w_2 \gamma_2 z}{\alpha_2} - \mu_2 - \delta_2 y\right) \\ & + p_3\left(a - bz - \frac{w_1 \gamma_1 x}{\alpha_1} - \frac{w_2 \gamma_2 y}{\alpha_2}\right)\end{aligned}$$

Now, since it is assumed that there are no periodic attractors in the boundary planes, then the only possible omega limit sets of the system (1) are the equilibrium points E_0, E_z, E_{xz} *and* E_{yz}. Thus according to the Gard technique [5] the proof is follows and the system is uniformly persists if we can proof that $\Psi(.) > 0$ at each of these points. Since

$$\Psi(E_0) = ap_3 - \mu_2 p_2 - \mu_1 p_1 \tag{6.3a}$$

$$\Psi(E_z) = \left(\frac{ae_1 w_1 \gamma_1 b}{a^2 + (a + \beta_1 b) b \gamma_1} - \mu_1\right) p_1 + \left(\frac{ae_2 w_2 \gamma_2 b}{a^2 + (a + \beta_2 b) b \gamma_2} - \mu_2\right) p_2 \tag{6.3b}$$

$$\Psi(E_{xz}) = \left(\frac{e_2 w_2 \gamma_2 \hat{z}}{\hat{\alpha}_2} - \mu_2\right) p_2 \tag{6.3c}$$

$$\Psi(E_{yz}) = \left(\frac{e_1 w_1 \gamma_1 \tilde{z}}{\tilde{\alpha}_1} - \mu_1\right) p_1 \tag{6.3d}$$

Obviously, $\Psi(E_0) > 0$ for the value of $p_3 > 0$ sufficiently large than $p_i; i = 1,2$. $\Psi(E_z) > 0$ for any positive constants $p_i; i = 1,2$ provided that conditions (4.1c)and (4.1d) hold. However, $\Psi(E_{xz})$ and $\Psi(E_{yz})$ are positive provided that the conditions (6.1) and (6.2) are satisfied respectively. Then strictly positive solution of system (1) do not have omega limit set and hence, system (1) is uniformly persistence. ■

7. Numerical Simulation

In this section the global dynamics of system (1) is investigated numerically. The system is solved numerically for different sets of parameters values and for different sets of initial conditions, and then the attracting sets and their time series are drown.

For the following set of parameters

$$a = 0.25, b = 0.2, w_1 = 1, w_2 = 1, \gamma_1 = 0.75, \gamma_2 = 0.75, \beta_1 = 2, \beta_2 = 2,$$
$$e_1 = 0.35, e_2 = 0.4, \delta_1 = 0.03, \delta_2 = 0.08, \mu_1 = 0.02, \mu_2 = 0.02.$$

(7.1)

The attracting sets along with their time series of system (1) are drown in Fig (1). Note that from now onward, in the time series figures, we will use the following representation: **blue color** represents the trajectory of the first predator, **green color** represents the trajectory of the second predator and the **red color** represents the trajectory of the prey.

