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Solution of a broad Class Singular Boundary Value Problem by Variational Iteration Method

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Abstract

In this paper an iterative method for finding a compatible solution to a class of singular second order differential equation of prescribed boundary values often observed common is considered by constructing a successive sequence of correction functional via variational theory. The analytical convergence of such iteratively generated sequential scheme is analyzed explicitly and duly discussed. Interestingly, the proposed method when applied on, over hither to widely quote numerical problems turns out to be quite encouraging and renders appropriate solution. May sometimes by this method the limiting value of functional sequence happens to be an exact solution too.

Keywords; singular problem, Variational iteration method, Convergence, sequence, smooth function Lagrange multiplier, linearization, discretization, transformation.

Introduction

The comprehensive behavior and the spectrum of all basic qualities systematically associated and properly distributed over to a class of events, situations or any other sudden precarious happening being observed on various fronts of all multidisciplinary sciences just like celestial bodies either internally or externally or both ways simultaneously may be completely realized or discerned and visualized abstractly by modeling it mathematically into a class of singular second order boundary value problem .To ascertain the inherent spectral characteristics in and around all thereof, a feasible and sustainable coherent solution either numerically appropriate or analytically of exact form is must and equally important to be worked out possible for such class accordingly by applying any consistent method of solution and after all applied so. The vital phenomenon of human physiology like tumor growth in a body, kinetics of oxygen uptake to name a few and many more phenomenons like transport processes and thermal explosions span and represent to a class of singular boundary value problems[3-5] of type

$$(x^{\alpha} g(x) y^{/})^{/} = x^{\alpha} g(x) f(x, y) \qquad 0 < x \leq 1 \ , \ 0 \leq \alpha < 1 \qquad (1.1)$$

$$y(0) = A \ , \ y(1) = B \qquad (1.11)$$

or

$$y(0) = A \ , \ a\, y(1) + b\, y^{/}(1) = B \qquad (1.12)$$

Where in (1.1) let p $(x)=x^{\alpha}$ g (x), g (0) ≠0 , 0≤ α < 1 and in (1.11), (1.12) a > 0 , b≥ 0

A, B are finite constants .The function f (x,y) is a real valued continuous function of two variables x *and* y such that $(x,y) \in \mathbb{R}\times\mathbb{R}$ and that $\frac{\partial f}{\partial y}$ is a nonnegative and continuous function in a domain **R** = $\{(x,y):(x,y)\in[0\ 1]\times\mathbb{R}\}$.The problem (1.1) is singular and x=0 is its singularity since real valued function p(x) =0 at x=o .The function p(x) further satisfies (i) p(x) >o ∀ x ∈(01], (ii) p(x)∈ C^1 [01]. Solution to such class of problems exists [6-7] .The class of problems (1.1) form a specific area of the field of differential equation and hither to been a matter of immense research and field of keen interest to learned authors. Several methods had had been applied on to such important class of boundary value problems [8-13].

Variation Iteration Method (VIM)

The variational iteration method, a modified Lagrange method [14] was originally proposed by He [15-17].It is a highly promising and profusely used method for solving problems of various manifolds in applied sciences as an optional method different from other existing methods of linearization, transformation and discretization. The proposed method has fared well and apply over a large class of mathematically modeled problems .Credit accrue to VIM for solving a class of distinguished and challenging problems like, nonlinear coagulation problem with mass loss , an approximate solution for one dimensional weakly nonlinear oscillations, nonlinear thermo elasticity, cubic nonlinear Schrodinger equation, nonlinear oscillators with discontinuities ,Burger's and coupled Burger's equation, multispecies Lotaka–Volterra equations, rational solution of Toda lattice equation, Helmholtz equation, generalized KdV equation[18-29].The basic virtues and fundamentals associated to variation iteration method may be synthesized by considering a general differential equation in operator form as follows:

$Dy(x) = g(x), x \in I \subseteq \mathbb{R}$, D, being usual differential operator (2.1)

