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# *- Completely Finite Quadrilateral $(A, B, C, D)$ and the Range of $\delta_{ABCD}(X)$

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**Abstract:**

In this paper, we define *-finite quadrilaterals and *-completely finite quadrilaterals for the operator equation

$$\delta_{ABCCD}(X) = A\,XB - CX^*\,D,$$

where $A, B, C, D \in \mathcal{B}(\mathcal{H})$. Also, we prove the following:

1- If $(A, B, C, D)$ is *-finite quadeteirial and $U$ is unitary operator then $(UAU^*, UBU^*, UCU^*, UDU^*)$ is *-finite quadeteirial .

2- Let $A, B, C$ and D $\in \mathcal{B}(\mathcal{H})$, and $(A, B, C, D)$ is *- finite quadrilateral, then no nonzero scalar operators contains in $\overline{R(\delta_{ABCD}(X)}$.

3- The quadrilateral $(A, B, C, D)$ is *-completely finite iff for every normal operator $N$ satisfies $ANB = CN^*D$, $\|N\| \le \|N + AXB - CX^*D\|$.

**Keywords:** Operator equation, *_ finite operators.

**Introduction:**

Let $\mathcal{B}(\mathcal{H})$ be the space of all bounded linear operators on the Hilbert space $\mathcal{H}$. Let $\delta_A(X) = AX - XA$ be the inner derivation. In this paper we deal with two basic concepts, first concept related to the concept that defined by Williams which is finite operator. The operator $A$ is finite if the distance between range $\delta_A(X)$ and the identity operator is equal or more than 1as in [5]. In recent years many authors modified the concept of finite operators one of them which introduce by Hammad in 2002 as follows: an operator $A \in \mathcal{B}(\mathcal{H})$ is *- finite operator if $0 \in \overline{W(AX - X^*A)}$ for $X \in \mathcal{B}(\mathcal{H})$. While, the second concept is completely finite operators which first defined by Elilami S. N. in [3], an operator $A \in \mathcal{B}(\mathcal{H})$ is called completely finite operator if $A_{|E}$ is finite for every nonzero reducing subspace $E$ of $A$.

This paper contains two sections: In §1 we motivate the definition of *-finite operators and give some properties. While in §2 we motivate the definition of completely finite operators and we omits the normal operators from the range of $\delta_{ABCD}$(X) as in theorems (2.5).

**§1 *- Finite quadrilaterals**

**Definition: (1.1)**

For $A, B, C$ and $D \in \mathcal{B}(\mathcal{H})$. The quadrilateral $(A, B, C, D)$ is *-finite quadrilateral if $0 \in \overline{W(AXB - CX^*D)}$ for each $X \in \mathcal{B}(\mathcal{H})$.

The following theorem is equivalent to definition of *- finite quadrilaterals.

**Preposition: (1.2)**

Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$. Then $(A, B, C, D)$ is *-finite quadrilateral iff $\|AXB - CX^*D - \lambda I\| \ge |\lambda|$ for each $X \in \mathcal{B}(\mathcal{H})$, and for each $\lambda \in \mathbb{C}$.

 

**Proof :**

We can proof easily, by theorem in [5] put the operator $AXB - CX^*D$ instead of A. We get that

$0 \in W_\circ(AXB - CX^*D)$ iff $\|AXB - CX^*D - \lambda I\| \geq |\lambda|$ for each $X \in \mathcal{B}(\mathcal{H})$, and $\lambda \in \mathbb{C}$. By [3] we get

$W_\circ(AXB - CX^*D) = \overline{W(AXB - CX^*D)}$. □

We will denoted to the set of all *- finite quadrilaterals by $K^*$.

**Theorem: (1.3)**

Let $A, B, C$ and D $\in \mathcal{B}(\mathcal{H})$. The following statements are equivalent

1) $(A, B, C, D) \in K^*$.
2) $\text{Inf}_X \|A X B - C X^*D - I\| = 1$.
3) There exists $f \in \rho$ such that $f(A X B) = f(C X^* D)$ $\forall X \in \mathcal{B}(\mathcal{H})$.

**Proof :**

(1) and (2) are equivalent by proposition (1.2) by taking $\lambda = 1$, then $\|A X B - C X^*D - I\| \geq 1$. Now, to prove (2) equivalent to (3), define a linear functional $f$ such that $f(I) = 1$, $\|f\| = \frac{1}{\text{Inf}\,\|A X B - C X^*D - I\|}$ and $f(A X B - C X^* D) = 0$. So, by Hahn- Banach theorem this functional can be extend to all $\mathcal{B}(\mathcal{H})$.Finaly to prove that (3) give (1) by the assumption $f(A X B) = f(C X^* D)$ then $f(A X B - C X^*D) = 0$. i.e., $0 \in \overline{W(A X B - C X^* D)}$. □

**Proposition : (1.4)**

If $(A, B, C, D) \in K^*$ then $(B^*, A^*, D^*, C^*) \in K^*$.