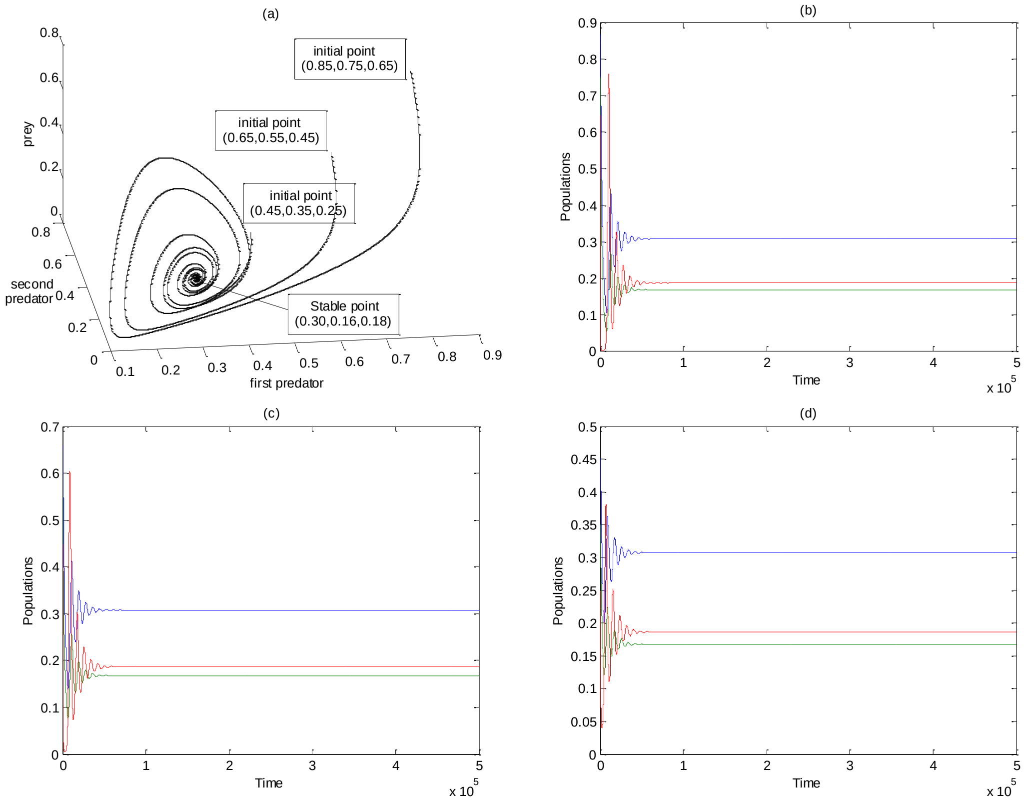

Fig. (1): The phase plot of system (1). (a) The solution of system (1) approaches asymptotically to stable positive point initiated at different initial points. (b) Time series of the attractor in (a) initiated at (0.85,0.75,0.65). (c) Time series of the attractor in (a) initiated at (0.65,0.55,0.45). (d) Time series of the attractor in (a) initiated at (0.45,0.35,0.25).

Obviously, these figure show that, the system (1) approaches to the globally asymptotically to coexistence equilibrium point $E_{xyz} = (0.30, 0.16, 0.18)$ in the $Int.R^3_+$ starting from different sets of initial conditions. However, for the set of parameters values (7.1) with $a = 0.5$, system (1) approaches to the globally asymptotically stable limit cycle in the $Int.R^3_+$ starting from different sets of initial conditions, see **Fig. (2).**