The solution function $y(x)$ is sufficiently smooth on some domain Ω and g (x) is an inhomogeneous real valued function. In order to start the generation of correction functional when L and N are linear and nonlinear differential operators respectively the relation (2.1) can be rewritten as

$L(y(x)) + N(y(x)) = g(x) \quad x \in I \subseteq \mathbb{R}$ (2.2)

Exclusively, the variation iteration method has natural error absorbing resilience and buoyancy with capability to generate a recursive sequence of correction functionals that really conserve power and merit potential in minimizing magnitude of processing error at every iterative step extracting out a just and acceptable solution to the given class of problems (1.1).The sequence of correctional functional for(2.2) is

$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\,((L\,(y_n(s)) + N\,(\widetilde{y_n(s)})) - g\,(s))\,ds$, n≥0 (2.3)

Where λ stands for Lagrange multiplier determined optimally satisfying all stationary conditions after the variational method is applied to (2.3). The resounding utility of method all over lies with the assumption and choice of considering the interrelated inconvenient highly nonlinear and complicated dependent variables as restricted variables thereby help minimizing the effort of simplification to the ensuing and evolving solution process to the general problem(1.1).As aforementioned, $\widetilde{y_n}$ is the restricted variation, which means $\delta\widetilde{y_n}$=0.Eventually, after λ is determined , a proper and suitable selective function may it be a linear one or otherwise with respect to (2.2) is assumed as an initial approximation for finding next successive iterative function by recursive sequence of correction functionals. Thereafter boundary conditions are imposed on the final or preferably on limiting value (as $n \to \infty$) of sequential approximations incurred after due process of iteration that continued and proceeded on. Moreover, in VIM the selective function is arbitrary and has flexibility of self adjusting Insertion choice for initial input solution generating process .Therefore this very method is again beneficial in reducing the burden of cumbersome component calculation in the opted methodology.

Variational Method and Lagrange Multiplier

The variational method and Lagrange multiplier are convoluted corresponding to (1.1) by the iterative and successive correction functional relation as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\,(s^\alpha g(s) y_n^{/}(s))^{/} - s^\alpha \widetilde{g(s)} f(s, y_n(s)))\,ds \qquad n\geq 0 \qquad (3.1)$$

Where $y_n(x)$ is n^{th} approximated iterative solution of (1.1). suppose optimal value of μ(s) is identified naturally by taking variation with respect to y_n(x) and subject to restricted variation $\delta\widetilde{y_n}$(x) =0.Then consequently from (3.1) we have,

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda(s)((s^\alpha g(s) y_n^{/})^{/} - s^\alpha \widetilde{g(s)} f(s, y_n(s))\,ds \qquad n\geq 0 \qquad (3.2)$$

Now integrating by parts and considering the restricted variation of y_n (i.e. δy_n=0) as well relation (3.2) simplifies to give,

$$\delta y_n(x) = (1 - \lambda^{/}(s) s^\alpha g(s))\,\delta y_n(x) + \delta(\lambda(s) s^\alpha g(s)\, y_n^{/}(s))\,|_{s=x} + \int_0^x (\lambda^{/}(s)\, s^\alpha g(s))^{/} \delta y_n(s)\,ds, \quad n \geq 0$$

Therefore, the stationary conditions are,

$1-\lambda^{/}(s)\ s^\alpha g(s) = 0,\ \lambda(x) = 0,\ (\lambda^{/}(s) s^\alpha g(s))^{/} = 0$ and that implies together to give,

$$\lambda(s) = \int_x^s \frac{1}{\zeta^\alpha g(\zeta)}\,d\zeta \qquad (3.3)$$

Therefore, from (3.1) the sequence of correction functionals is given by

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(\int_x^s \frac{1}{p(\zeta)}\,d\zeta\right) ((s^\alpha g(s) y_n(s))^{/} - s^\alpha g(s) \widetilde{f(s, y_n(s))}\,ds \qquad n\geq 0 \qquad (3.4)$$

Now it may be deduced from (3.4) that the limit of the convergent iterative sequence $\{y_n\}_{n=1}^{\infty}$ if it converges on, satisfying given boundary conditions is the exact solution to (1.1).