**Poof :**

Since for any operator $A \in \mathcal{B}(\mathcal{H})$, $(A^*)^* = A$. And $\|A X B - C X^* D - I\| \geq 1, \forall X \in \mathcal{B}(\mathcal{H})$. The map $f(X) = X^*$ is surjective so, $\|A X^* B - C X D - I\| \geq 1, \forall X \in \mathcal{B}(\mathcal{H})$.

Therefore

$\|B^* X A^* - D^*X^*C^* - I\| = \|(A X^* B - C X D)^* - I\| \geq 1 \; \forall X \in \mathcal{B}(\mathcal{H})$. □

**Proposition : (1.5)**

If $(A, B, C, D)$ is a *- finite quadrilateral then $(\lambda A, B, \lambda C, D)$ is also *-finite quadrilateral for each $\lambda \in \mathbb{C}$.

**Proof :**

If $\lambda = 0$ then it is clear that $(\lambda A, B, \lambda C, D)$ is a * - finite quadrilateral.

Now, let $0 \neq \lambda \in \mathbb{C}$ given nonzero number $\varepsilon$ then for each operator $X \in \mathcal{B}(\mathcal{H})$ there exist vector $y \in \mathcal{H}$ such that

$$|\langle (A X B - C X^* D)y, y\rangle| < \left( {}^{\varepsilon}/_{\lambda} \right)$$

So,

$|\langle (\lambda A X B - \lambda C X^* D)y, y\rangle| < \varepsilon$. i.e. , $0 \in \overline{W(\lambda A X B - \lambda C X^* D)}$. □

**Proposition : (1.6)**

Let $A, B, C$ and D $\in \mathcal{B}(\mathcal{H})$ are defined in following form:

$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$, $B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$, $C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$ and D = $\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$ on $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$. Such that $(A_1, B_1, C_1, D_1) \in K^*_{\mathcal{H}_1}$ or $(A_2B_2, C_2D_2) \in K^*_{\mathcal{H}_2}$ then $(A, B, C, D) \in K^*_{\mathcal{H}}$.

**Proof :**

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} - \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}\begin{bmatrix} X_1{}^* & X_3{}^* \\ X_2{}^* & X_4{}^* \end{bmatrix}\begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} - \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}$$

$$= \begin{bmatrix} A_1X_1B_1 & A_1X_2B_2 \\ A_2X_3B_1 & A_2X_4B_2 \end{bmatrix} - \begin{bmatrix} C_1X_1{}^*D_1 & C_1\,X_3{}^*\,D_2 \\ A_2\,X_2{}^*B_1 & C_4\,X_4{}^*\,D_2 \end{bmatrix} - \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}$$

$$= \begin{bmatrix} A_1X_1B_1 - C_1X_1{}^*D_1 - I_1 & A_1X_2B_2 - C_1\,X_3{}^*\,D_2 \\ A_2X_3B_1 - A_2X_2{}^*B_1 & A_2X_4B_2 - C_4\,X_4{}^*\,D_2 - I_2 \end{bmatrix}$$

Then

$$\|A\,X\,B - C\,X^*D - I\| = \left\| \begin{bmatrix} A_1X_1B_1 - C_1X_1{}^*D_1 - I_1 & A_1X_2B_2 - C_1\,X_3{}^*\,D_2 \\ A_2X_3B_1 - A_2X_2{}^*\,B_1 & A_2X_4B_2 - C_4\,X_4{}^*\,D_2 - I_2 \end{bmatrix} \right\|,$$

Then

$$\|A\,X\,B - C\,X^*D - I\| \geq \|A_1X_1B_1 - C_1X_1{}^*D_1 - I_1\| \geq 1,$$

or

$$\|A\,X\,B - C\,X^*D - I\| \geq \|A_2X_4B_2 - C_4\,X_4{}^*\,D_2 - I_2\| \geq 1.$$

Therefore, $\|A\,X\,B - C\,X^*D - I\| \geq 1$.

**Theorem : (1.7)**

If $(A, B, C, D)$ is *-finite quadeteirial and $U$ is unitary operator then $(UAU^*, UBU^*, UCU^*, UDU^*)$ is *-finite quadeteirial.