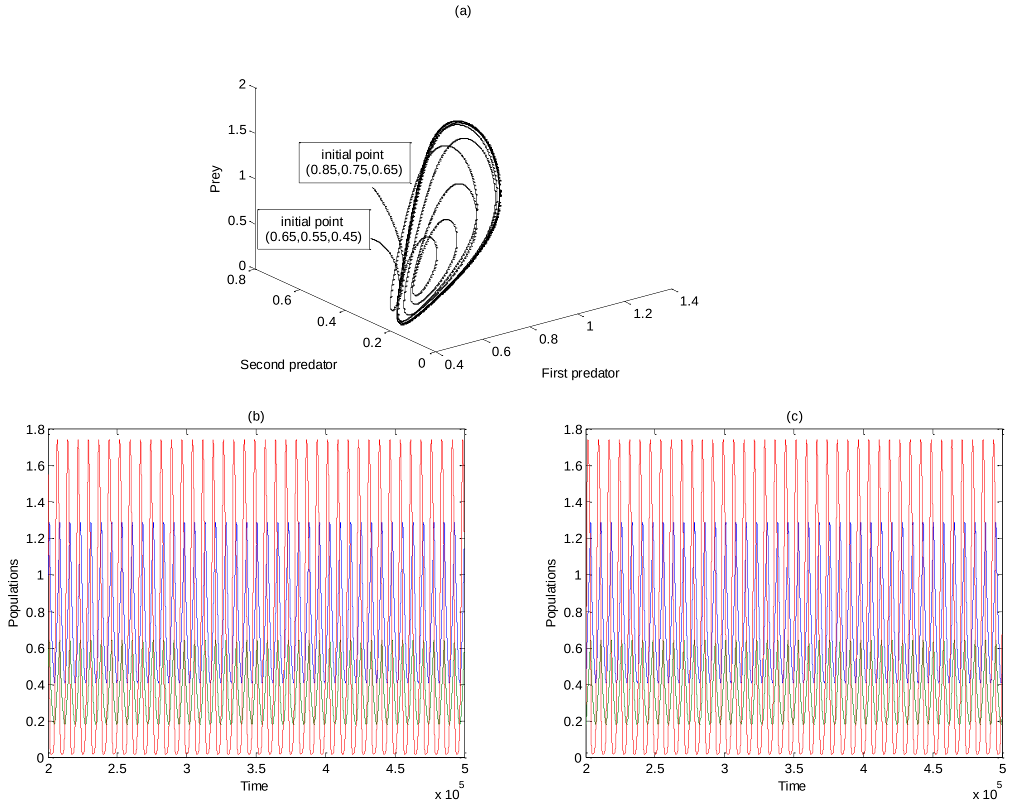

Fig. (2): The phase plot of system (1). (a) The solution of system (1) approaches asymptotically to stable limit cycle initiated at different initial points. (b) Time series of the attractor in (a) initiated at (0.85,0.75,0.65). (c) Time series of the attractor in (a) initiated at (0.65,0.55,0.45).

Further analysis for the role of changing in the value of the parameter a keeping the rest of parameters values as in Eq. (7.1), it observed that for $a \leq 0.02$ and $a \geq 2.72$, system (1) approaches asymptotically to stable point $E_z = (0,0,\frac{a}{b})$, as shown in **Fig.(3)**, while for $0.05 \leq a \leq 0.43$, the solution of system (1) has a globally asymptotically stable positive point, however for $0.44 \leq a \leq 0.67$ the solution approaches to periodic dynamic in the $Int.R^3_+$, further for $0.68 \leq a \leq 2.35$ the solution of system (1) approaches asymptotically to positive point, finally $2.43 \leq a \leq 2.71$, the solution of system (1) approaches to $E_{yz} = (0,\tilde{y},\tilde{z})$ in the interior of positive quadrant of $yz-$plane.

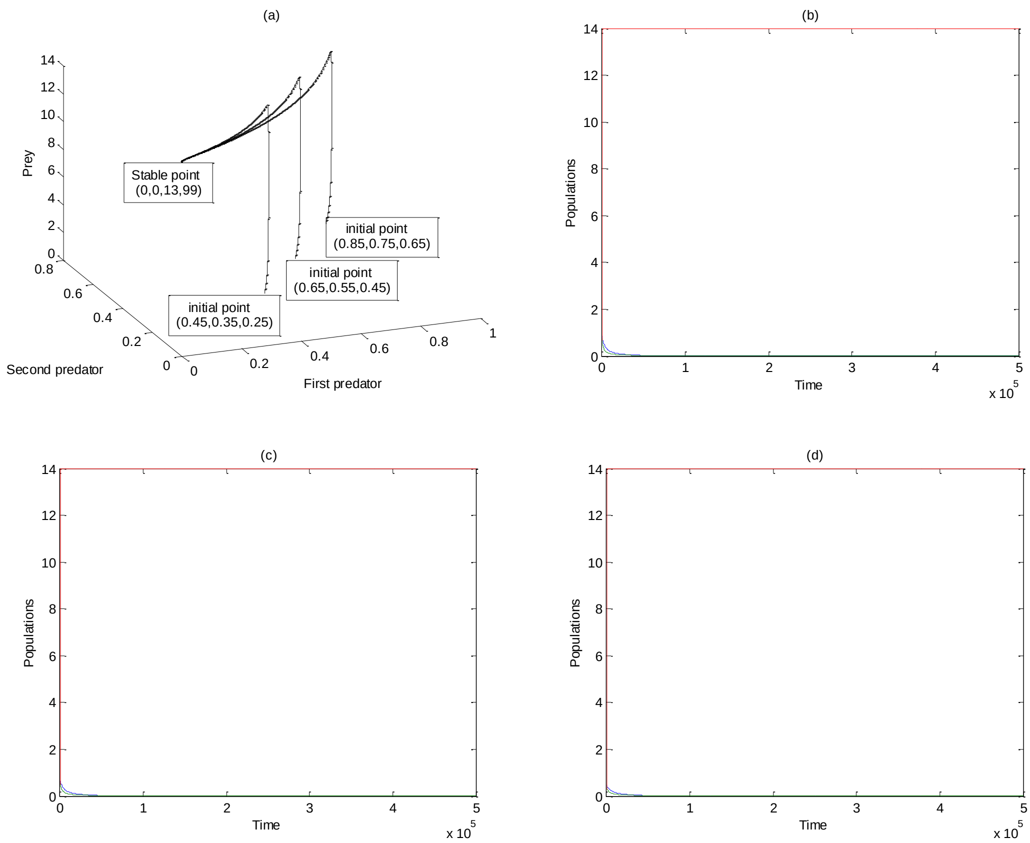

Fig. (3): The phase plot of system (1). (a) The solution of system (1) approaches asymptotically to stable equilibrium point E_Z initiated at different initial points. (b) Time series of the attractor in (a) initiated at (0.85,0.75,0.65). (c) Time series of the attractor in (a) initiated at (0.65,0.55,0.45). (d) Time series of the attractor in (a) initiated at (0.45,0.35,0.25).