Convergence of Iterative Sequence

At the outset, the convergence of the sequence of correctional functionals generated on by execution of VIM with respect to given class (1.1) in view of (3.1) may be established by observing and considering that

$y_{n+1(x)} = y_n(x) + \sum_{k=0}^{n-1}(y_{k+1}(x) - y_k(x))$ is the n^{th} partial sum of the infinite series
$$y_0(x) + \sum_{k=0}^{\infty}(y_{k+1}(x) - y_k(x)) \quad (4.1)$$

And the convergence of infinite series (4.1) necessarily implies the convergence of iterative sequence $\{y_n(x)\}_{n=1}^{\infty}$ of partial sums of the series (4.1).Having considered so and p $(x) = x^{\nu}$ g (x), where g(0) ≠0 and for every $\gamma \epsilon [01)$ .Let $y_0(x)$ is the assumed initial selective function for the solution process then the first successive functional iterate is given by

$$y_1(x) = \int_0^x \lambda(s)((p(s)y_0^{/}(s))^{/} \_ p(s) f(s, y_0(s)))\, ds \quad (4.2)$$

Integrating by parts in sequel and applying the existing stationary conditions, we may deduce that,

$$|y_1(x) - y_0(x)| = |\int_0^x (y_0^{/}(s) + \lambda(s)p(s)f(s, y_0(s))ds| \quad (4.3)$$

or
$$|y_1(x) - y_0(x)| = \int_0^x |(y_0^{/}(s)| + |\lambda(s)||p(s)||f(s, y_0(s)|ds \quad (4.4)$$

Again pursuing on similar procedures as in (4.2) and adopting usual stationary conditions likewise, relation (3.4) imply that

$$|y_2(x) - y_1(x)| = |\int_0^x \lambda(s)\ p(s)(f(s), y_1(s)) - f(s, y_0(s))\, ds| \quad (4.5)$$

or, $|y_2(x) - y_1(x)| \le \int_0^s |\lambda(s)p(s)\ (f(s), y_1(s)) - f(s, y_0(s))|ds$

or, $|y_2(x) - y_1(x)| \le \int_0^x |\lambda(s)|p(s)|\ (f(s), y_1(s)) - f(s, y_0(s))\ |ds \quad (4.6)$

And, above all

$$|y_{n+1}(x) - y_n(x)| = |\int_0^x \lambda(s)\ p(s)\ (f(s, y_n(s)) - f(s, y_{n-1}(s)))ds \quad (4.7)$$

or, $|y_{n+1}(x) - y_n(x)| \le \int_0^x |\lambda(s)||p(s)|\ (f(s, y_n(s)) - f(s, y_{n-1}(s)))\ |ds \quad \forall \quad n \ge 2 \quad (4.8)$

Now, since f(x, y) and $\frac{\partial f(x,y)}{\partial y}$ are continuous on R, therefore for fix sϵ [0 1] and by virtue of mean value theorem ∃ $(s, \theta_n^0(s)) \in R$ satisfying (say, $y_{n-1}(s) < \theta_n^0(s) < y_n(s)$),

$\forall n \in IN$, $s \le x \le 1$, such that

$$|f(s, y_n(s)) - f(s, y_{n-1}(s))| = |\frac{\partial f(s, \theta_{n+1}^0(s))}{\partial y}||y_n(s) - y_{n-1}(s)| \quad \forall \quad n \ge 2 \quad (4.9)$$

Now, suppose $M_{\infty}^1 = \sup\ (|y_0^{/}(s)| + |\lambda(s)||p(s|| f(s, y_0(s))|$ (4.10)