Proof :

Suppose $A, B, C, D \in \mathcal{B}(\mathcal{H})$, let $U$ be a unitary operator and $X \in \mathcal{B}(\mathcal{H})$ then by assumption $0 \in \overline{W(A(U^*XU)B - C(U^*XU)^*D)}$. Thus there exists a sequence $\{y_n\}$ in $\mathcal{H}$ such that

$$\langle (A(U^*XU)B - C(U^*XU)^*D)y_n, y_n\rangle \longrightarrow 0,$$

$$\langle A(U^*XU)By_n,\ y_n\rangle - \langle C(U^*XU)^*Dy_n, y_n\rangle \longrightarrow 0,$$

$$\langle X(UBU^*)Uy_n, (\,UA^*U^*)Uy_n\rangle - \langle X^*(UDU^*)Uy_n, (U\,C^*U^*)Uy_n\rangle \longrightarrow 0.$$

Thus,

$$\langle \big((UAU^*)X(UBU^*) - (UCU^*)X^*(UDU^*)\big)Uy_n, Uy_n\,\rangle \longrightarrow 0.$$

Which means that $0 \in \overline{W(UAU^*)X(UBU^*) - (UCU^*)X^*(UDU^*))}$ for each $X \in \mathcal{B}(\mathcal{H})$, note that $\|Uy_n\| = 1$ for all $n$ when $\|y_n\| = 1$.

Now, we give the relation between *-finite quadrilateral $(A, B, C, D)$ and range$\delta_{ABCD}(X)$.

**Proposition: (1.8)**

Let $A, B, C$ and D $\in \mathcal{B}(\mathcal{H})$, and $(A\,, B\,, C, D)$ is *- finite quadrilateral, then no nonzero scalar operators contains in $\overline{R(\delta_{ABCD}(X)}$.

**Proof:**

Suppose that $\lambda\,I \in \overline{R(\delta_{ABCD}(X))}$ then $\exists\, X_n \in \mathcal{B}(\mathcal{H})$ such that

$$\|A\,X_n\,B - C\,X_n{}^*\,D - \lambda\,I\,\| \longrightarrow 0$$

but $(\,A, B, C, D)$ is *- finite quadrilateral then

$$|\lambda| \leq \|A\,X_n\,B - C\,X_n{}^*\,D - \lambda\,I\,\| \longrightarrow 0$$

This is a contradiction. □

**Proposition: (1.9)**

Let $A, B, C$ and D $\in \mathcal{B}(\mathcal{H})$ and $R\ (\delta_{ABCD}(X))$ has no invertible operator then $(A\ , B\ , C\ , D)$ is *- finite quadrilateral**.**

**Proof:**

Let $F\ \in R\ (\delta_{ABCD}(X))$ then by assumption $F$ is not invertible then $\|F - I\| \geq 1$ then by definition ( $A, B\ , C, D)$ is *- finite quadrilateral**.** □

**Proposition: (1.10)**

If the bounded linear operators $A, B, C,$ and $D$ are compact operators then $(A\ , B, C, D)$ is *-finite quadrilateral.

**Proof:**

Let $A, B, C, D$ be a compact operators then clearly $R(\delta_{ABCD}(X))$ consists of compact operators. Since any compact operator defined on an infinite dimensional Hilbert space is not invertible . Hence by proposition (1.9), we deduce that $(A\ , B, C, D)$ is *-finite quadrilateral. □

**§ 2 *- Finite quadrilateral and range $\boldsymbol{\delta_{ABCD}}$(X)**

**Definition: (2. 1)**

An quadrilateral $(A, B, C, D)$ is *-completely finite quadrilateral if $(A_{|M}, B_{|M}, C_{|M}, D_{|M})$ is *-finite quadrilateral for every nonzero reducing subspace $M$ of $A, B, C$ and $D$.

**Lemma : (2.2)**

For $A, B, C, D\ \in \mathcal{B}(\mathcal{H})$, the quadrilateral $(A, B, C, D)$ is *-completely finite quadrilateral if $\{P_1, \ldots, P_n\}$ is a set of projection that satisfied $APB = CPD$ and $\lambda_1, \ldots, \lambda_n$ are scalars, then

$$\max\ \{\ |\lambda_i| : i = 1, \ldots, n\}\ \leq\ \|\textstyle\sum_{i=1}^{n} \lambda_i P_i +\ AXB -\ CX^* D\|$$

