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Bayesian Estimation for Parameters of Power Function Distribution under Various Priors

Saman Hanif, Saeed Dobbah Al-Ghamdi, Khushnoor Khan*, Muhammad Qaiser Shahbaz

Department of Statistics, Faculty of Sciences, King Abdulaziz University, P.O Box 80203, Jeddah 21589, Saudi Arabia

Abstract

Although the idea of Bayesian inference dates back to the late 18th century, its use by statisticians has been rare until recently. But due to advancement in the simulation techniques Bayesian inference and estimation is gaining currency. This paper seeks to focus on the Bayesian estimates of the Power Function distribution using Weibull and Generalized Gamma distributions as priors for the unknown parameters. Furthermore, the statistical performance of the obtained estimators is compared with the Maximum likelihood of Power Function distribution and the Bayesian estimator of Gamma distribution as prior of the unknown parameter. The comparison has been done using Monte Carlo simulation using MSE as yardstick of the comparison.

Keywords: Squared error loss function, Bayesian estimator, Prior distribution, Monte Carlo simulation.

1. Introduction

The Power function distribution is a simple yet very powerful distribution and is used to model insurance related data. The distribution is just the inverse of the Pareto distribution and appears as a special case of the Beta distribution and has the density function

$$f\left(x;\theta\right)=\theta x^{\theta-1};0<x<1,\theta>0\,; \qquad (1)$$

where θ is shape parameter of the distribution. The distribution has also been used in reliability theory; see Zarrin et al. (2013).

Estimation of parameters for the Power function distribution has been done by various authors. Mood et al. (1974) has discussed the maximum likelihood estimation for the parameters of the distribution. Zaka and Akhtar (2013) have discussed various methods, including methods of moments and least squares, for parameter estimation of the distribution.

Bayesian estimation for parameters of power function distribution has also been considered by number of authors. Rahman et al. (2012) have considered Bayesian estimation for the distribution under conjugate prior and under various loss functions. Omer and Low (2012) have discussed Bayesian estimation of generalized Power function distribution under non-informative and informative priors.

This paper will focus on the Bayesian estimation for parameters of Power function distribution under two different priors which has not been considered so far. Parameter estimation has been done under squared loss function.

The rest of the paper is structured as follows: Section 2 discusses various available estimators of Power function distribution: Section 3, discusses the posterior distribution of θ under two different priors namely Weibull distribution and Generalized Gamma Distribution.: Section 4, discusses the Bayes estimators under the squared loss functions for both Weibull and GG distribution: Section 5, empirically compares the estimates of the parameter θ using Monte Carlo simulation. Finally Section 6 concludes the paper with brief discussion.

2. Various Available Estimators

The density function of Power function distribution is given in (1). The maximum likelihood estimator for parameter θ is

$$\hat{\theta}_1=-\frac{n}{\sum_{i=1}^{n}\ln x_i}. \qquad (2)$$

Rahman et al. (2012) have obtained the Bayesian estimator of parameter θ under square loss function using Gamma prior.

The estimator is:

$$\hat{\theta}_2 = \frac{\alpha + n}{\left(\beta - \sum_{i=1}^{n} \ln x_i\right)}; \quad (3)$$

where α is shape parameter and β is scale parameter of the prior distribution.
Omer and Low (2012) have obtained Bayes estimate for parameter of generalized Power function distribution. The estimator is given as:

$$\hat{\theta}_2 = -\frac{(\alpha + n)}{\sum_{i=1}^{n} \ln(x_i + a) - \beta}; \quad (4)$$

where a is location parameter of generalized Power function distribution.
The proceeding section will give posterior distribution for parameter θ under different priors.

3. Posterior Distribution of θ

The density function of Power function distribution is given in (1). In this section we derive the posterior distribution of parameter θ under two different priors, namely the Weibull distribution and the Generalized Gamma distribution. The Weibull distribution is found to be useful in diverse fields ranging from engineering to medical sciences (see Lawless (2002), Martz and Waller (1982)). Habib, Roy and Atik-ur-rehman (2012) have studied the power function using different loss functions. Generalized Gamma (GG) Distribution was introduced by Stacy &Mihram (1965). Despite its long history and growing use in various applications, the GG family and its properties have been remarkably presented in different papers.
The density function of the Weibull distribution is given as:

$$f(\theta) = \alpha\beta\theta^{\beta-1}\exp\left(-\alpha\theta^{\beta}\right); \theta > 0$$

and the generalized Gamma distribution with density function is given as:

$$f(\theta) = \frac{\beta}{\Gamma(\alpha)}\theta^{\alpha\beta-1}\exp\left(-\theta^{\beta}\right); \theta > 0 .$$