$S \le x \le 1$

and $M_{00}^2 = \sup\ (|\lambda(s)||p(s)||\frac{\partial f(s, \theta_n^0(s))}{\partial y}|)$ (4.11)

$s \le x \le 1$,n∈ IN

Again to start with assume M=sup (M_∞^1,M_∞^2) (4.12)

Now, again observe and proceed to establish the truthfulness of the inequality

$$|y_{n+1}(\mathrm{s}) - y_n(\mathrm{s})| \le \frac{M^{n+1}x^{n+1}}{n+1!} \qquad \forall n \in IN \tag{4.13}$$

Obviously, relations (4.4), (4.10), (4.9) and (4.12) together imply that

$$|\, y_1(\mathrm{x})\text{-}y_0(\mathrm{x})\,| \le \int_o^x M_1 \,\mathrm{ds} \;\le \int_0^x M \,\mathrm{ds} \;= \mathrm{M}x \tag{4.14}$$

As well as, $|y_2(x) - y_1(\mathrm{x})| \le \underset{s\le x\le 1,\, n\in IN}{sup}|\lambda(\mathrm{s})||\mathrm{p(s)}||\frac{\partial f(s,\theta_1^0(s))}{\partial y}| \int_0^x |(y_1(\mathrm{s})) - y_0(\mathrm{s}))|\mathrm{ds}$

or, $|y_2(x) - y_1(\mathrm{x})| \le \underset{s\le x\le 1,\, n\in IN}{\sup}(|\lambda(s)||p(s)||\frac{\partial f\left(s,\theta_1^0(s)\right)}{\partial y}|\; \int_0^x |(y_1(\mathrm{s})) - y_0(\mathrm{s}))|\mathrm{ds} = \mathrm{M}\int_0^x M \,\mathrm{ds} = \frac{M^2x^2}{2}$

Thus, the statement (4.13) is true for natural number n=1

As usual, suppose that $|y_n(\mathrm{s}) - y_{n-1}(\mathrm{s})| \le \frac{M^n x^n}{n!}$ holds for some, $n \in IN$

Then, again relations (4.8), (4.9) and (4.12) altogether imply that

$$|y_{n+1}(x) - y_n(\mathrm{x})| \le \int_0^x |\lambda(\mathrm{s})||\mathrm{p(s)}|| \,(\underset{S\le x\le 1,\, n\in IN}{\sup}|\frac{\partial f(s,\theta_n^0(s))}{\partial y}|)|y_n(\mathrm{s}) - y_{n-1}(\mathrm{s})|\mathrm{ds}$$

$$\text{or, } |y_{n+1}(\mathrm{x}) - y_n(\mathrm{x})| \le \sup(|\lambda(s)||p(s)|\left|\frac{\partial f\left(s,\theta_{n+1}^0(s)\right)}{\partial y}\right| \int_0^x |y_n(s) - y_{n-1}(s)|\mathrm{ds}$$

$$\le \mathrm{M}\int_0^x \frac{M^n s^n}{n!}\mathrm{ds} = \frac{M^{n+1}x^{n+1}}{n+1!}$$

Therefore, by Principle of Induction

$|y_{n+1}(x) - y_n(x)| \le \frac{M^{n+1}x^{n+1}}{n+1!}$ holds $\forall x \epsilon [0\ 1]$ and $\forall n \in IN$

So the series (4.1) converges both absolutely and uniformly for all $x \in [0\ 1]$

Since, $|y_0(\mathrm{x})| + \sum_{n=0}^{\infty} |y_{n+1}(\mathrm{x}) - y_n(\mathrm{x})| \le |y_0(\mathrm{x})| + \sum_{n=0}^{\infty} \frac{M^{n+1}x^{n+1}}{n+1!} = |y_0(\mathrm{x})| + (e^{Mx} - 1)$ $\forall x \in [01]$

Asserting that the series $y_0(x) + \sum_{k=0}^{\infty}(y_{k+1}(\mathrm{x}) - y_k(x))$ converges uniformly $\forall x \in [01]$ and hence the sequence of its partial sums $\{y_n(x)\}_{n=0}^{\infty}$ converges to a limit function as the solution.