**Proof :**

Let $P =\ \lambda_1 P_1 + \cdots + \lambda_n P_n$, let $m$ in $\{1,\ldots, n\}$ and $X\ \in\ \mathcal{B}(\mathcal{H})$ the orthogonal projection $P_m$ commute with each $A, B, C, D$ and $P$. So on $\mathcal{H} = R(P_m) \oplus R(P_m)^{\perp}$, we can write

$$A = \begin{bmatrix} T & 0 \\ 0 & * \end{bmatrix}, B = \begin{bmatrix} T & 0 \\ 0 & * \end{bmatrix}, C = \begin{bmatrix} T & 0 \\ 0 & * \end{bmatrix}, D = \begin{bmatrix} T & 0 \\ 0 & * \end{bmatrix} \text{ and } X = \begin{bmatrix} Y & * \\ * & * \end{bmatrix}.$$

So, we get

$$\|P + (AXB - CX^* D)\| = \left\| \begin{bmatrix} \lambda_m + TYT - TY^* T & * \\ * & * \end{bmatrix} \right\|$$
$$\geq \|\lambda_m + TYT - TY^* T\|\ \geq |\lambda_m|.$$

The last inequality is true since $T$ is the restriction of $A, B, C, D$ to $R(P_m)$ therefore $(T, T, T, T)$ is a *-finite quadrilateral. □

**Theorem : (2.3)**

The quadrilateral $(A, B, C, D)$ is *-completely finite iff for every normal operator $N$ satisfies $ANB = CN^* D$, $\|N\| \leq\ \|N + AXB - CX^* D\|$.

**Proof :**

Suppose that the quadrilateral $(A\ , B, C, D)$ is *-completely finite and let $E$ be the resolution of identity of the normal operator $N$ where $N$ satisfies $ANB = CN^* D$. if $\{\sigma_1, \ldots, \sigma_n\}$ is a family of Borel sets that form a partition of the spectrum of N, and if $\lambda_i \in\ \sigma_i$ for $i = 1, \ldots, n$, by lemma (2.2) we get

$\max\{|\lambda_i|: i = 1, \ldots, n\} \leq \|\sum_{i=1}^{n} \lambda_i E_i + AXB - CX^*D\|$ where $X \in \mathcal{B}(\mathcal{H})$. But we can always choose $\lambda = \lambda_1 \in \sigma_1$ with $\lambda \in \partial\sigma(N)$ and $|\lambda| = \|N\|$. Hence

$\|N\| \leq \|\lambda_1 E_1 + \cdots + \lambda_n E_n + AXB - CX^*D\|$. By spectral theorem for normal operator, we get that

$$\|N\| \leq \|N + AXB - CX^*D\|.$$

Conversely, let $E$ be a nonzero reducing subspace of $A, B, C$ and $D$. Since $E$ is closed subspace of $\mathcal{H}$, so $\mathcal{H} = E \oplus E^{\perp}$ according to the decomposition of $\mathcal{H}$, we can write $A = F \oplus G, B = F \oplus M, C = F \oplus U, D = F \oplus L$ where $F = A_{|E} = B_{|E} = C_{|E} = D_{|E}$ and $N = I_{|E} \oplus 0$ is normal on $\mathcal{H}$ and commutes with $A, B, C$ and $D$. So, the operator$X = Y \oplus 0$ and $X^* = Y^* \oplus 0$ on $\mathcal{H}$, therefore

$$1 = \|N\| \leq \|N + AXB - CX^*D\| = \left\|(I_{|E} + F\,XF - FX^*F\,) \oplus 0\right\|$$
$$= \left\|(I_{|E} + F\,XF - FX^*F\,\right\|.$$

Last argument lead to $(F, F, F, F)$ is *-finite quadrilateral in $E$ which means that the quadrilateral $(A_{|E}, B_{|E}, C_{|E}, D_{|E})$ is *-completely finite quadrilateral. □

As a view of the concept kernel orthogonal that first introduce by Anderson see [1], we give the following.

**Proposition : (2.4)**

If $(A, B, C, D)$ be *-finite quadrilateral, then $R(\delta_{ABCD}(X))$ is orthogonal to set of scalars operators.

Proof :

Since $(A, B, C, D)$ be * -finite quadrilateral. So, $\|AXB - CX^*D - \lambda I\| \geq \|\lambda I\|, \forall \lambda \in \mathbb{C}, X \in \mathcal{B}(\mathcal{H})$. thus by [1] we deduce that $R(\delta_{ABCD}(X))$ is orthogonal to set of scalars operators. □

For considering the property of kernel orthogonal on the operator equation $AXB - CX^*D = F$ we used the introduced tool which is * -completely finite as in the following.

**Remark : (2.5)**

Theorem (2.3) can be rewritten in the following form:

Let $A, B \in \mathcal{B}(\mathcal{H})$ and $N$ be normal operator then the following be equivalent

1- The quadrilateral $(A, B, C, D)$ is *-completely finite ,

2- kernel $\delta_{ABCD}(X)$ is orthogonal to range $\delta_{ABCD}(X)$.

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