For the parameters values given in Eq. (7.1) with varying e_1 in the range $e_1 \le 0.14$, the solution approaches to $E_{yz} = (0, \tilde{y}, \tilde{z})$ in the interior of positive quadrant of yz – plane, as shown in **Fig.(4)**, however for $0.15 \le e_1 \le 0.66$, the solution approaches to a positive equilibrium point, finally for $e_1 \ge 0.67$, system (1) approaches asymptotically to the equilibrium point $E_{xz} = (\hat{x}, 0, \hat{z})$ in the interior of positive quadrant of xz – plane, as shown in **Fig.(5)**.

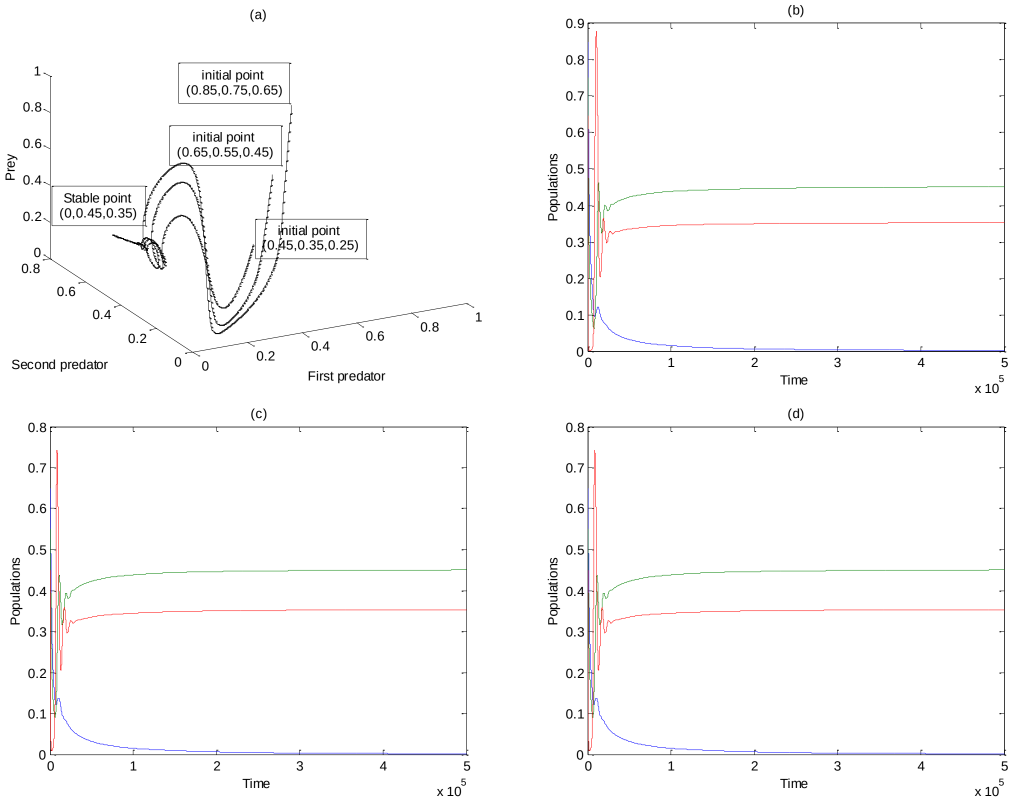

Fig. 4: The phase plot of system (1) with $e_1 = 0.14$. (a) The solution of system (1) approaches asymptotically to $E_{yz} = (0,0.45,0.35)$ initiated at different initial points. (b) Time series of the attractor in (a) initiated at (0.85,0.75,0.65). (c) Time series of the attractor in (a) initiated at (0.65,0.55,0.45). (d) Time series of the attractor in (a) initiated at (0.45,0.35,0.25).