Numerical Problems

To begin with implementation and realization of scope of VIM we consider and apply this very method to find the solution of some linear and nonlinear problems often referred, discussed , and solved numerically and differently by methods like Cubic B-Spline, Adomian method and finite difference technique of solution

Example -1: Consider the boundary value problem [11, 13]

$$y^{//}+\frac{y^{/}}{x}+y=4-9x+x^2-x^3 \text{ , and } y(0)=y(1)=0. \quad (5.1)$$

Solution: Since inhomogeneous function in (5.1) is a polynomial of degree three having zero and one as its real roots, therefore the selective function may be taken as

$$y_0(x)=x(1-x)(a+bx)=ax+(b-a)x^2-bx^3 \quad (5.2)$$

The correction functional corresponding to (5.9) by VIM is given by

$$y_{n+1}(x)=y_n(x)+\int_0^x \lambda(s)\,(-(sy_n^{/}(s))^{/}-y_n(s)s+4s-9s^2+s^3-s^4)ds \qquad n=0, 1, 2\ldots$$

Therefore, first iterative approximant is

$$y_1(x)=y_0(x)+\int_0^x \lambda(s)\,(-(sy_0^{/}(s))^{/}-y_0(s)s+4s-9s^2+s^3-s^4)\,ds \quad (5.3)$$

Where λ (s) is the optimally identified Lagrange multiplier.

Now, using (5.2) in (5.3) then after integral simplifications we get

$$y_1(x)=x^2-\frac{(9+a)}{9}x^3+(a-b+1)\frac{x^4}{16}+(b-1)\frac{x^5}{25} \quad (5.4)$$

Clearly, $y(0)=0$ is self imposed on (5.12) and enforcement of next boundary condition $y(1)=0$ implies that a=0and b=1.That is how $y_1(x)=x^2-x^3$ gives the exact solution to the problem improving over the results as obtained in [11, 13].

Example- 2: Consider the boundary value problem [12]

$$y^{//}(x)+\frac{\alpha}{x}y^{/}(x)=-x^{1-\alpha}\cos x-(2-\alpha)\,x^{-\alpha}\sin x$$

$$y(0)=0, y(1)=\cos 1 \quad (5.5)$$

Solution: To solve (5.5) we construct correction functional as follows

$$y_{n+1}(x)=y_n(x)+\int_0^x \lambda(s)\,((-s^{\alpha}y_n^{/}(s))^{/}-s\cos s-(2-\alpha)\sin s)\,ds \qquad n\geq 0$$

Where λ(s) Is optimally identified Lagrange multiplier similar to (3.3). Then the first iterative solution is given by

$$y_1(x) = y_o(x) + \int_0^x \lambda(s)\ ((-s^\alpha y_0^/(s))^/ - s\cos s - (2-\alpha)\sin s)\ ds$$

Since the selective function y_0 (x) is arbitrary for simplicity and easiness in process of calculation we may require to choose y_0 (x) = $a_0 \ x^{1-\alpha}$, so that $(-s^\alpha\ y_0^/)^/$) = 0 to yield

$$y_1(x)= a_0 x^{1-\alpha}+\int_0^x \lambda(s)\ (-s\cos s-(2-\alpha)\sin s)ds$$

Now term by term series integration and required intermediary simplifications it again yields,

$$y_1(x) = a_0\ x^{1-\alpha} - [\sum_0^\infty (-1)^n \frac{x^{2n+3}}{(2n+3-\alpha)(2n+1)!} + (1-\alpha)\sum_{n=1}^\infty (-1)^{n+1}\frac{x^{2n+1-\alpha}}{(2n+1-\alpha)(2n)!}]$$

or, $y_1(x) = a_0\ x^{1-\alpha} + x^{1-\alpha}\sum_{n=1}^\infty (-1)^n \frac{x^{2n}}{(2n)!}$

or, $y_1(x) = a_0\ x^{1-\alpha} + x^{1-\alpha}\sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{(2n)!} - x^{1-\alpha}$

or, $y_1(x) == (a_0 - 1)\ x^{1-\alpha} + x^{1-\alpha}\ \cos x$ (5.6)