Generally the posterior distribution of parameter θ is expressed as

$$f(\theta \mid \mathbf{x}) = \frac{\prod_{i=1}^{n} f(x_i \mid \theta) g(\theta)}{\int_{\Re} \prod_{i=1}^{n} f(x_i \mid \theta) g(\theta) d\theta} .$$

The posterior distribution when prior distribution of θ is Weibull is obtained as under:

$$f_1(\theta \mid \mathbf{x}) = \frac{\left\{\prod_{i=1}^{n}\left(\theta x_i^{\theta-1}\right)\right\}\alpha\beta\theta^{\beta-1}\exp\left(-\alpha\theta^{\beta}\right)}{\int_{\Re}\left\{\prod_{i=1}^{n}\left(\theta x_i^{\theta-1}\right)\right\}\alpha\beta\theta^{\beta-1}\exp\left(-\alpha\theta^{\beta}\right)}$$

$$= \frac{\alpha\beta\theta^{n+\beta-1}\exp\left\{(\theta-1)\sum_{i=1}^{n}\ln x_i\right\}\exp\left(-\alpha\theta^{\beta}\right)}{\int_{0}^{\infty}\alpha\beta\theta^{n+\beta-1}\exp\left\{(\theta-1)\sum_{i=1}^{n}\ln x_i\right\}\exp\left(-\alpha\theta^{\beta}\right)d\theta}$$

$$= \frac{\sum_{r=0}^{\infty}\frac{1}{r!}\left(\sum_{i=1}^{n}\ln x_i\right)^{r}\theta^{n+\beta+r-1}\exp\left(-\alpha\theta^{\beta}\right)}{\sum_{r=0}^{\infty}\frac{1}{r!}\left(\sum_{i=1}^{n}\ln x_i\right)^{r}\int_{0}^{\infty}\theta^{n+\beta+r-1}\exp\left(-\alpha\theta^{\beta}\right)d\theta}$$

$$= \frac{\beta}{k}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\sum_{i=1}^{n}\ln x_i\right)^{r}\theta^{n+\beta+r-1}\exp\left(-\alpha\theta^{\beta}\right); \quad (5)$$

where $$k=\sum_{r=0}^{\infty}\frac{1}{r!}\left(\sum_{i=1}^{n}\ln x_i\right)^r\alpha^{-(n+\beta+r)/\beta}\Gamma\left(\frac{n+r}{\beta}+1\right).$$

Similarly, the posterior distribution of θ when prior distribution is generalized Gamma is

$$f_2(\theta\,|\,\mathbf{x})=\frac{\left\{\prod_{i=1}^{n}\left(\theta x_i^{\theta-1}\right)\right\}\frac{\beta}{\Gamma(\alpha)}\theta^{\alpha\beta-1}\exp\left(-\theta^{\beta}\right)}{\int_{\Re}\left\{\prod_{i=1}^{n}\left(\theta x_i^{\theta-1}\right)\right\}\frac{\beta}{\Gamma(\alpha)}\theta^{\alpha\beta-1}\exp\left(-\theta^{\beta}\right)}$$

$$=\frac{\sum_{r=0}^{\infty}\frac{1}{r!}\left(\sum_{i=1}^{n}\ln x_i\right)^r\theta^{n+\alpha\beta+r-1}\exp\left(-\theta^{\beta}\right)}{\sum_{r=0}^{\infty}\frac{1}{r!}\left(\sum_{i=1}^{n}\ln x_i\right)^r\int_0^{\infty}\theta^{n+\alpha\beta+r-1}\exp\left(-\theta^{\beta}\right)d\theta}$$

$$=\frac{\beta}{k_1}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\sum_{i=1}^{n}\ln x_i\right)^r\theta^{n+\alpha\beta+r-1}\exp\left(-\theta^{\beta}\right); \qquad (6)$$

where $$k_1=\sum_{r=0}^{\infty}\frac{1}{r!}\left(\sum_{i=1}^{n}\ln x_i\right)^r\Gamma\left(\frac{n+r}{\beta}+\alpha\right).$$

The proceeding section will focus on the Bayesian estimator for the parameter of Power function distribution.