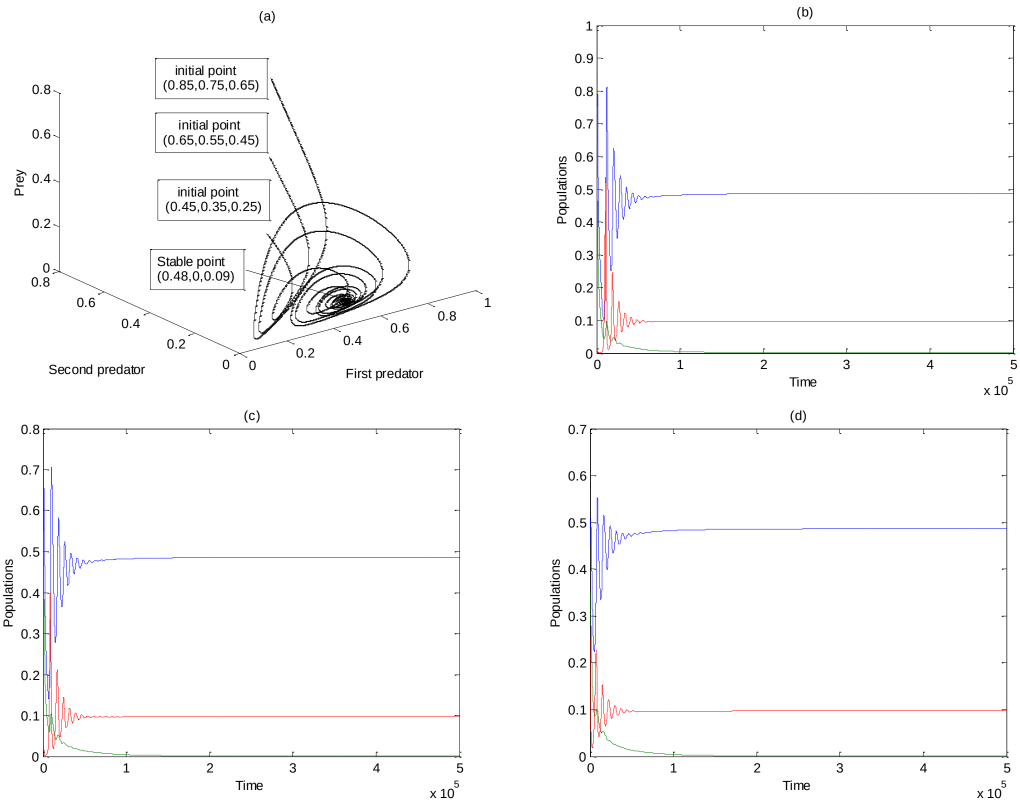

Fig. 5: The phase plot of system (1) with $e_1 = 0.75$. (a) The solution of system (1) approaches asymptotically to $E_{xz} = (0.48, 0, 0.09)$ initiated at different initial points. (b) Time series of the attractor in (a) initiated at (0.85,0.75,0.65). (c) Time series of the attractor in (a) initiated at (0.65,0.55,0.45). (d) Time series of the attractor in (a) initiated at (0.45,0.35,0.25).

8. Conclusions and Discussion

In this paper, a mathematical model consisting of a Holling type IV prey predator model has proposed and analyzed. The model consists of three non-linear autonomous differential equations that describe the dynamics of three different population namely first predator x, second predator y, prey z. The boundedness of the system (1) has been discussed. The dynamical behavior of system (1) has been investigated locally as well as globally.

To understand the effect of varying parameter on the global dynamics of system (1) and to confirm our obtained analytical results, system (1) has been solved numerically and the following results are obtained:

1. For the set of hypothetical parameters values given Eq. (7.1), the system (1) approaches asymptotically to globally stable positive equilibrium point $E_{xyz} = (x^*, y^*, z^*)$.
2. The intrinsic growth rate of system (1) plays a vital role on the persistence of the system. In fact, for the small values and large values of the parameter a the predator facing extinction. However for suitable choice of

this parameter, the system (1) still persists and has either stable point or else periodic dynamics.

3. Finally, the conversion rate e_1 decreases keeping other parameters as in Eq. (7.1) then the first predator will faces extinction and the solution of system (1) approaches asymptotically to the equilibrium point $E_{yz} = (0, \tilde{y}, \tilde{z})$. However, increasing e_1 causes extinction in the second predator and the solution of system (1) approaches to the equilibrium point $E_{xz} = (\hat{x}, 0, \hat{z})$.

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