In order to match the boundary condition, $y(1) = \cos 1$ taking limit as $(x \to 1)$ we find that $a_0 = 1$, i.e. only the first iterate giving the exact solution as $y(x)= y_1(x) = x^{1-\alpha}\ \cos x$

Example-3: Consider the boundary value problem **[**11, 12, 30]

$$(x^\alpha y^/)^/ = \beta x^{\alpha+\beta-2}\ ((\alpha+\beta-1)+\beta x^\beta)\ y$$

Y (0) =1 , y (1) =exp (1) (5.7)

Solution : The correction functional for the problem (5.7) is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\ ((s^\alpha y_n^/)^/ - \beta\ (\alpha+\beta-1)s^{\alpha+\beta-2} - \beta^2 s^{\alpha+\beta-2})\ y_n(s)\ ds \quad (5.8)$$

λ(s) Is optimally identified Lagrange multiplier similar as in (3.2)

Inserting, $y(0) = y_0(x) = 1$ to (5.8) when n=1, as selective initial approximation function we process out following induced successive iterative approximate solutions as

$$y_1(x) = 1+x^\beta+\beta\frac{x^{2\beta}}{2(\alpha+\beta-1)}$$

$$y_2(x) = 1+x^\beta+\frac{x^{2\beta}}{2.1}+\beta\frac{x^{3\beta}}{3(\alpha+3\beta-1)}$$

$$y_3(x) = 1+x^\beta+\frac{x^{2\beta}}{2.1}+\frac{x^{3\beta}}{3.2.1}+\beta\frac{x^{4\beta}}{4.2(\alpha+4\beta-1)}$$

$$y_4(x) = 1+x^\beta+\frac{x^{2\beta}}{2.1}+\frac{x^{3\beta}}{3.2.1}+\frac{x^{4\beta}}{4.3.2.1}+\beta\frac{x^{5\beta}}{5.3.2(\alpha+5\beta-1)}$$

$$y_5(x) = 1+x^\beta+\frac{x^{2\beta}}{2.1}+\frac{x^{3\beta}}{3.2.1}+\frac{x^{4\beta}}{4.3.2.1}+\frac{x^{5\beta}}{5.4.3.2.1}+\beta\frac{x^{6\beta}}{6.4.3.2(\alpha+6\beta-1)}$$

Similarly, continuing in like manner inductively we may find the general term of the sequence

$$y_n(x) = 1+x^\beta+\frac{x^{2\beta}}{2.1}+\frac{x^{3\beta}}{3.2.1}+\frac{x^{4\beta}}{4.3.2.1}+\frac{x^{5\beta}}{5.4.3.2.1}+\frac{x^{6\beta}}{6.5.4.3.2.1}+\ldots\ldots\ldots\ldots+\frac{x^{n\beta}}{n!}+\frac{n\beta x^{(n+1)\beta}}{n+1!(\alpha+(n+1)\beta-1)}$$

i.e. $y_n(x) = \sum_{k=0}^{n} \frac{x^{k\beta}}{k!} + \frac{n\beta x^{(n+1)\beta}}{n+1!(\alpha+(n+1)\beta-1)}$ (5.9)

Now, we observe that $T_n = \frac{n\beta x^{(n+1)\beta}}{n+1!(\alpha+(n+1)\beta-1)}$ (say), is the general term of a convergent

Series $\sum_{n=0}^{\infty} \frac{n\beta x^{(n+1)\beta}}{n+1!(\alpha+(n+1)\beta-1)}$. Therefore, lim (n→ ∞) $\frac{n\beta x^{(n+1)\beta}}{n+1!(\alpha+(n+1)\beta-1)}$ = 0 and (5.9) facilitates the exact solution to (5.7) as y(x) =lim (n→ ∞)($\sum_{k=0}^{n} \frac{x^{k\beta}}{k!}$) =exp (x^{β}).