4. Bayesian Estimator of the Parameter θ

The posterior distribution of parameter θ under different priors was obtained in previous section. We now obtain the Bayesian estimator of the parameter θ under squared error loss function. Under the squared error loss function the Bayes estimator is simply expected value of the posterior distribution. The posterior distribution of θ when prior distribution is Weibull is given in (5) as;

$$f_1(\theta\,|\,\mathbf{x})=\frac{\beta}{k}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\sum_{i=1}^{n}\ln x_i\right)^r\theta^{n+\beta+r-1}\exp\left(-\alpha\theta^{\beta}\right).$$

The expected value of the posterior distribution is:

$$\hat{\theta}_{BW}=\int_0^{\infty}\theta f_1(\theta\,|\,\mathbf{x})d\theta=\frac{\beta}{k}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\sum_{i=1}^{n}\ln x_i\right)^r\int_0^{\theta}\theta^{n+\beta+r-1}\exp\left(-\alpha\theta^{\beta}\right)d\theta$$

or $$\hat{\theta}_{BW}=\int_0^{\infty}\theta f_1(\theta\,|\,\mathbf{x})d\theta=\frac{\beta}{k}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\sum_{i=1}^{n}\ln x_i\right)^r\int_0^{\theta}\theta^{n+\beta+r-1}\exp\left(-\alpha\theta^{\beta}\right)d\theta$$

$$\hat{\theta}_{BW}=\frac{\beta}{k}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\sum_{i=1}^{n}\ln x_i\right)^r\alpha^{-(n+\beta+r+1)/\beta}\Gamma\left(\frac{n+r+1}{\beta}+1\right). \qquad (7)$$

Similarly, the Bayesian estimator of θ under generalized Gamma prior is expected value of the posterior distribution given in (6) and is expressed as;

$$\hat{\theta}_{BGG}=\frac{\beta}{k_1}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\sum_{i=1}^{n}\ln x_i\right)^r\Gamma\left(\frac{n+r+1}{\beta}+\alpha\right). \qquad (8)$$

Empirical comparison of Bayesian estimator of parameter θ under various priors will be discussed in the following section.

5. Empirical Comparison

This section provides the empirical comparison of Bayesian estimators of the parameter θ under various prior distributions. The empirical study has been conducted by generating 10000 random samples of different sizes from the power function distribution by using various values of θ. After drawing the samples, the Bayes estimator has been computed by using different values of parameters of prior distribution. The mean square error has been computed to see the performance of the estimates. Results of the empirical study for various values of parameters of prior distribution have been given in tables below.

Table 1: Estimated Value of Parameter θ under various Prior Distribution for $\alpha = 2.5, \beta = 1.5 \, and \, \theta = 1.5$

n	Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{BG}$	$\hat{\theta}_{BW}$	$\hat{\theta}_{BGG}$
5	Estimated Value	1.873	1.699	2.415	2.203
	MSE	1.136	0.266	6.057	18.576
10	Estimated Value	1.665	1.645	8.012	26.167
	MSE	0.351	0.192	87.039	21.599
15	Estimated Value	1.603	1.599	14.003	27.678
	MSE	0.195	0.137	1.673	0.184
20	Estimated Value	1.579	1.580	1.524	2.843
	MSE	0.138	0.107	0.137	0.00005
25	Estimated Value	1.567	1.570	1.567	2.918
	MSE	0.110	0.090	0.00001	0.00003
30	Estimated Value	1.553	1.557	1.607	2.993
	MSE	0.087	0.074	0.00001	0.00002

Table 2: Estimated Value of Parameter θ under various Prior Distribution for $\alpha = 3.0, \beta = 2.0 \, and \, \theta = 2.5$

n	Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{BG}$	$\hat{\theta}_{BW}$	$\hat{\theta}_{BGG}$
5	Estimated Value	3.149	2.103	2.141	4.612
	MSE	3.158	0.196	0.0006	0.011
10	Estimated Value	2.789	2.267	2.777	5.709
	MSE	1.011	0.229	0.000006	0.922
15	Estimated Value	2.674	2.333	3.321	9.914
	MSE	0.551	0.209	0.030	3.125
20	Estimated Value	2.633	2.376	3.852	19.531
	MSE	0.391	0.189	0.386	2.146
25	Estimated Value	2.602	2.398	2.565	4.143
	MSE	0.290	0.163	0.105	0.992
30	Estimated Value	2.582	2.413	2.812	16.485
	MSE	0.239	0.148	0.458	0.129

Table 2: Estimated Value of Parameter θ under various Prior Distribution for $\alpha = 3.5, \beta = 2.5 \, and \, \theta = 3.0$

n	Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{BG}$	$\hat{\theta}_{BW}$	$\hat{\theta}_{BGG}$
5	Estimated Value	3.751	2.104	1.421	4.867
	MSE	4.439	0.123	0.000003	0.0005
10	Estimated Value	3.343	2.392	1.298	5.5.10
	MSE	1.416	0.179	0.000009	0.00002
15	Estimated Value	3.211	1.852	1.101	6.076
	MSE	0.790	0.100	0.000001	0.007
20	Estimated Value	3.164	2.636	3.599	6.669
	MSE	0.552	0.184	0.0000002	0.249
25	Estimated Value	3.127	2.695	3.909	8.603
	MSE	0.433	0.177	0.00004	1.378
30	Estimated Value	3.102	2.737	3.191	6.986
	MSE	0.343	0.163	0.0002	1.398

The empirical results have also been plotted below for immediate comparison. In these plots we have given the mean square errors of various estimators given in the above tables.

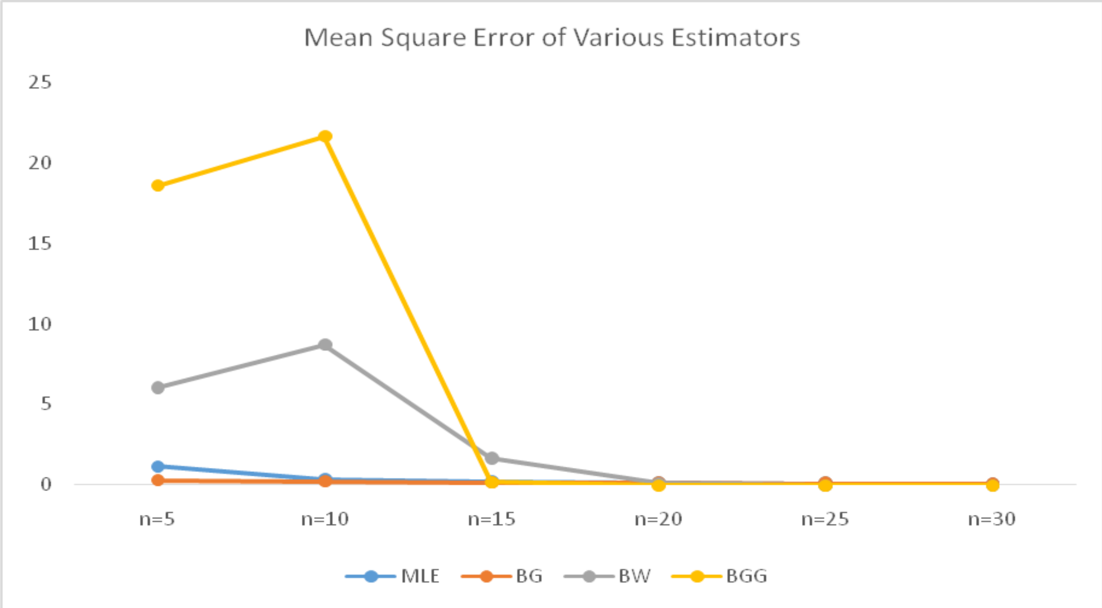

Figure 1: Mean Square Errors for $\alpha = 2.5, \beta = 1.5\, and\, \theta = 1.5$