Example-4: Let the boundary value problem [8] be

$(x^{\alpha}y^{/})^{/} = \frac{\beta x^{\alpha}}{4+x^{\beta}}(\beta x^{\beta}e^{y} - (\alpha+\beta-1))$

y (0) =ln $\frac{1}{4}$, $y(1)$ = ln $\frac{1}{5}$ (5.10)

Solution: Let, $y_0 = y(0)$ = ln $\frac{1}{4}$, be the selective initial approximation function .Then by VIM,

First iterative approximate solution to (5.10) simplifies to

$$y_1(x) = \ln\frac{1}{4} + \int_0^x \frac{\lambda(s)}{4+x^{\beta}}\left(\frac{\beta^2 s^{\alpha+2\beta-2}}{4} - (\alpha+\beta-1)\,\beta^{\alpha} s^{\alpha+\beta-2}\right)ds \quad (5.11)$$

Whereas $\lambda(s) is$ optimally identified Lagrange multiplier as in (3.3) and after simplifying (5.11) the required first approximate solution to (5.10) satisfying the given boundary condition $y(0)$ = ln $\frac{1}{4}$ is the following,

$$y_1(x) = \ln\frac{1}{4} - \frac{x^{\beta}}{4} + \frac{1}{2}\left(\frac{x^{\beta}}{4}\right)^2 + \sum_{n=3}^{\infty}\left(\frac{\alpha+2\beta-1}{\alpha+n\beta-1}\right)\left(\frac{(-1)^n}{n}\right)\left(\frac{x^{\beta}}{4}\right)^n \quad (5.12)$$

Now, we observe in (5.12) that the first three terms of the first approximate iterative solution of (5.10) match the first three terms of the expanded Taylor's series of exact solution even without imposition of right-side boundary condition so for. However, the way $y(1)$ = ln $\frac{1}{5}$ could match (5.12) if every term of the sequence$\{(\frac{\alpha+2\beta-1}{\alpha+n\beta-1})\}_{n=3}^{\infty}$ reduces to the sequence $\{1\}_{n=3}^{\infty}$. this may be done by allowing the arbitrarily parameter β to approach zero explicitly and independently only in the coefficient $(\frac{\alpha+2\beta-1}{\alpha+n\beta-1})$ of $\left(\frac{(-1)^n}{n}\right)(\frac{x^{\beta}}{4})^n$ .Therefore, allowing the process to do shoot and satisfy the right boundary condition so that the first iterate mends its way to improvise and produce finally an exact solution to (5.10) given by

$$y(x) = y_1(x) = \ln\frac{1}{4} - \frac{x^{\beta}}{4} + \frac{1}{2}\left(\frac{x^{\beta}}{4}\right)^2)^n + \sum_{n=3}^{\infty}\left(\frac{(-1)^n}{n}\right)\left(\frac{x^{\beta}}{4}\right)^n = \ln\frac{1}{4+x^{\beta}}$$

Example-5: Consider the boundary value problem [30]

$$(x^{\alpha}(1+x^2)y^{/})^{/} = 5(1+x^2)\, x^{\alpha+3}(5x^5 + (\alpha+4) + \frac{2x^2}{1+x^2}) \qquad (5.13)$$

$y(0)$=1 $\qquad y(1)$ = exp (1)

Solution: The correction functional corresponding to (5.9) by VIM is given by

$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\,(-(s^{\alpha}(1+s^2)y_n^{/}(s))^{/} + 5(1+s^2)s^{\alpha+3}(5s^5 + (\alpha+4) + \frac{2s^2}{1+s^2})ds$, n=0, 1, 2…