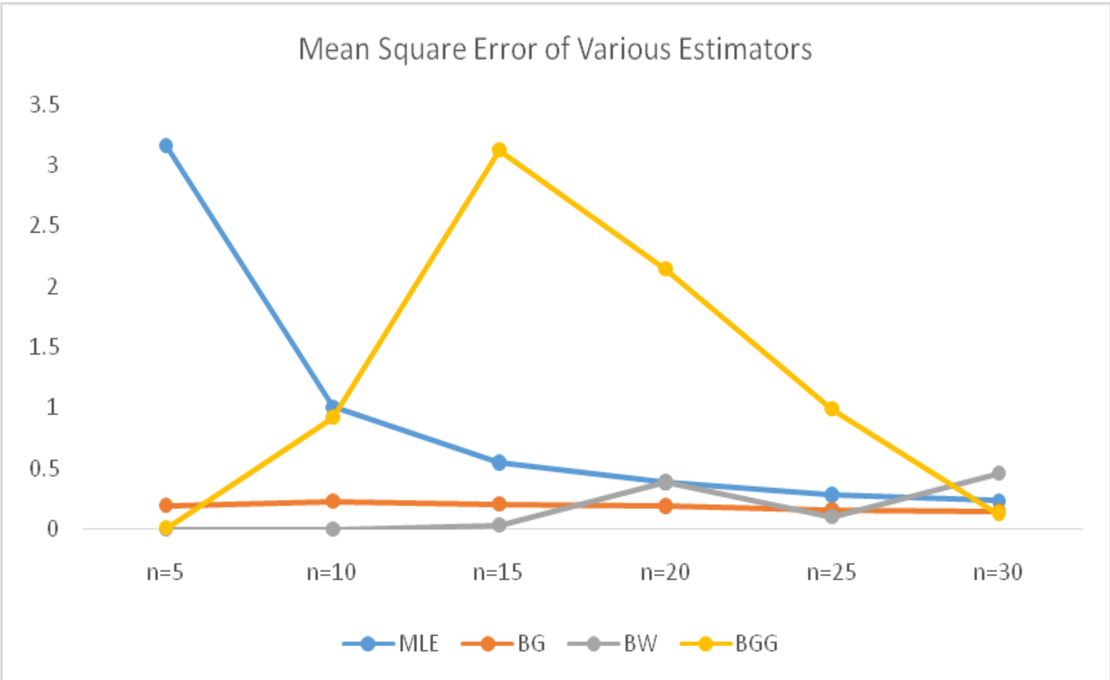

Figure 2: Mean Square Errors for $\alpha = 3.0, \beta = 2.0\, and\, \theta = 2.5$

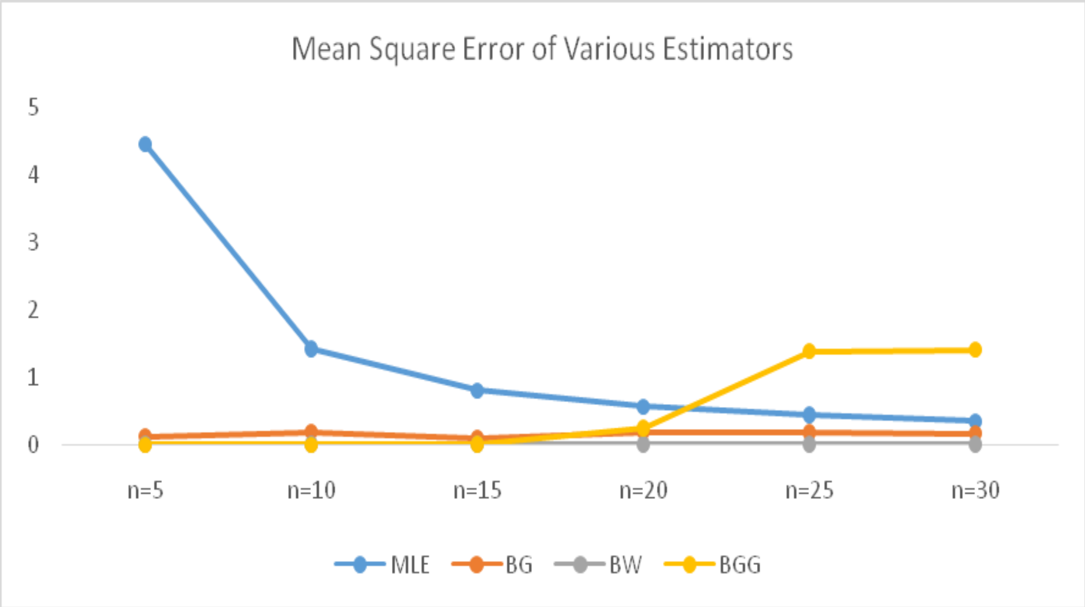

Figure 3: Mean Square Errors for $\alpha = 3.5, \beta = 2.5 \, and \, \theta = 3.0$

The results of empirical study are given in above tables. From the tables we can see that the Bayes estimate of the parameter θ under Weibull prior has better performance as compared to other methods as it has smallest mean square error.

6. Conclusion

The present paper studied Bayesian Estimation for parameters of Power function distribution under various priors. The authors worked out the Bayes estimators under two different priors – Weibull and Generalized Gamma Distribution. Monte Carlo simulation was used to compare the efficiency of estimators. Results of the findings revealed that the performance of Bayes estimator with Weibull prior outclass other competing method for $\theta = 3.0$ irrespective of sample size. For $\theta = 2.5$ Bayes estimator with Gamma prior performs better than the other estimators for large sample sizes. Again for small sample sizes the Bayes estimator with Weibull prior performed better as compared to other estimators. For $\theta = 1.5$ Bayes estimator with Gamma prior performs better for small sizes for large sample sizes the behavior pattern for all the estimators is the same.

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