Where λ (s) is the optimally identified Lagrange multiplier. Therefore, for n=1 we have

$$y_1(x) = y_0(x) + \int_0^x \lambda(s)\,(-(s^{\alpha}(1+s^2)y_0^{/}(s))^{/} + 5(1+s^2)s^{\alpha+3}(5s^5 + (\alpha+4) + \frac{2s^2}{1+s^2})ds \qquad (5.14)$$

Let us suppose initially the selective function be y_{01}= 1, then the first approximant is $y_1(x) = 1 + x^5 + \frac{5}{2(\alpha+11)}x^{10} + \frac{25}{(\alpha+9)(\alpha+11)}[ln(1+x^2) - x^2 + \frac{x^4}{2} - \frac{x^6}{3} + \frac{x^8}{4}] = 1 + x^5 + 25\int_0^x (\frac{s^9}{(\alpha+9)} + \frac{s^{11}}{(\alpha+11)})\frac{1}{(1+s^2}ds$

Now, assuming the arbitrariness of selective function, with a motive to avoid over do simplifications we may revise selective function as y_{02}= 1+x^5 then the solution approximant by correctional functional is

$$y_2(x) = 1 + x^5 + \frac{x^{10}}{2} + \frac{25}{2}\int_0^x (\frac{s^{14}}{(\alpha+14)} + \frac{s^{16}}{(\alpha+16)})\,\frac{1}{(1+s^2)}ds$$

If again we revise the selective function as y_{03}= 1+x^5+$\frac{x^{10}}{2}$then the next approximate solution is

$$y_3(x) = 1 + x^5 + \frac{x^{10}}{2} + \frac{x^{15}}{3.2} + \frac{25}{3.2}\int_0^x (\frac{s^{19}}{(\alpha+19)} + \frac{s^{21}}{(\alpha+21)})\,\frac{1}{(1+s^2)}ds$$

Thus repeated internal revision in selective function at n^{th} stage by like manner must produce the n^{th} Successive approximate solution to (5.13) as

$$y_n(x) = \sum_0^n \frac{x^{5n}}{n!} + \frac{1}{n!}\int_0^x (\frac{s^{5n+4}}{(5n+\alpha+4)} + \frac{s^{5n+6}}{(5n+\alpha+6)})\,\frac{1}{(1+s^2)}ds \qquad (5.15)$$

Let $\mathbb{I}_n = \frac{1}{n!}\int_0^x (\frac{s^{5n+4}}{(5n+\alpha+4)} + \frac{s^{5n+6}}{(5n+\alpha+6)})\,\frac{1}{(1+s^2)}ds$

$$|\,\mathbb{I}_n| = |\frac{1}{n!}\int_0^x (\frac{s^{5n+4}}{(5n+\alpha+4)} + \frac{s^{5n+6}}{(5n+\alpha+6)})\,\frac{1}{(1+s^2)}ds\,| \le \frac{2}{n!(5n+\alpha+4)} \quad (\to 0 \text{ as } n\to\infty)$$

Henceforth,$|\,y_n(x) - \sum_0^n \frac{x^{5n}}{n!}| \to 0$ as n→ ∞)

1. Thus we find that $y(x)$ =lim (n→ ∞) y_n(x) = $\sum_0^\infty \frac{x^{5n}}{n!}$=exp (x^5) an exact solution to (5.13) **Conclusion**

This is to mention that He's variation iteration method successfully applies to a linear as well as to a nonlinear class of boundary value problems of type (1.1) with convergent iterative scheme of solution. The proposed method provides an exact solution or any other solution of high accuracy to some of the frontier examples available in literature by exploiting (i) arbitrariness of selective function (ii) internal improvisation of selective function and sometime (iii) a careful and maneuvered imposition of boundary condition on iterative approximant solution.

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